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Cartan equivalence problem for third-order differential operators

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Abstract: This article is dedicated to solving the equivalence problem for a pair of third-order differential operators on the line under general fiber-preserving transformation using the Cartan method of equivalence. We will treat 2 versions of equivalence problems: first, the direct equivalence problem, and second, an equivalence problem to determine conditions on 2 differential operators such that there exists a fiber-preserving transformation mapping one to the other according to gauge equivalence.

Key words: Differential operator, Cartan equivalence, gauge equivalence, invariant, pseudogroup, Lie algebra

1. Introduction

The classification of linear differential equations is a special case of the general problem of classifying differential operators, which has a variety of important applications, including quantum mechanics and the projective geometry of curves [9]. The general equivalence problem is to recognize when 2 geometrical objects are mapped on each other by a certain class of diffeomorphisms. E. Cartan developed the general equivalence problem and provided a systematic procedure for determining the necessary and sufficient conditions [1, 2]. In Cartan's approach, the conditions of equivalence of 2 objects must be reformulated in terms of differential forms. We associate a collection of one-forms to an object under investigation in the original coordinates; the corresponding object in the new coordinates will have its own collection of one-forms. Once an equivalence problem has been reformulated in the proper Cartan form, in terms of a coframe ω on the m-dimensional base manifold M, along with a structure group $G \subset \mathrm{GL}(m)$, we can apply the Cartan equivalence method. The goal is to normalize the structure group valued coefficients in a suitably invariant manner, and this is accomplished through the determination of a sufficient number of invariant combinations thereof [9]. Kamran and Olver have solved the equivalence problem for second-order differential operator with 2 versions of the equivalence problem [7]. In this attempt, we shall solve the local equivalence problem by 2 versions of the equivalence problem for the class of linear third-order operators on the line. For simplicity, we shall only deal with the local equivalence problem for scalar differential operators in a single independent variable, although these problems are important for matrix-valued and partial differential operators, as well.

There are some recent works on solving equivalence problems on third-order ordinary differential equations [8, 10, 6]. The problems here are related to the more general equivalence problem for third-order ordinary differential equations, which Cartan studied under point transformations [3], while Chern turned his attention to the problem under contact transformations [4].

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2. Equivalence of third-order differential operators

Our starting point is a third-order differential operator:

$$\mathscr{D} = \sum_{i=0}^{3} f_i(x)\, D^i, \tag{2.1}$$

where f_0, f_1, f_2, and f_3 are analytic functions of the real variable x; $D^i = d^i/dx^i$; and $D^0 = \mathrm{Id}$ is the identity operator. Applying $\mathscr{D}$ on a scalar-valued function $u(x)$, we obtain the following expression:

$$\mathscr{D}[u] = \sum_{i=0}^{3} f_i(x)\, D^i u. \tag{2.2}$$

We discuss the problem of equivalence under general fiber-preserving transformations, which are linear in the dependent variable.

$$\bar{x} = \xi(x), \qquad \bar{u} = \varphi(x)\, u, \tag{2.3}$$

where $\varphi(x) \neq 0$. The total derivative operators are related by the chain rule formula:

$$\bar{D} = \frac{d}{d\bar{x}} = \frac{1}{\xi'(x)}\, \frac{d}{dx} = \frac{1}{\xi'(x)}\, D. \tag{2.4}$$

We first consider the *direct equivalence problem*, which identifies the 2 linear differential functions

$$\sum_{i=0}^{3} f_i(x)\, D^i u = \mathscr{D}[u] = \bar{\mathscr{D}}[\bar{u}] = \sum_{i=0}^{3} \bar{f}_i(\bar{x})\, \bar{D}^i \bar{u}. \tag{2.5}$$

under change of variables (2.3). This induces the transformation rule

$$\bar{\mathscr{D}} = \mathscr{D} \cdot \frac{1}{\varphi(x)} \qquad \text{when} \qquad \bar{x} = \xi(x) \tag{2.6}$$

on the differential operators themselves, and we try to find explicit conditions on the coefficients of the 2 differential operators that guarantee that they satisfy (2.5) for some change of variables of the form (2.3).

