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Bifurcations and parametric representations of traveling wave solutions for the Green–Naghdi equations

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Abstract: By using the bifurcation theory of dynamical systems to study the dynamical behavior of the Green–Naghdi equations, the existence of solitary wave solutions along with smooth periodic traveling wave solutions is obtained. Under different regions of parametric spaces, various sufficient conditions to guarantee the existence of the above solutions are given. Some exact and explicit parametric representations of traveling wave solutions are constructed.

Key words: Green–Naghdi equations, bifurcation theory of dynamical systems, bifurcation curves, solitary waves, periodic waves

1. Introduction

In this paper, we study the dynamical behavior of the Green–Naghdi (GN) equations [3, 4]. Specifically, we determine traveling wave solutions and new solitary wave solutions for the GN equations:

$$\eta_t + (u\eta)_x = 0, \tag{1.1a}$$

$$u_t + uu_x + \eta_x = \frac{1}{3\eta}\left(\eta^2 \frac{d}{dt}(\eta u_x)\right)_x. \tag{1.1b}$$

The Green–Naghdi equations were derived for both free-surface and interfacial-surface waves under the assumption of long wavelengths. Here, η and u represent the surface disturbance and the mean horizontal velocity, respectively. The GN equations were originally developed by Green and Naghdi in 1974 to analyze some nonlinear free-surface flows. After the successful application of the GN equations to nonlinear ship wave-making problems [2], the method was applied to many nonlinear water wave problems. Later, the model was extended to deep-water waves by Webster and Kim and by Xu et al. [11, 13] in 2 and 3 dimensions, respectively. In the work of Wu and Chen [12], a wave equation model and the finite element method (WE/FEM) were adopted to solve the GN equations. In the work of Li [9, 10], the author showed the linear stability for solitary waves of small amplitudes for system (1.1).

Based on shallow-water theory, solitary waves hitting a ship will lead to periodic heaving and pitching motions and increase the drag on the ship dramatically. Which features will occur when the homoclinic orbit (or periodic orbit) intersects with the singular straight line? We are thus interested in an important question

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that should be investigated: if the equilibrium points $S_{1,2}(-\phi_*, \pm\sqrt{3})$ in the straight line $\nu = -\phi_*$ of (1.9) are node points, what is the dynamical behavior of orbits of the vector field defined by (1.6)? We hope to answer this problem in this paper.

Let $\omega = u\eta$, then (1.1) becomes

$$\eta_t + \omega_x = 0, \tag{1.2a}$$

$$\omega_t = -\left(\frac{\omega^2}{\eta}\right)_x - \eta\eta_x + \frac{1}{3}\left(\eta^2 \frac{d}{dt}\left(\eta\left(\frac{\omega}{\eta}\right)_x\right)\right)_x. \tag{1.2b}$$

To find traveling wave solutions of the GN equations, we set

$$\eta = \phi(\xi) = \phi(x - ct),\ \omega = \psi(\xi) = \psi(x - ct), \tag{1.3}$$

where c is the wave speed. Substituting (1.3) into (1.2a) and integrating with respect to ξ leads to

$$\psi = c\phi - g_1, \tag{1.4a}$$

where g_1 is an integral constant. Substituting (1.4a) and (1.3) into (1.2b) and integrating with respect to ξ leads to

$$g_1^2\left[\phi\phi'' - (\phi')^2\right] + 3g_1^2 + 3(g_2 + cg_1)\phi + \frac{3}{2}\phi^3 = 0, \tag{1.4b}$$

where g_2 is an integral constant.

Assume that $g_1 \neq 0$ because otherwise we only get the trivial solutions of (1.4b), and then (1.4b) is equivalent to the following 2-dimensional systems:

$$\frac{d\phi}{d\xi} = y,\quad \frac{dy}{d\xi} = \frac{1}{\phi}\left[y^2 - 3 - \frac{3(g_2 + cg_1)}{g_1^2}\phi - \frac{3}{2g_1^2}\phi^3\right]. \tag{1.5}$$

We make the transformation
$$\nu = \phi(\xi) - \phi_*,\ \xi = \xi,\quad y = y,$$

where $\phi_* \neq 0$ is an arbitrary constant that satisfies the equation

$$1 + \frac{g_2 + cg_1}{g_1^2}\phi + \frac{1}{2g_1^2}\phi^3 = 0.$$

Then system (1.5) becomes

$$\frac{d\nu}{d\xi} = y,\quad \frac{dy}{d\xi} = \frac{1}{\nu + \phi_*}\left[y^2 - \frac{3}{2g_1^2}\nu\left(-\frac{2g_1^2}{\phi_*} + 2\phi_*^2 + 3\phi_*\nu + \nu^2\right)\right], \tag{1.6}$$

which has the first integral

$$y^2 = (\nu + \phi_*)^2\left[-\frac{3}{g_1^2}\left(-\frac{g_1^2}{(\nu + \phi_*)^2} + \frac{2g_1^2 + \phi_*^3}{\phi_*(\nu + \phi_*)} + \nu\right) + h\right], \tag{1.7}$$

i.e.

$$H(\nu, y) = \frac{y^2}{(\nu + \phi_*)^2} + \frac{3}{g_1^2}\left(-\frac{g_1^2}{(\nu + \phi_*)^2} + \frac{2g_1^2 + \phi_*^3}{\phi_*(\nu + \phi_*)} + \nu\right) = h \tag{1.8}$$

h is a Hamiltonian constant.

