

Attractors for parabolic problems in weighted spaces

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Abstract: The purpose of this paper is to investigate the asymptotic behavior of the solutions of parabolic equations with singular initial data in weighted spaces $L^r_{\delta(x)}(\Omega)$ where $\delta(x)$ is the distance to the boundary. We first establish the existence of the attractor for that equation in $L^r_{\delta(x)}(\Omega)$ and then show the existence of the exponential attractor in $L^2_{\delta(x)}(\Omega)$. In contrast to our previous results, we get the existence of attractors in weak topology spaces.

Key words: Attractor, parabolic equation, singular initial data, weighted space

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a sufficiently smooth boundary. Consider the following nonlinear reaction-diffusion equation:

$$\begin{cases} u_t - \Delta u + f(x, u) = g(x) & \text{in } \Omega, \quad t > 0, \\ u = 0 & \text{on } \partial\Omega, \\ u(0) = u_0, \end{cases} \quad (1.1)$$

where the nonlinear term $f(x, u) \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ satisfies

$$f(x, 0) = 0, \quad (1.2)$$

$$|f(x, u) - f(x, v)| \leq C |a(x)| |u - v| (|u|^{\rho-1} + |v|^{\rho-1} + 1) \quad (1.3)$$

with $\rho > 1$ and $a(x) \in L^\beta_{\delta(x)}(\Omega)$, $\beta > 1$. For $1 \leq r < \infty$, the weighted Lebesgue spaces $L^r_{\delta(x)}(\Omega)$ are defined by

$$L^r_{\delta} := L^r_{\delta(x)}(\Omega) := L^r(\Omega; \delta(x)dx),$$

where $\delta(x) = \text{dist}(x, \partial\Omega)$. Evidently, $L^r_{\delta(x)}(\Omega)$ is a Banach space endowed with the norm

$$\|u\|_{L^r_{\delta}} = \left(\int_{\Omega} |u|^r \delta(x) dx \right)^{\frac{1}{r}}.$$

Suppose that $g(x) \in L^r_{\delta(x)}(\Omega)$.

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Elliptic and parabolic problems with nonlinearity analogous with that of (1.1) have drawn much attention, and a review of these studies was given in [13]. In contrast to the regular data problems, the well-posedness of evolution equations with singular initial data can be obtained in critical cases. Thus, it plays an important role in the study of evolution equations, especially in the long-time behavior of them. There has been a great deal of study done on parabolic problems with singular initial data. Authors in [6] studied problem (1.1) with $f(x, u) = -|u|^{p-1}u$ and $g(x) = 0$, where $p > 1$ and $\Omega \subset \mathbb{R}^N$ is a bounded domain. They obtained that if $r \geq \frac{N}{2}(p-1)$ and $u_0 \in L^r(\Omega)$, there exists a unique solution $u \in C([0, T], L^r(\Omega)) \cap L_{loc}^\infty((0, T), L^\infty(\Omega))$. Further study was done in [10]. More precisely, the authors in [10] obtained that if $r \geq \frac{N+1}{2}(p-1)$ and $u_0 \in L_{\delta(x)}^r(\Omega)$, there exists a unique solution $u \in C([0, T], L_{\delta(x)}^r(\Omega)) \cap L_{loc}^\infty((0, T), L^\infty(\Omega))$ of (1.1) with $f(x, u) = -|u|^{p-1}u$ and $g(x) = 0$. In [1], the authors studied the abstract parabolic problem

$$\begin{cases} u_t = Au + f(t, u), & t > t_0, \\ u(t_0) = u_0, \end{cases} \tag{1.4}$$

where the linear operator $A : D(A) \subset X^0 \rightarrow X^0$ satisfies that $-A$ is a sectorial operator in the Banach space X^0 , and $f(t, u)$ satisfies some local Lipschitz condition. They obtained that there exists a unique solution (ϵ -regular mild solution) $u(t) \in C([t_0, \tau]; X^1) \cap C((t_0, \tau]; X^{1+\epsilon})$ of (1.4), where X^α ($\alpha > 0$) is the fractional power spaces associated to A , and they applied their abstract results to (1.1) with $f(x, u) = -|u|^{p-1}u$ and $g(x) = 0$. If $f(x, u) = -a(x)u^q - b(x)u^p$, $0 < q \leq 1 < p$, $a(x) \in L^\alpha(\Omega)$, $b(x) \in L^\beta(\Omega)$, $\alpha, \beta \geq 1$, and $g(x) = 0$, it has been proven in [14] that there exists a unique positive solution $u \in C([0, T]; L^r(\Omega)) \cap L_{loc}^\infty((0, T); L^\infty(\Omega))$ of (1.1) with $u_0 \in L^r(\Omega)$, $1 \leq r < \infty$, and $u_0 \geq \gamma\delta(x)$, where γ is a positive constant.