The transformation rule (2.6) preserves neither the eigenvalue problem $\mathscr{D}[u] = \lambda u$ nor the Schrödinger equation $iu_t = \mathscr{D}[u]$, since we are missing a factor of $\varphi(x)$. To rectify this problem, we need to multiply by $\varphi(x)$ and use the *gauge equivalence* with the following transformation rule:

$$\bar{\mathscr{D}} = \varphi(x) \cdot \mathscr{D} \cdot \frac{1}{\varphi(x)} \qquad \text{when} \qquad \bar{x} = \xi(x). \tag{2.7}$$

In quantum mechanics, equivalence plays an important role since it preserves the solution set to the associated Schrödinger equation, or its stationary counterpart, the eigenvalue problem.

The appropriate space to work in will be the third jet space J^3, which has local coordinates $\Upsilon = \{(x, u, p, q, r) \in \mathrm{J}^3 : p = u_x, q = u_{xx}, r = u_{xxx}\}$, and our goal is to construct an appropriate coframe on J^3, which will encode the relevant transformation rules to our problem. Note first that a point transformation will be in the desired linear form (2.3) if and only if, for some pair of functions α, β, one-form equations

$$d\bar{x} = \alpha\, dx, \tag{2.8}$$

$$\frac{d\bar{u}}{\bar{u}} = \frac{du}{u} + \beta\, dx \tag{2.9}$$

hold on the subset of J^3 where $u \neq 0$. Indeed, (2.8) implies that $\bar{x} = \xi(x)$, with $\alpha = \xi_x$, while (2.9) necessarily requires $\bar{u} = \varphi(x)u$, with $\beta = \varphi_x/\varphi$.

Second, in order that the derivative coordinates p, q, and r transform correctly, we must prolong the transformation (2.3) and need to preserve the contact ideal $\mathcal{I} = \langle du - p\, dx, dp - q\, dx, dq - r\, dx\rangle$ on J^3 as the following form:

$$d\bar{u} - \bar{p}\, d\bar{x} = a_1(du - p\, dx), \tag{2.10}$$

$$d\bar{p} - \bar{q}\, d\bar{x} = a_2(du - p\, dx) + a_3(dp - q\, dx), \tag{2.11}$$

$$d\bar{q} - \bar{r}\, d\bar{x} = a_4(du - p\, dx) + a_5(dp - q\, dx) + a_6(dq - r\, dx), \tag{2.12}$$

where a_{ij}s are functions on J^3. The combination of the first contact condition (2.10) with the linearity conditions (2.8) and (2.9) constitutes part of an overdetermined equivalence problem. Since

$$\begin{pmatrix} d\bar{x} \\ d\bar{u} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ \dfrac{\frac{p}{u} + \beta}{\alpha} & \dfrac{1}{\lambda u} \end{pmatrix}}_{2\times 2 \text{ matrix } A} \begin{pmatrix} \alpha dx \\ \lambda(du - p\, dx) \end{pmatrix}, \tag{2.13}$$

we determine that the entries of A matrix in (2.13) are invariant for the overdetermined problem, and therefore we should normalize $\beta = -p/u,\ \lambda = 1/u$. Hence, the one-form

$$\frac{d\bar{u} - \bar{p}\, d\bar{x}}{\bar{u}} = \frac{du - p\, dx}{u} \tag{2.14}$$

is invariant, and (2.14) be replaced with both (2.9) and (2.10). Therefore, we may choose 4 elements of our coframe as form

$$\omega^1 = dx, \quad \omega^2 = \frac{du - p\, dx}{u}, \quad \omega^3 = dp - q\, dx, \quad \omega^4 = dq - r\, dx, \tag{2.15}$$

which are defined on the third jet space J^3 locally parameterized by (x, u, p, q, r), with the transformation rules

$$\bar{\omega}^1 = a_1\omega^1, \quad \bar{\omega}^2 = \omega^2, \quad \bar{\omega}^3 = a_2\omega^2 + a_3\omega^3, \quad \bar{\omega}^4 = a_4\omega^2 + a_5\omega^3 + a_6\omega^4, \tag{2.16}$$

where a_i with $i = 1, \cdots, 6$ are functions on J^3. According to (2.5), the function $I(x, u, p, q, r) = \mathscr{D}[u] = f_3(x)r + f_2(x)q + f_1(x)p + f_0(x)u$ is an invariant for the problem, and thus its differential,

$$\omega^5 = dI = f_3 dr + f_2 dq + f_1 dp + f_0 du + (f_3' r + f_2' q + f_1' p + f_0' u)dx, \tag{2.17}$$

is an invariant one-form. We thus take it as the final element of our coframe.