Because of the difficulty of directly studying the singular system (1.6), we make the "time scale" transformation $d\xi = (\nu + \phi_*)d\zeta$ to remove the singularity[5] and reduce the singular system (1.6) to the following regular system:

$$\frac{d\nu}{d\zeta} = (\nu + \phi_*)y, \quad \frac{dy}{d\zeta} = y^2 - \frac{3}{2g_1^2}\nu(-\frac{2g_1^2}{\phi_*} + 2\phi_*^2 + 3\phi_*\nu + \nu^2). \tag{1.9}$$

Then, except on the singular straight line $\nu = -\phi_*$, the system (1.6) has the same topological phase portraits as (1.9). Moreover, the straight line $\nu = -\phi_*$ is an invariant integral straight line of (1.9). We call system (1.9) the associated regular system of (1.6) [6].

System (1.6) is a 2-parameter planar dynamical system depending on the parameter group (g_1, ϕ_*), and we shall investigate the bifurcations of phase portraits of (1.6) in the phase plane (ν, y) as the parameters g_1, ϕ_* are changed.

Suppose that $\nu(\xi)$ is a continuous solution of (1.6) for $\xi \in (-\infty, \infty)$ and $\lim_{\xi\to+\infty}\nu(\xi) = \alpha$, $\lim_{\xi\to-\infty}\nu(\xi) = \beta$. Recall that (i) $\nu(x,t)$ is called a solitary wave solution if $\alpha = \beta$; (ii) $\nu(x,t)$ is called a kink or anti-kink solution if $\alpha \neq \beta$. Usually, a solitary wave solution of (1.1) corresponds to a homoclinic orbit of (1.6); a kink (or anti-kink) wave solution (1.1) corresponds to a heteroclinic orbit (or the so-called connecting orbit) of (1.6). Similarly, a periodic orbit of (1.6) corresponds to a periodically traveling wave solution of (1.1). Thus, to investigate all possible bifurcations of solitary waves and periodic waves of (1.1), we need to find all periodic annuli and homoclinic orbits of (1.6), which depend on the system parameters. The bifurcation theory of dynamical systems [1, 6, 7, 8] plays an important role in our study.

It is easy to see that the right-hand side of the second equation in (1.6) is not continuous when $\nu = -\phi_*$. In other words, on such a straight line in the phase plane (ν, y), the function ν''_ξ is not well defined, which implies that (1.1) sometimes has non-smooth traveling wave solutions.

The paper is organized as follows. In Section 2, we consider the different dynamical behavior of orbits of vector fields defined by (1.6) and (1.9) and the phase portraits of systems (1.6) in different parameter regions. In Section 3, we give some exact solitary traveling wave solutions of system (1.1) in different parameter regions. Finally, the summary is given in Section 4.

2. Dynamical behavior and phase portraits (1.6) and (1.9)

Clearly, system (1.9) has the same dynamics for both cases $g_1 > 0$ and $g_1 < 0$. Without loss of generality, we assume that $g_1 > 0$, and therefore we only investigate the dynamics of (1.9) in the (g_1, ϕ_*)-right half parameter plane.

Denote that

$$f(\nu) = -\frac{2g_1^2}{\phi_*} + 2\phi_*^2 + 3\phi_*\nu + \nu^2, \tag{2.1}$$

$$\Delta = \phi_*^2 + \frac{8g_1^2}{\phi_*}. \tag{2.2}$$

It follows from $\Delta = \phi_*^2 + \frac{8g_1^2}{\phi_*} = 0$ that there is a bifurcation curve

$$L_1: \quad \phi_* = -2(g_1)^{\frac{2}{3}}.$$

Thus, we have the following conclusions.

(i) If $\Delta > 0$ and $\Delta \neq 9\phi_*^2$, there exist 3 equilibrium points at $O(0,0)$, $A_1(\nu_1,0)$, and $A_2(\nu_2,0)$ on the ν-axis of (1.9), where $\nu_{1,2} = \frac{-3\phi_* \pm \sqrt{\Delta}}{2}$; if $\Delta = 9\phi_*^2$, there exist 2 equilibrium points at $O(0,0)$ and $B(-3\phi_*,0)$ on the ν-axis of (1.9). Thus, from $\Delta = 9\phi_*^2$, we can get another bifurcation curve:

$$L_2: \quad \phi_* = (g_1)^{\frac{2}{3}}.$$

(ii) If $\Delta = 0$, there exist 2 equilibrium points at $O(0,0)$ and $E(-\frac{3\phi_*}{2},0)$ on the ν-axis of (1.9).