In this paper, we establish the existence of a global attractor for (1.1) in $L_{\delta(x)}^r(\Omega)$ and exponential attractor for (1.1) in $L_{\delta(x)}^2(\Omega)$. Based on the well-posedness of (1.1) obtained in [12] and under some dissipative conditions, we show that the semigroup corresponding to (1.1) with $\frac{1}{\beta} + \frac{p-1}{r} < \frac{2}{N+1}$ is well defined on $L_{\delta(x)}^r(\Omega)$, and we prove that it possesses a compact attractor in $L_{\delta(x)}^r(\Omega)$. Since $L^r(\Omega) \subset L_{\delta(x)}^r(\Omega)$, we mention here that in contrast to the results in [13], we get the existence of attractors in weak topology spaces. It is known that the global attractor has 2 essential defaults: being very sensitive to the perturbation and attracting the orbits at a slow rate. To overcome these defaults, exponential attractors were introduced in [8]. It is noticed that, in contrast to the global attractor, the exponential attractor is not unique. Applying the abstract results of [3, 7, 9], we prove that there exists an exponential attractor for (1.1) in $L_{\delta(x)}^2(\Omega)$.

2. Preliminaries and main results

It is well-known that if Ω has a C^2 smooth boundary, there exist constants $c_1, c_2 > 0$ such that

$$c_1\delta(x) \leq \phi_1(x) \leq c_2\delta(x), \quad x \in \Omega,$$

where $\phi_1(x)$ is the first eigenfunction of $-\Delta$ in $H_0^1(\Omega)$. It then follows that $L_{\delta(x)}^r(\Omega) = L_{\phi_1(x)}^r(\Omega)$, $1 \leq r \leq \infty$, and the 2 norms are equivalent. In this case, from [10] we know that the operator Δ generates a C_0 semigroup $\{T(t)\}_{t \geq 0}$ on $L_{\delta(x)}^r(\Omega)$, $1 \leq r < \infty$. For nonnegative integer m and $1 \leq p < \infty$, define $W_{\delta(x)}^{m,p}(\Omega)$ by

$$W_{\delta(x)}^{m,p}(\Omega) = \{u \in \mathcal{D}'(\Omega) : \|u\|_{W_{\delta(x)}^{m,p}(\Omega)} = \sum_{|\alpha| \leq m} \left(\int_{\Omega} |D^\alpha u(x)|^p \delta(x) dx \right)^{\frac{1}{p}} < \infty\},$$

where $\mathcal{D}'(\Omega)$ is the set of all distributions over Ω . By the interpolation theory in [16], one can define $W_{\delta(x)}^{s,p}(\Omega)$ for $0 < s < \infty$. Denote by $W_{0,\delta(x)}^{s,p}(\Omega)$ the completion of $C_0^\infty(\Omega)$ in $W_{\delta(x)}^{s,p}(\Omega)$.

Lemma 2.1 (See [12]) *Let $1 \leq p \leq q \leq \infty$, $0 \leq r \leq s < \infty$, and $T(t) : W_{0,\delta(x)}^{r,p}(\Omega) \rightarrow W_{0,\delta(x)}^{s,q}(\Omega)$ for $t > 0$. Then there exists a constant $C > 0$ such that*

$$\| T(t)u_0 \|_{W_{0,\delta(x)}^{s,q}} \leq Ct^{-\frac{N+1}{2}(\frac{1}{p}-\frac{1}{q})+\frac{1}{2}(r-s)} \| u_0 \|_{W_{0,\delta(x)}^{r,p}}, \tag{2.5}$$

for all $t > 0$ and $u_0 \in W_{0,\delta(x)}^{r,p}(\Omega)$.