In the second problem (2.7), for the extra factor of φ, the invariant is

$$I(x,u,p,q,r)=\frac{\mathscr{D}[u]}{u}=\frac{f_3(x)r+f_2(x)q+f_1(x)p}{u}+f_0(x). \tag{2.18}$$

Thus, we take

$$\omega^5=dI=\frac{f_3}{u}\,dr+\frac{f_2}{u}\,dq+\frac{f_1}{u}\,dp-\frac{f_3r+f_2q+f_1p}{u^2}\,du+\left\{\frac{f_3'r+f_2'q+f_1'p}{u}+f_0'\right\}dx \tag{2.19}$$

as a final element of the coframe for the second equivalence problem (2.7). In both cases, the set of one-forms $\{\omega^1,\omega^2,\omega^3,\omega^4,\omega^5\}$ is a coframe on the subset

$$\Omega^*=\Big\{(x,u,p,q,r)\in \mathrm{J}^3\,\Big|\,u\neq 0 \text{ and } f_3(x)\neq 0\Big\}. \tag{2.20}$$

We restrict our attention to a connected component $\Omega\subset\Omega^*$ of subset (2.20) such that the signs of $f_0(x)$ and u are fixed. In both the first and second problems, since $\omega^5=dI$ is a closed invariant one-form, the last coframe element agrees up to

$$\bar{\omega}^5=\omega^5. \tag{2.21}$$

Proposition 2.1 *Suppose $\mathscr{D}$ and $\bar{\mathscr{D}}$ are third-order differential operators. Let $\{\omega^1,\omega^2,\omega^3,\omega^4,\omega^5\}$, and $\{\bar{\omega}^1,\bar{\omega}^2,\bar{\omega}^3,\bar{\omega}^4,\bar{\omega}^5\}$ be the corresponding coframes, on open subsets Ω and $\bar{\Omega}$ of the third jet space, given by (2.15) and (2.17) or (2.19), the choice of ω^5 and $\bar{\omega}^5$ depending on the equivalence problem under consideration. The differential operators are equivalent under the pseudogroup (2.3) according to the respective transformation rules (2.6) and (2.7) if and only if there is a diffeomorphism Φ that satisfies*

$$\Phi^*(\bar{\omega}_i)=\sum_{j=1}^{5} g_{ij}\,\omega_j, \tag{2.22}$$

for $i=1,\cdots,5$, where $g=(g_{ij})$ is a G–valued function on J^3,

$$G=\left\{\begin{pmatrix} a_1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & a_2 & a_3 & 0 & 0\\ 0 & a_4 & a_5 & a_6 & 0\\ 0 & 0 & 0 & 0 & 1\end{pmatrix} : a_i\in\mathbb{R};\ i=1,\cdots,6,\quad a_1a_3a_6\neq 0\right\} \tag{2.23}$$

and Φ^* denotes the pull-back map on differential forms.

In order to apply Cartan's reduction algorithm for direct equivalence and gauge equivalence problems so as to prescribe invariant normalizations of the 6 group parameters $a_1,a_2,\cdots,a_6$, we must lift coframes to the space

$J^3 \times G$. The *lifted coframes* have the forms

$$\begin{aligned}
\theta^1 &= a_1\omega^1, \\
\theta^2 &= \omega^2, \\
\theta^3 &= a_2\omega^2 + a_3\omega^3, \\
\theta^4 &= a_4\omega^2 + a_5\omega^3 + a_6\omega^4, \\
\theta^5 &= \omega^5.
\end{aligned} \tag{2.24}$$

3. The direct equivalence problem

Theorem 3.1 *The final structure equations for the direct equivalence problem with (2.15) and (2.17) coframes are*

$$\begin{aligned}
d\theta^1 &= \frac{1}{3}\,\theta^1 \wedge \theta^2, \\
d\theta^2 &= \theta^1 \wedge \theta^3, \\
d\theta^3 &= \theta^1 \wedge \theta^4 + \frac{1}{3}\theta^2 \wedge \theta^3, \\
d\theta^4 &= -I\theta^1 \wedge \theta^2 + \frac{1}{9}I_2\theta^1 \wedge \theta^3 + I_1\theta^1 \wedge \theta^4 + \theta^1 \wedge \theta^5 + \frac{2}{3}\theta^2 \wedge \theta^4, \\
d\theta^5 &= 0,
\end{aligned} \tag{3.1}$$

where the functions I, I_1*, and* I_2 *are*

$$\begin{aligned}
I &= f_3r + f_2q + f_1p + f_0u, \\
I_1 &= \frac{(f_3' - f_2)u - 2f_3p}{3\sqrt[3]{(f_3u)^2}}, \\
I_2 &= \frac{1}{f_3u\sqrt[3]{f_3u}}\Big[(3f_3'f_2 + 3f_3f_3'' - 4f_3'^2 - 9f_1f_3)u^2 + 5f_3^2p^2 - 24f_3^2uq + (7f_3'f_3 - 15f_3f_2)up\Big],
\end{aligned} \tag{3.2}$$

the fundamental structure invariants of the problem.

Proof First, we take the initial 4 one-forms (2.15) and (2.17) as our final coframe constituents. The next step is to calculate the differentials of lifted coframe elements (2.24). An explicit computation leads to the structure equations

$$\begin{aligned}
d\theta^1 &= \alpha^1 \wedge \theta^1, \\
d\theta^2 &= T^2_{12}\theta^1 \wedge \theta^2 + T^2_{13}\theta^1 \wedge \theta^3, \\
d\theta^3 &= \alpha^2 \wedge \theta^2 + \alpha^3 \wedge \theta^3 + T^3_{12}\theta^1 \wedge \theta^2 + T^3_{13}\theta^1 \wedge \theta^3 + T^3_{14}\theta^1 \wedge \theta^4, \\
d\theta^4 &= \alpha^4 \wedge \theta^2 + \alpha^5 \wedge \theta^3 + \alpha^6 \wedge \theta^4 + T^4_{12}\theta^1 \wedge \theta^2 + T^4_{13}\theta^1 \wedge \theta^3 + T^4_{14}\theta^1 \wedge \theta^4 + T^4_{15}\theta^1 \wedge \theta^5, \\
d\theta^5 &= 0,
\end{aligned} \tag{3.3}$$

with

$$\alpha^1 = \frac{da_1}{a_1}, \qquad \alpha^2 = \frac{a_3 da_2 - a_2 da_3}{a_3}, \qquad \alpha^3 = \frac{da_3}{a_3},$$
$$\alpha^4 = \frac{a_3 a_6 da_4 + (a_2 a_5 - a_3 a_4) da_6 - a_2 a_6 da_5}{a_3 a_6}, \quad \alpha^5 = \frac{a_3 da_5 - a_5 da_6}{a_3 a_6}, \quad \alpha^6 = \frac{da_6}{a_6},$$

forming a basis for the right-invariant *Maurer–Cartan forms* on the Lie group G. The torsion coefficients in the structure equations (3.3) are explicitly given by

$$\begin{aligned}
T^2_{12} &= -\frac{a_2 + a_3 p}{a_1 a_3 u}, \quad T^2_{13} = \frac{1}{a_1 a_3 u}, \quad T^3_{12} = -\frac{a_2 a_6 a_3 p + a_2^2 a_6 - a_2 a_5 a_3 u + a_3^2 a_4 u}{a_1 a_3 a_6 u},\\
T^3_{13} &= \frac{a_2 a_6 - a_3 a_5 u}{a_1 a_3 a_6 u}, \quad T^3_{14} = \frac{a_3}{a_1 a_6},\\
T^4_{12} &= -\frac{(a_3 a_4 a_6 p + a_2 a_4 a_6 + a_4 a_3 a_5 - a_2 a_5^2) f_3 + (a_2 a_5 a_6 - a_4 a_3 a_6) f_2 - a_2 a_6^2 f_1 + a_6^2 a_3 u^2 f_0}{a_1 a_3 a_6 f_3 u},\\
T^4_{13} &= \frac{(a_4 a_6 - a_5^2 u) f_3 + a_5 a_6 u f_2 - a_6^2 u f_1}{a_1 a_3 a_6 f_3 u}, \quad T^4_{14} = \frac{a_5 f_3 - a_6 f_2 - a_6 a_9}{a_1 a_6 f_3}, \quad T^4_{15} = \frac{a_6}{a_1 f_3}.
\end{aligned} \tag{3.4}$$