Theorem 2.2 (See [12]) *Assume that $f(x, u)$ satisfies (1.2)-(1.3) with $\rho > 1$ and $a(x) \in L_{\delta(x)}^\beta(\Omega)$, $1 < \beta \leq \infty$. Let $\frac{1}{\beta} + \frac{\rho-1}{r} < \frac{2}{N+1}$ (resp. $\frac{1}{\beta} + \frac{\rho-1}{r} = \frac{2}{N+1}$), $1 \leq r < \infty$ (resp. $r > 1$), and $g(x) \in L_{\delta(x)}^r(\Omega)$. Then if $u_0 \in L_{\delta(x)}^r(\Omega)$, there exists $T = T(u_0) > 0$ such that there is a unique solution $u \in C([0, T], L_{\delta(x)}^r(\Omega)) \cap L_{loc}^\infty((0, T); L^\infty(\Omega))$ of (1.1). Furthermore, the solution depends continuously on the initial data $u_0 \in L_{\delta(x)}^r(\Omega)$.*

To investigate the long-time behavior of solution of (1.1) with initial data in $L_{\delta(x)}^r(\Omega)$, we need the following dissipative condition:

$$f(x, u)u \geq -D_0 | u |^2 - D_1 | u |, \tag{2.6}$$

where D_0 and D_1 are positive constants such that $D_0 < \frac{\lambda_1}{r}$, λ_1 is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$. Similar dissipative conditions were also introduced in [2, 3, 5, 13, 17].

The first main result of this paper is described as:

Theorem 2.3 *Assume that $f(x, u)$ satisfies (1.2)-(1.3) with $\rho > 1$, $a(x) \in L_{\delta(x)}^\beta(\Omega)$, $1 < \beta \leq \infty$, and $g(x) \in L_{\delta(x)}^r(\Omega)$. Let $\frac{1}{\beta} + \frac{\rho-1}{r} < \frac{2}{N+1}$, $1 < r < \infty$, and (2.2) hold. Then problem (1.1) possesses a compact attractor \mathcal{A} in $L_{\delta(x)}^r(\Omega)$.*

To overcome the defaults of global attractor, *exponential attractors* (or inertial sets) were introduced in [8]. By definition, an exponential attractor is a compact semiinvariant set of the phase space, has the finite fractal dimension, and attracts exponentially the trajectories. In order to construct the exponential attractor, we present the definition of the squeezing property for the discrete map.

Definition 2.4 (See [8]) *Let H be a separable Hilbert space, $B \subseteq H$, a map $S : B \rightarrow B$ is said to satisfy the (discrete) squeezing property if there exists an orthogonal projection P_N of rank N such that for every u and v in B ,*

$$\| P_N(Su - Sv) \| \leq \| (I - P_N)(Su - Sv) \| \implies \| Su - Sv \| \leq \frac{1}{8} \| u - v \|,$$

where I is the identity map on H .

Theorem 2.5 (See [3]) *Let H be a separable Hilbert space, $\{S(t)\}_{t \geq 0}$ be a semigroup acting on H , and let B be a closed bounded subset of H such that $S(t)$ maps B into itself. Let $\{S(t)\}_{t \geq 0}$ have a global attractor on*

B. Suppose that for a fixed $t_* > 0$, S_{t_*} is Lipschitz on B and satisfies the discrete squeezing property on B . Then S_{t_*} has an exponential attractor \mathcal{M}^* on B .

In order to get the existence of the exponential attractor, we impose a little stronger dissipative condition than (2.2) on $f(x, u)$: Suppose that $f(x, u) \in C^{0,1}(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ such that

$$\frac{\partial f(x, u)}{\partial u} \geq -D_0. \tag{2.7}$$

The second main result of this paper is described as:

Theorem 2.6 *Assume that $f(x, u)$ satisfies (1.2)-(1.3) with $\rho > 1$, $a(x) \in L^\beta(\Omega)$, and $g(x) \in L^2_{\delta(x)}(\Omega)$. Let $\frac{1}{\beta} + \frac{\rho-1}{2} < \frac{2}{N+1}$ and (2.3) hold. Then problem (1.1) possesses an exponential attractor in $L^2_{\delta(x)}(\Omega)$.*

3. Proofs of the main results

We first show that the semigroup corresponding to (1.1) is well defined on $L^r_{\delta(x)}(\Omega)$.

Lemma 3.1 *Assume that the assumptions of Theorem 2.3 hold. Then the solution of (1.1) satisfies*

$$\| u(t) \|_{L^r_{\delta(x)}} \leq \frac{C_2}{C_1} \| u(0) \|_{L^r_{\delta(x)}} e^{-C_1 t} + C_2(1 + \| g(x) \|_{L^r_{\delta(x)}}),$$

where C_1 and C_2 are positive constants.