In the absorption process, we replace each Maurer–Cartan form in the structure equations with a linear combination of coframe elements, with $\alpha^\kappa \mapsto \sum_{j=1}^5 z^\kappa_j \theta^j$, where coefficients z^κ_j are allowed to depend on both the base variables x, u, p, q, r and the group parameters $a_1, a_2, \ldots, a_6$. Therefore, we have

$$\begin{aligned}
\Theta^1 &= -z^1_2 \theta^1 \wedge \theta^2 - z^1_3 \theta^1 \wedge \theta^3 - z^1_4 \theta^1 \wedge \theta^4 - z^1_5 \theta^1 \wedge \theta^5,\\
\Theta^2 &= T^2_{12} \theta^1 \wedge \theta^2 + T^2_{13} \theta^1 \wedge \theta^3,\\
\Theta^3 &= (z^2_1 + T^3_{12}) \theta^1 \wedge \theta^2 + (z^3_1 + T^3_{13}) \theta^1 \wedge \theta^3 + T^3_{14} \theta^1 \wedge \theta^4 + (z^3_2 - z^2_3) \theta^2 \wedge \theta^3\\
&\quad - z^2_4 \theta^2 \wedge \theta^4 - z^3_4 \theta^3 \wedge \theta^4 - z^2_5 \theta^2 \wedge \theta^5 - z^3_5 \theta^3 \wedge \theta^5,\\
\Theta^4 &= (z^4_1 + T^4_{12}) \theta^1 \wedge \theta^2 + (z^5_1 + T^4_{13}) \theta^1 \wedge \theta^3 + (z^6_1 + T^4_{14}) \theta^1 \wedge \theta^4 + (z^5_2 - z^4_3) \theta^2 \wedge \theta^3\\
&\quad + (z^6_2 - z^4_4) \theta^2 \wedge \theta^4 + (z^6_3 - z^5_4) \theta^3 \wedge \theta^4 + T^4_{15} \theta^1 \wedge \theta^5 - z^4_5 \theta^2 \wedge \theta^5 - z^5_5 \theta^3 \wedge \theta^5 - z^6_5 \theta^4 \wedge \theta^5,\\
\Theta^5 &= 0.
\end{aligned} \tag{3.5}$$

Some coefficients of $\theta^j \wedge \theta^k$ in (3.5) that happen to be independent of the parameters z^κ_j are invariants of the problem, and so one can normalize to reduce the structure group. In the above, the essential torsion components are $T^2_{12}, T^2_{13}, T^3_{14}, T^4_{15}$, as given in (3.4), which is able to absorb all the torsion components except them. By direct inspection of the structure equations (3.3), we deduce that any torsion component in $d\theta^2$ is essential because there are no Maurer–Cartan forms in it and since the Maurer–Cartan forms in $d\theta^3$ multiply either θ^2, or θ^3, and $d\theta^4$ multiply θ^2, θ^3, and θ^4, they can never produce a multiple of the two-form $\theta^1 \wedge \theta^4$ and $\theta^1 \wedge \theta^5$ upon replacement, respectively.

Since the essential torsion coefficients all depend on the group parameters, the next step in the process is to normalize them to as simple a form as possible. We first normalize $T^2_{12} = 0$ by setting $a_2 = -a_3 p$, thereby eliminating the group parameter a_2. Second, we can normalize $T^3_{14} = T^4_{15} = 1$ by setting $a_3 = a_1 a_6$, $a_6 = a_1 f_3$. With these 3 normalizations, the fourth essential torsion coefficient becomes $T^2_{13} = 1/(a_1^3 f_3 u)$. By assumption

$f_3(x)u \neq 0$, and because of using real-valued functions, $T^2_{13} = 1$. Therefore, we normalize $a_1 = (f_3u)^{-1/3}$. The group parameter normalizations are

$$a_1 = \frac{1}{\sqrt[3]{f_3u}}, \quad a_2 = -\sqrt[3]{\frac{f_3}{u^2}}\, p, \quad a_3 = \sqrt[3]{\frac{f_3}{u^2}}, \quad a_6 = \sqrt[3]{\frac{f_3}{u}}. \tag{3.6}$$