Proof Multiplying (1.1) by $|u|^{r-2} u \phi_1$ and integrating with respect to x , we obtain

$$\begin{aligned} \int_{\Omega} u_t |u|^{r-2} u \phi_1 dx + \int_{\Omega} (-\Delta u) |u|^{r-2} u \phi_1 dx + \int_{\Omega} f(x, u) |u|^{r-2} u \phi_1 dx \\ = \int_{\Omega} g(x) |u|^{r-2} u \phi_1 dx. \end{aligned} \tag{3.8}$$

Note that

$$\int_{\Omega} u_t |u|^{r-2} u \phi_1 dx = \frac{1}{r} \frac{d}{dt} \int_{\Omega} |u|^r \phi_1 dx. \tag{3.9}$$

By the fact

$$\int_{\Omega} \nabla u \cdot \nabla \phi_1 |u|^{r-2} u dx = \frac{\lambda_1}{r} \int_{\Omega} |u|^r \phi_1 dx,$$

we get that

$$\begin{aligned} \int_{\Omega} (-\Delta u) |u|^{r-2} u \phi_1 dx &= \int_{\Omega} \nabla u \cdot \nabla (|u|^{r-2} u) \phi_1 dx + \int_{\Omega} \nabla u \cdot \nabla \phi_1 |u|^{r-2} u dx \\ &= (r-1) \int_{\Omega} |\nabla u|^2 |u|^{r-2} \phi_1 dx + \frac{\lambda_1}{r} \int_{\Omega} |u|^r \phi_1 dx \\ &= \frac{4(r-1)}{r^2} \int_{\Omega} |\nabla |u|^{\frac{r}{2}}|^2 \phi_1 dx + \frac{\lambda_1}{r} \int_{\Omega} |u|^r \phi_1 dx. \end{aligned} \tag{3.10}$$

By (2.2), Hölder’s inequality and Young’s inequality, we get that

$$\begin{aligned} \int_{\Omega} f(x, u) |u|^{r-2} u \phi_1 dx &\geq -D_0 \int_{\Omega} |u|^r \phi_1 dx - D_1 \int_{\Omega} |u|^{r-1} \phi_1 dx \\ &\geq -D_0 \int_{\Omega} |u|^r \phi_1 dx - D_1 \left(\int_{\Omega} |u|^r \phi_1 dx \right)^{\frac{r-1}{r}} \left(\int_{\Omega} \phi_1 dx \right)^{\frac{1}{r}} \\ &\geq -(D_0 + \epsilon) \int_{\Omega} |u|^r \phi_1 dx - \frac{D_1^r}{r} \left(\frac{\epsilon r}{r-1} \right)^{-(r-1)} \int_{\Omega} \phi_1 dx, \end{aligned} \tag{3.11}$$

$$\begin{aligned} \int_{\Omega} g(x) |u|^{r-2} u \phi_1 dx &\leq \left(\int_{\Omega} |g(x)|^r \phi_1 dx \right)^{\frac{1}{r}} \left(\int_{\Omega} |u|^r \phi_1 dx \right)^{\frac{r-1}{r}} \\ &\leq \epsilon \int_{\Omega} |u|^r \phi_1 dx + \frac{1}{r} \left(\frac{\epsilon r}{r-1} \right)^{-(r-1)} \int_{\Omega} |g(x)|^r \phi_1 dx, \end{aligned} \tag{3.12}$$

where ϵ is small enough such that $D_0 + 2\epsilon < \frac{\lambda_1}{r}$. Therefore, it follows from (3.1)-(3.5) that

$$\begin{aligned} \frac{d}{dt} \|u(t)\|_{L^r_{\phi_1}}^r + (\lambda_1 - r(D_0 + 2\epsilon)) \|u(t)\|_{L^r_{\phi_1}}^r + \frac{4(r-1)}{r} \int_{\Omega} |\nabla |u|^{\frac{r}{2}}|^2 \phi_1 dx \\ \leq D_1^r \left(\frac{\epsilon r}{r-1} \right)^{-(r-1)} \int_{\Omega} \phi_1 dx + \left(\frac{\epsilon r}{r-1} \right)^{-(r-1)} \|g(x)\|_{L^r_{\phi_1}}^r. \end{aligned} \tag{3.13}$$

Neglecting the third term on the left-hand side of (3.6) and applying Gronwall’s lemma, we get the result. This completes the proof. \square

By Theorem 2.2 and Lemma 3.1, the semigroup $S(t)u_0 = u(t)$ is defined on $L^r_{\delta(x)}(\Omega)$, where $u(t)$ is the solution of (1.1). Moreover, the set

$$B_0 = \{u \in L^r_{\delta(x)}(\Omega) : \|u\|_{L^r_{\delta}} \leq R_0\}$$

is a bounded absorbing set for the semigroup $\{S(t)\}_{t \geq 0}$, where

$$R_0^r = 2C_2(1 + \|g(x)\|_{L^r_{\delta}}^r).$$

Let $T_1 = T_1(B_0) > 0$ such that for all $t \geq T_1$, $S(t)B_0 \subset B_0$.

In order to get the existence of the global attractor, we decompose the solution $u(t)$ of (1.1) into the sum $u(t) = v(t) + k(t)$, where $v(t)$ solves the following equation:

$$\begin{cases} \partial_t v - \Delta v = 0 & \text{in } \Omega, \\ v(x, t) = 0 & \text{on } \partial\Omega, \\ v(0) = u_0(x), \end{cases} \tag{3.14}$$

and the remainder $k(t)$ satisfies

$$\begin{cases} \partial_t k - \Delta k + f(x, k + v) = g & \text{in } \Omega, \\ k(x, t) = 0 & \text{on } \partial\Omega, \\ k(0) = 0. \end{cases} \tag{3.15}$$

As the proof of Lemma 3.1, we get that the solution of (3.7) is globally defined and exponentially decay:

$$\|v(t)\|_{L^r_{\delta(x)}} \leq \frac{C_2}{C_1} e^{-\lambda_1 t} \|u_0\|_{L^r_{\delta(x)}}. \tag{3.16}$$

For the solution of (3.8), we have:

Lemma 3.2 *Assume that the assumptions of Theorem 2.3 hold. Then, for every $0 < \epsilon \leq \epsilon_0$, $\epsilon_0 < \min\{1, 2 - (N + 1)(\frac{1}{\beta} + \frac{\rho-1}{r})\}$, and any $t > T_1$ and $u_0 \in B_0$, the solution of (3.8) satisfies*

$$\|k(t)\|_{W^{\epsilon,r}_{0,\delta(x)}} \leq C(t, \|a(x)\|_{L^\beta_{\delta(x)}}, \epsilon, R_0, \|g(x)\|_{L^r_{\delta(x)}}).$$

Proof From the proof of Theorem 3.1 in [12], we get that the solution of (3.8) satisfies

$$k(t) = \int_0^t e^{\Delta(t-s)} [-f(x, k(s) + v(s)) + g(x)] ds.$$

Let $\frac{1}{G_1} = \frac{1}{\beta} + \frac{\rho}{r}$. By Lemma 2.1 and 3.1 we have

$$\begin{aligned} & \int_0^t \|e^{\Delta(t-s)} f(x, k(s) + v(s))\|_{W^{\epsilon,r}_{0,\delta(x)}} ds \\ & \leq C \int_0^t \|e^{\Delta(t-s)} |a(x)| (1 + |u(s)|^\rho)\|_{W^{\epsilon,r}_{0,\delta(x)}} ds \\ & \leq C_3 \int_0^t (t-s)^{-\frac{\epsilon}{2} - \frac{N+1}{2}(\frac{1}{G_1} - \frac{1}{r})} \| |a(x)| (1 + |u(s)|^\rho)\|_{L^{G_1}_{\delta(x)}} ds \\ & \leq C_4 \int_0^t (t-s)^{-\frac{\epsilon}{2} - \frac{N+1}{2}(\frac{1}{\beta} + \frac{\rho-1}{r})} \|a(x)\|_{L^\beta_{\delta(x)}} (1 + \|u(s)\|_{L^r_{\delta(x)}}^\rho) ds. \end{aligned}$$