Now substituting normalizations (3.6) in the lifted coframe (2.24), in the second loop through the equivalence procedure, we calculate the differentials of the new invariant coframe, and so the revised structure equations are

$$\begin{aligned}
d\theta^1 &= \frac{1}{3}\,\theta^1 \wedge \theta^2,\\
d\theta^2 &= \theta^1 \wedge \theta^3,\\
d\theta^3 &= T^3_{12}\theta^1 \wedge \theta^2 + T^3_{13}\theta^1 \wedge \theta^3 + \theta^1 \wedge \theta^4 + \frac{1}{3}\theta^2 \wedge \theta^3,\\
d\theta^4 &= \alpha^1 \wedge \theta^2 + \alpha^2 \wedge \theta^3 + T^4_{12}\theta^1 \wedge \theta^2 + T^4_{13}\theta^1 \wedge \theta^3 + T^4_{14}\theta^1 \wedge \theta^4 + T^4_{23}\theta^2 \wedge \theta^3\\
&\qquad -\frac{1}{3}\theta^2 \wedge \theta^4 + \theta^1 \wedge \theta^5,\\
d\theta^5 &= 0,
\end{aligned} \tag{3.7}$$

where α^1 and α^2 are the Maurer–Cartan forms on the structure group G. The essential torsion coefficients are

$$T^3_{12} = -a_4 - \sqrt[3]{\frac{f_3^2}{u}}\, q, \quad T^3_{13} = \frac{uf_3' - 5pf_3}{3\sqrt[3]{(f_3u)^2}} - a_5, \quad T^4_{14} = a_5 + \frac{2f_3'u - 3f_2u - f_3p}{3\sqrt[3]{(f_3u)^2}}. \tag{3.8}$$

Since in (3.7) the other torsion coefficients can be absorbed by the Maurer–Cartan forms, we just normalize the essential torsion coefficients (3.8) and we find the following parameters:

$$a_4 = -\sqrt[3]{\frac{f_3^2}{u}}\, q, \qquad a_5 = \frac{f_3'u - 5f_3p}{3\sqrt[3]{(f_3u)^2}}. \tag{3.9}$$

The normalizations (3.9) have the effect of reducing the original structure group G to a one-parameter subgroup and we have finally normalized all the group parameters. Inserting their prescribed values (3.6) and (3.9) into (2.24), the invariant coframe is now given by:

$$\begin{aligned}
\theta^1 &= \frac{dx}{\sqrt[3]{f_3u}},\\
\theta^2 &= \frac{du - p\,dx}{u},\\
\theta^3 &= \sqrt[3]{\frac{f_3}{u^2}}\Big[(dp - q\,dx) - \frac{p}{u}(du - p\,dx)\Big],\\
\theta^4 &= -\sqrt[3]{\frac{f_3^2}{u}}\big(\frac{qdu - pq\,dx}{u}\big) + \frac{f_3'u - 5f_3p}{3\sqrt[3]{(f_3u)^2}}\big(dp - q\,dx\big) + \sqrt[3]{\frac{f_3}{u}}\big(dq - r\,dx\big),\\
\theta^5 &= f_3dr + f_2dq + f_1dp + f_0du + (f_3'r + f_2'q + f_1'p + f_0'u)dx.
\end{aligned} \tag{3.10}$$

Therefore, the structure equations (3.1) with fundamental invariants coefficients (3.2) are obtained. □

In a local coordinate Υ on J^3, the coframe can be written in terms of the coordinate coframe, such that $\theta^i = \sum_j a^i_j(x)dx^j$, where $A = (a^i_j(x))$ is a nonsingular $m\times m$ matrix of functions. The differential of a function can be rewritten in the coframe-adapted form:

$$dF = \sum_{j=1}^{5} \frac{\partial F}{\partial \theta^j}\theta^j, \tag{3.11}$$

where one will refer to the resulting coefficients $\partial F/\partial\theta^j = \sum_i b^i_j(x)\partial F/\partial x^i$ as *coframe derivatives* of F where $B = (b^i_j(x)) = A^{-1}$. Comparing (3.11) and the formulae (3.10) for the invariant coframe, we find that coframe derivatives of F are given explicitly by