Thus,

$$\begin{aligned} & \|k(t)\|_{W^{\epsilon,r}_{0,\delta(x)}} \\ & \leq \int_0^t \|e^{\Delta(t-s)} [-f(x, k(s) + v(s)) + g(x)]\|_{W^{\epsilon,r}_{0,\delta(x)}} ds \\ & \leq \int_0^t \|e^{\Delta(t-s)} f(x, k(s) + v(s))\|_{W^{\epsilon,r}_{0,\delta(x)}} ds + \int_0^t \|e^{\Delta(t-s)} g(x)\|_{W^{\epsilon,r}_{0,\delta(x)}} ds \\ & \leq C_4 \|a(x)\|_{L^\beta_{\delta(x)}} \int_0^t (t-s)^{-\frac{\epsilon}{2} - \frac{N+1}{2}(\frac{1}{\beta} + \frac{\rho-1}{r})} (\|u(s)\|_{L^r_{\delta(x)}}^\rho + 1) ds + C_5 \|g(x)\|_{L^r_{\delta(x)}} t^{1-\frac{\epsilon}{2}} \\ & \leq C_6 \|a(x)\|_{L^\beta_{\delta(x)}} t^{1-\frac{\epsilon}{2} - \frac{N+1}{2}(\frac{1}{\beta} + \frac{\rho-1}{r})} + C_5 \|g(x)\|_{L^r_{\delta(x)}} t^{1-\frac{\epsilon}{2}}. \end{aligned}$$

□

Proof of Theorem 2.3 By Lemma 3.1-3.2 and the standard theory (e.g., see [4, 11, 15]) of dynamical systems, we get that there exists a global attractor for (1.1) in $L^r_{\delta(x)}(\Omega)$. □

In order to get the existence of the exponential attractor for (1.1) in $L^2_{\delta(x)}(\Omega)$, the higher regularity of solution of (3.16) (see below) than that of (3.8) is needed, which guarantees the discrete solution semigroup

satisfies the squeezing property. As the proof of Lemma 3.1, we get that there exist positive constants C_7 and C_8 such that for any $u_0 \in L^2_{\delta(x)}(\Omega)$, the solution of (1.1) satisfies

$$\| u(t) \|_{L^2_{\delta(x)}}^2 \leq \frac{C_2}{C_1} \| u_0 \|_{L^2_{\delta(x)}}^2 e^{-C_7 t} + C_8(1 + \| g(x) \|_{L^2_{\delta(x)}}^2). \tag{3.17}$$

Therefore, the solution semigroup $\{S(t)\}_{t \geq 0}$ of (1.1) is well defined on $L^2_{\delta(x)}(\Omega)$ and the set

$$B_1 = \{u \in L^2_{\delta(x)}(\Omega) : \| u \|_{L^2_{\delta(x)}} \leq R_1\}$$

is the absorbing set for the semigroup $\{S(t)\}_{t \geq 0}$, where

$$R_1^2 = 2C_8(1 + \| g(x) \|_{L^2_{\delta(x)}}^2).$$

Let $T_2 = T_2(B_1) > 0$ such that for all $t \geq T_2$, $S(t)B_1 \subset B_1$. Let

$$B_2 = \overline{\bigcup_{t \geq 0} S(t + T_2)B_1}^{L^2_{\delta(x)}}.$$

We note that B_2 is a closed bounded positively invariant set and $B_2 \subset B_1$.

Lemma 3.3 *Assume that the assumptions of Theorem 2.5 hold. Then for any $t \geq 0$, the $S(t)$ is Lipschitz continuous.*

Proof Let $u_1(t) = S(t)u_1(0)$ and $u_2(t) = S(t)u_2(0)$ be 2 solutions of (1.1) with initial values $u_1(0)$ and $u_2(0)$, respectively. Setting $w(t) = u_1(t) - u_2(t)$, we note that $w(t)$ satisfies

$$\begin{cases} w_t - \Delta w + f(x, u_1) - f(x, u_2) = 0 & \text{in } \Omega, \quad t > 0, \\ w = 0 & \text{on } \partial\Omega, \\ w(0) = u_1(0) - u_2(0). \end{cases} \tag{3.18}$$

Multiplying (3.11) by $w\phi_1$ and integrating over Ω , we have

$$\frac{1}{2} \frac{d}{dt} \| w \|_{L^2_{\phi_1}}^2 + \| w \|_{W^{1,2}_{0,\phi_1}}^2 + \frac{\lambda_1}{2} \| w \|_{L^2_{\phi_1}}^2 + \int_{\Omega} (f(x, u_1) - f(x, u_2))w\phi_1 = 0. \tag{3.19}$$

For the fourth term on the left-hand side of (3.12), by (2.3) we have

$$\begin{aligned} \int_{\Omega} (f(x, u_1) - f(x, u_2))w\phi_1 dx &= \int_{\Omega} \int_0^1 f'_u(x, u_2 + \theta(u_1 - u_2))d\theta | w |^2 \phi_1 dx \\ &\geq -D_0 \int_{\Omega} | w |^2 \phi_1 dx. \end{aligned} \tag{3.20}$$

Therefore, by (3.13), it follows from (3.12) that

$$\frac{d}{dt} \| w \|_{L^2_{\phi_1}}^2 \leq (2D_0 - \lambda_1) \| w \|_{L^2_{\phi_1}}^2,$$

which implies that

$$\| w(t) \|_{L^2_{\delta(x)}}^2 \leq \frac{c_2}{c_1} e^{(2D_0 - \lambda_1)t} \| w(0) \|_{L^2_{\delta(x)}}^2. \tag{3.21}$$

This completes the proof. □

Let $t_* = \frac{8}{\lambda_1} \ln 2$.

Lemma 3.4 *Assume that the assumptions of Theorem 2.5 hold. Then the discrete semigroup S_{t_*} satisfies the squeezing property on B_2 .*

Proof Split the solution $w(t)$ of (3.11) as follows: $w(t) = w_1(t) + w_2(t)$, where $w_1(t)$ solves the following problem

$$\begin{cases} \frac{\partial w_1}{\partial t} - \Delta w_1 = 0 & \text{in } \Omega, \\ w_1 = 0 & \text{on } \partial\Omega, \\ w_1(0) = w(0) = u_1(0) - u_2(0), \end{cases} \tag{3.22}$$

and $w_2(t)$ satisfies

$$\begin{cases} \frac{\partial w_2}{\partial t} - \Delta w_2 + f(x, u_1) - f(x, u_2) = 0 & \text{in } \Omega, \\ w_2 = 0 & \text{on } \partial\Omega, \\ w_2(0) = 0. \end{cases} \tag{3.23}$$

For the equation (3.15), we get that

$$\| w_1(t) \|_{L^2_{\delta(x)}}^2 \leq \frac{c_2}{c_1} e^{-\lambda_1 t} \| w(0) \|_{L^2_{\delta(x)}}^2. \tag{3.24}$$

Note that

$$w_2(t) = \int_0^t e^{\Delta(t-s)} [f(x, u_1) - f(x, u_2)] ds.$$

By Theorem 2.2 we know that the solution of (1.1) belongs to $L^\infty_{loc}(0, T; L^\infty(\Omega))$. Notice that $\frac{1}{\beta} + \frac{\rho-1}{2} < \frac{2}{N+1}$, which implies that $\frac{1}{\beta} - \frac{1}{2} < \frac{1}{N+1}$. Thus, using (2.1) and from above, we have

$$\begin{aligned} & \|w_2(t)\|_{W^{1,2}_{0,\delta(x)}} \\ & \leq C \int_0^t \| e^{\Delta(t-s)} | a(x) | (1 + | u_1(s) |^{\rho-1} + | u_2(s) |^{\rho-1}) | w(s) \|_{W^{1,2}_{0,\delta(x)}} ds \\ & \leq C_9 \int_0^t \| e^{\Delta(t-s)} | a(x) | (1 + | u_1(s) |^\rho + | u_2(s) |^\rho) \|_{W^{1,2}_{0,\delta(x)}} ds \\ & \leq C_{10} \int_0^t (t-s)^{-\frac{1}{2} - \frac{N+1}{2}(\frac{1}{\beta} - \frac{1}{2})} \| | a(x) | (1 + | u_1(s) |^\rho + | u_2(s) |^\rho) \|_{L^\beta_{\delta(x)}} ds \\ & \leq C_{11} \int_0^t (t-s)^{-\frac{1}{2} - \frac{N+1}{2}(\frac{1}{\beta} - \frac{1}{2})} \| a(x) \|_{L^\beta_{\delta(x)}} ds \\ & \leq C_{12} t^{\frac{1}{2} - \frac{N+1}{2}(\frac{1}{\beta} - \frac{1}{2})}. \end{aligned} \tag{3.25}$$

Thus,

$$\| w_2(t_*) \|_{W_{0,\delta(x)}^{1,2}}^2 \leq (C_{12} t_*^{\frac{1}{2} - \frac{N+1}{2}(\frac{1}{\beta} - \frac{1}{2})})^2 := M_1^2. \tag{3.26}$$