$$\begin{aligned}
\frac{\partial F}{\partial\theta^1} &= \sqrt[3]{f_3 u}\,\widehat{D}_x F,\\
\frac{\partial F}{\partial\theta^2} &= u\frac{\partial F}{\partial u} + p\frac{\partial F}{\partial p} + q\frac{\partial F}{\partial q} - \Big(\frac{f_2 q + f_1 p + f_0 u}{f_3}\Big)\frac{\partial F}{\partial r},\\
\frac{\partial F}{\partial\theta^3} &= \frac{u}{\sqrt[3]{f_3 u}}\frac{\partial F}{\partial p} + \Big(\frac{5pf_3 - f_3' u}{3f_3\sqrt[3]{f_3 u}}\Big)\frac{\partial F}{\partial q} - \Big(\frac{5f_2 f_3 p - f_2 f_3' u + 3f_1 f_3 u}{3f_3^2\sqrt[3]{f_3 u}}\Big)\frac{\partial F}{\partial r},\\
\frac{\partial F}{\partial\theta^4} &= \frac{u}{\sqrt[3]{(f_3 u)^2}}\frac{\partial F}{\partial q} - \frac{f_2 u}{f_3\sqrt[3]{(f_3 u)^2}}\frac{\partial F}{\partial r},\\
\frac{\partial F}{\partial\theta^5} &= \frac{1}{f_3}\frac{\partial F}{\partial r},
\end{aligned} \tag{3.12}$$

where

$$\widehat{D}_x = \frac{\partial}{\partial x} + p\frac{\partial}{\partial u} + q\frac{\partial}{\partial p} + r\frac{\partial}{\partial q} + R\frac{\partial}{\partial r}, \quad R = -\frac{f_2 r + f_1 q + f_0 p + f_3' r + f_2' q + f_1' p + f_0' u}{f_3}.$$

The Jacobi identities for the coframe derivatives are found by reapplying the exterior derivative to the structure equations (3.1). An easy calculation shows that $d^2\theta^1 = d^2\theta^2 = d^2\theta^3 = 0$ automatically, while the identity $d^2\theta^4 = 0$ implies the following syzygy among our fundamental invariants and the derived invariants:

$$\frac{\partial I_1}{\partial\theta^2} = -I_1, \qquad \frac{\partial I_1}{\partial\theta^3} = 2, \qquad \frac{\partial I_2}{\partial\theta^2} = \frac{2}{3}I_2, \qquad \frac{\partial I_2}{\partial\theta^3} = 15I_1, \qquad \frac{\partial I_2}{\partial\theta^4} = -24,$$

and also we have $dI = \theta^5$, meaning that $\partial I/\partial\theta^5 = 1$ and $\partial I/\partial\theta^i = 0$ for $i = 1,2,3,4$.

4. The gauge equivalence problem

Theorem 4.1 *The final structure equations for gauge equivalence with (2.15) and (2.19) coframes are*

$$\begin{aligned}
d\theta^1 &= 0,\\
d\theta^2 &= \theta^1\wedge\theta^3,\\
d\theta^3 &= \theta^1\wedge\theta^4,\\
d\theta^4 &= I_1\theta^1\wedge\theta^3 + I_2\theta^1\wedge\theta^4 + \theta^1\wedge\theta^5,\\
d\theta^5 &= 0,
\end{aligned} \tag{4.1}$$

where the coefficients I_1 *and* I_2,

$$
\begin{aligned}
I_1 &= \frac{(f_3 f_3'' - 3f_1 f_3 + f_2 f_3' - \frac{4}{3} f_3'^2)u + 3(f_3 f_3' - 2f_2 f_3)p - 9f_3^2 q}{3f_3 \sqrt[3]{f_3}\, u}, \\
I_2 &= \frac{f_3' u - 3f_3 p - f_2 u}{3\sqrt[3]{f_3^2}\, u},
\end{aligned} \tag{4.2}
$$

are the fundamental invariants of the problem.

Proof We determine the solution to the problem of gauge equivalence of third-order differential operators by a similar computation of the previous section. The Cartan formulation of this problem will use the same initial 4 one-forms (2.15), but now the final coframe element is (2.19). In the first loop through the second equivalence problem procedure, the structure group (2.23) is exactly the structure group of direct equivalence, and then the equivalence method has the same intrinsic structure (3.3) by the essential torsion coefficients

$$
T^2_{12} = -\frac{a_2 + a_3 p}{a_1 a_3 u}, \quad T^2_{13} = \frac{1}{a_1 a_3 u}, \quad T^3_{14} = \frac{a_3}{a_1 a_6}, \quad T^4_{15} = \frac{a_6 u}{a_1 f_3}. \tag{4.3}
$$