Note that the left-hand side of (3.19) can be written in the form $(Lw_2, w_2) \leq M_1^2$, where L is the operator determined by the quadratic form. Since the set defined by (3.19) is compact in $L_{\delta(x)}^2(\Omega)$, we get that L^{-1} is compact on $L_{\delta(x)}^2(\Omega)$. Therefore, by spectral theory (e.g., see [18]), we know that (3.19) defines an ellipsoid B_3 in $L_{\delta(x)}^2(\Omega)$, and it can be rewritten as

$$B_3 = \{w_2 : \sum_{j=1}^{\infty} \mu_j (w_2, e_j)_{L_{\delta(x)}^2}^2 \leq M_1^2\},$$

where $\{e_j\}$ is an orthonormal basis in $L_{\delta(x)}^2(\Omega)$, e_j is an eigenfunction of L , and $\mu_j \rightarrow +\infty$ monotonely as $j \rightarrow +\infty$.

Choose N large enough such that

$$\mu_N \geq \frac{256M_1^2}{\| w(0) \|_{L_{\delta(x)}^2}}, \tag{3.27}$$

and let $E_N = span\{e_1, e_2, \dots, e_N\}$. Obviously, if $w_2(t_*) \in B_3$, it holds that

$$\| (I - P_N)w_2(t_*) \|_{L_{\delta(x)}^2}^2 = \sum_{j=N+1}^{\infty} (w_2(t_*), e_j)_{L_{\delta(x)}^2}^2 \leq \frac{M_1^2}{\mu_N}.$$

Therefore, by (3.17) and (3.20), we get that if

$$\| P_N(S_{t_*} u_1(0) - S_{t_*} u_2(0)) \|_{L_{\delta(x)}^2} \leq \| (I - P_N)(S_{t_*} u_1(0) - S_{t_*} u_2(0)) \|_{L_{\delta(x)}^2},$$

then

$$\begin{aligned} & \| S_{t_*} u_1(0) - S_{t_*} u_2(0) \|_{L_{\delta(x)}^2}^2 \\ &= \| P_N(S_{t_*} u_1(0) - S_{t_*} u_2(0)) \|_{L_{\delta(x)}^2}^2 + \| (I - P_N)(S_{t_*} u_1(0) - S_{t_*} u_2(0)) \|_{L_{\delta(x)}^2}^2 \\ &\leq 2 \| (I - P_N)(S_{t_*} u_1(0) - S_{t_*} u_2(0)) \|_{L_{\delta(x)}^2}^2 \\ &\leq 2 \left[\frac{1}{256} \| w(0) \|_{L_{\delta(x)}^2}^2 + \frac{M_1^2}{\mu_N \| w(0) \|_{L_{\delta(x)}^2}^2} \| w(0) \|_{L_{\delta(x)}^2}^2 \right] \\ &\leq \frac{1}{64} \| w(0) \|_{L_{\delta(x)}^2}^2. \end{aligned}$$

This completes the proof. □

Proof of Theorem 2.5 By Theorem 2.3-2.4 and Lemma 3.3-3.4, we get that S_{t_*} has an exponential attractor \mathcal{M}^* on B_2 . Let

$$\mathcal{M} = \bigcup_{0 \leq t \leq t_*} S(t)\mathcal{M}^*.$$

By (3.14), we get that the map $F(t, u(0)) = S(t)u_0$ is Lipschitz from $[0, t_*] \times B_2$ into B_2 . Therefore, it follows from Theorem 3.1 of [9] that \mathcal{M} is an exponential attractor for $\{S(t)\}_{t \geq 0}$ acting on B_2 and

$$d_F(\mathcal{M}) = d_F(\mathcal{M}^*) + 1,$$

where the d_F denotes the fractal dimension. □

Remark 3.5 By (3.14) we can choose $t_* > 0$ such that the discrete semigroup $S(t_*)$ is α -contraction. Thus, we can also prove Theorem 2.5 by using Theorem 2.1 in [7]. On the other hand, the results of Theorem 2.3 and 2.5 can be used for reaction-diffusion equations with specific nonlinearities, e.g., $f(x, u) = a(x) |u|^{p-1} u$ with $a(x) \in L^{\beta}_{\delta(x)}(\Omega)$, $1 < \beta \leq \infty$, $a(x) > 0$ for a.e. $x \in \Omega$, and $p > 1$.

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