One can normalize the group parameters by setting

$$
a_1 = \frac{1}{\sqrt[3]{f_3}}, \quad a_2 = -\sqrt[3]{\frac{f_3}{u^3}}\, p, \quad a_3 = \sqrt[3]{\frac{f_3}{u^3}}, \quad a_6 = \sqrt[3]{\frac{f_3^2}{u^3}}. \tag{4.4}
$$

In the second loop of the present equivalence problem, we substitute the normalizations (4.4) in lifted coframe (2.22) and calculate differentials of the new invariant coframe to obtain revised structure equations. Now we normalize the essential torsion components (4.3) by the remaining parameters

$$
a_4 = -\sqrt[3]{\frac{f_3^2}{u^3}}\, q, \qquad a_5 = \frac{f_3' u - 6 f_3 p}{3u \sqrt[3]{f_3^2}}. \tag{4.5}
$$

Thus, the final invariant coframe is now given by

$$
\begin{aligned}
\theta^1 &= \frac{dx}{\sqrt[3]{f_3}}, \\
\theta^2 &= \frac{du - p\, dx}{u}, \\
\theta^3 &= \frac{\sqrt[3]{f_3}}{u^2}\Big[(p^2 - qu)\, dx - p\, du + u\, dp\Big], \\
\theta^4 &= -\frac{1}{3\sqrt[3]{f_3}\, u^3}\Big[(3f_3 u^2 r + f_3' u^2 q - f_3' u p^2 - 9 f_3 u p q + 6 f_3 p^3)\, dx + \\
&\qquad (f_3' u p + 3 f_3 u q - 6 f_3 p^2)\, du + (6 f_3 p - f_3' u) u\, dp - 3 f_3 u^2\, dq\Big], \\
\theta^5 &= \frac{f_3' r + f_2' q + f_1 p + f_0' u}{u}\, dx - \frac{f_3 r + f_2 q + f_1 p}{u^2}\, du + \frac{f_1}{u}\, dp + \frac{f_2}{u}\, dq + \frac{f_3}{u} dr.
\end{aligned} \tag{4.6}
$$

Then the final structure equations (4.1) with fundamental invariant coefficients (4.1) are obtained. □

Note that the original invariant I, given in (2.18), does not appear among the structure functions of the adapted coframe. Nor can it appear among the derived invariants, since the coframe derivatives are

$$\begin{aligned}
\frac{\partial F}{\partial \theta^1} &= \sqrt[3]{f_3}\,\widehat{D}_x F,\\
\frac{\partial F}{\partial \theta^2} &= u\frac{\partial F}{\partial u} + p\frac{\partial F}{\partial p} + q\frac{\partial F}{\partial q} + r\frac{\partial F}{\partial r},\\
\frac{\partial F}{\partial \theta^3} &= \frac{u}{\sqrt[3]{f_3}}\frac{\partial F}{\partial p} + \Big(\frac{6f_3p - f_3'u}{3f_3\sqrt[3]{f_3}}\Big)\frac{\partial F}{\partial q} + \Big(\frac{(f_2f_3' - 3f_1f_3)u - 6f_2f_3p}{3f_3^2\sqrt[3]{f_3}}\Big)\frac{\partial F}{\partial r},\\
\frac{\partial F}{\partial \theta^4} &= \frac{u}{\sqrt[3]{f_3^2}}\frac{\partial F}{\partial q} - \frac{f_2u}{f_3\sqrt[3]{f_3^2}}\frac{\partial F}{\partial r},\\
\frac{\partial F}{\partial \theta^5} &= \frac{u}{f_3}\frac{\partial F}{\partial r},
\end{aligned} \tag{4.7}$$

where

$$\widehat{D}_x = \frac{\partial}{\partial x} + p\frac{\partial}{\partial u} + q\frac{\partial}{\partial p} + r\frac{\partial}{\partial q} + R\frac{\partial}{\partial r}, \tag{4.8}$$

$$R = -\frac{f_2ru + f_1qu - f_1p^2 - f_2pq - f_3pr + f_3'ru + f_2'qu + f_1'p + f_0'u^2}{f_3u}. \tag{4.9}$$

The identity $d^2\theta^4 = 0$ leads to the following syzygy among fundamental invariants and the derived invariants:

$$\frac{\partial I_1}{\partial \theta^4} = -\frac{\partial I_2}{\partial \theta^3} = -3, \qquad \frac{\partial I_1}{\partial \theta^3} = -2I_2. \tag{4.10}$$

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