(iii) If $\Delta < 0$, (1.9) has a unique equilibrium point at $O(0,0)$ on the ν-axis.

(iv) On the straight line $\nu = -\phi_*$, (1.9) has 2 equilibrium points at $S_1(-\phi_*,\sqrt{3})$ and $S_2(-\phi_*,-\sqrt{3})$.

It is easy to see that $0 < -\phi_* < \nu_2 < \nu_1$ for $\phi_* < -2(g_1)^{\frac{2}{3}}$; $0 < -\phi_* < \nu_2 = \nu_1$ for $\phi_* = -2(g_1)^{\frac{2}{3}}$; and $0 < -\phi_*$ for $-2(g_1)^{\frac{2}{3}} < \phi_* < 0$. Similarly, $\nu_2 < -\phi_* < 0 < \nu_1$ for $0 < \phi_* < (g_1)^{\frac{2}{3}}$; $\nu_2 = \nu_1 < -\phi_* < 0$ for $\phi_* = (g_1)^{\frac{2}{3}}$; and $\nu_2 < -\phi_* < \nu_1 < 0$ for $(g_1)^{\frac{2}{3}} < \phi_*$.

Let $M(\nu_e,y_e)$ be the coefficient matrix of the linearized system of (1.9) at the equilibrium point (ν_e,y_e) and $J(\nu_e,y_e) = detM(\nu_e,y_e)$; then we have:

$$J(\nu_1,0) = \frac{3}{8g_1^2}(\sqrt{\Delta} - \phi_*)(\Delta - 3\phi_*\sqrt{\Delta}),$$

$$J(\nu_2,0) = -\frac{3}{8g_1^2}(\sqrt{\Delta} + \phi_*)(\Delta + 3\phi_*\sqrt{\Delta}),$$

$$J(0,0) = \frac{6}{2g_1^2}(\phi_*^3 - g_1^2), \ J(-\phi_*,\pm\sqrt{3}) = 6 > 0,$$

$$Trace(M(-\phi_*,\pm\sqrt{3})) = \pm 3\sqrt{3},$$

and

$$(Trace(M(-\phi_*,\pm\sqrt{3})))^2 - 4J(-\phi_*,\pm\sqrt{3}) = 3 > 0.$$

By the theory of planar dynamical system, we know that for an equilibrium point of a planar integrable system, the equilibrium point is a saddle point if $J < 0$; the equilibrium point is a center if $J > 0$ and $Trace(M) = 0$; and the equilibrium point is a node point (a proper node) if $J > 0$ and $(Trace(M(\nu_i,y_i))^2 - 4J(\nu_i,y_i) > 0(= 0)$. Obviously, $S_{1,2}(-\phi_*,\pm\sqrt{3})$ both are nodes. Moreover, S_1 is unstable and S_2 is stable.

From the first integral defined by (1.8), we denote that

$$h_0 = H(0,0) = \frac{3(g_1^2 + \phi_*^3)}{g_1^2\phi_*^2}, \ h_s = H(-\phi_*,\pm\sqrt{3}) = \infty,$$

$$h_1{=}H(\nu_1,0){=}\frac{3[-16\phi_*g_1^2{-}4\phi_*^4{-}3\phi_*^3{+}\sqrt{\Delta}(8g_1^2{+}4\phi_*^3{+}7\phi_*^2){-}5\phi_*\Delta{+}\Delta^{\frac{3}{2}}]}{2g_1^2\phi_*(-\phi_* + \sqrt{\Delta})^2},$$

$$h_2{=}H(\nu_2,0){=}\frac{3[-16\phi_*g_1^2{-}4\phi_*^4{-}3\phi_*^3{-}\sqrt{\Delta}(8g_1^2{+}4\phi_*^3{+}7\phi_*^2){-}5\phi_*\Delta{-}\Delta^{\frac{3}{2}}]}{2g_1^2\phi_*(\phi_* + \sqrt{\Delta})^2}.$$

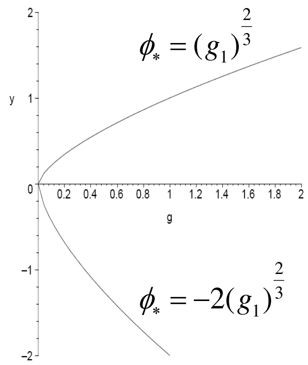


Figure 1. The partition of (g_1, ϕ_*)-right half plane.

We next use the above statements to consider the bifurcations of the phase portraits of (1.9). The curves L_1, L_2 and the straight line $\phi_* = 0$ divide the right half (g_1, ϕ_*)-parameter plane into 4 subregions, shown in Figure 1. The bifurcations of the phase portraits of (1.9) are shown in Figure 2.

We notice that the orbits of system (1.9) have the following dynamical behavior.

1. Case I: $\phi_* < -2(g_1)^{\frac{2}{3}}$ (see Figure 2a).

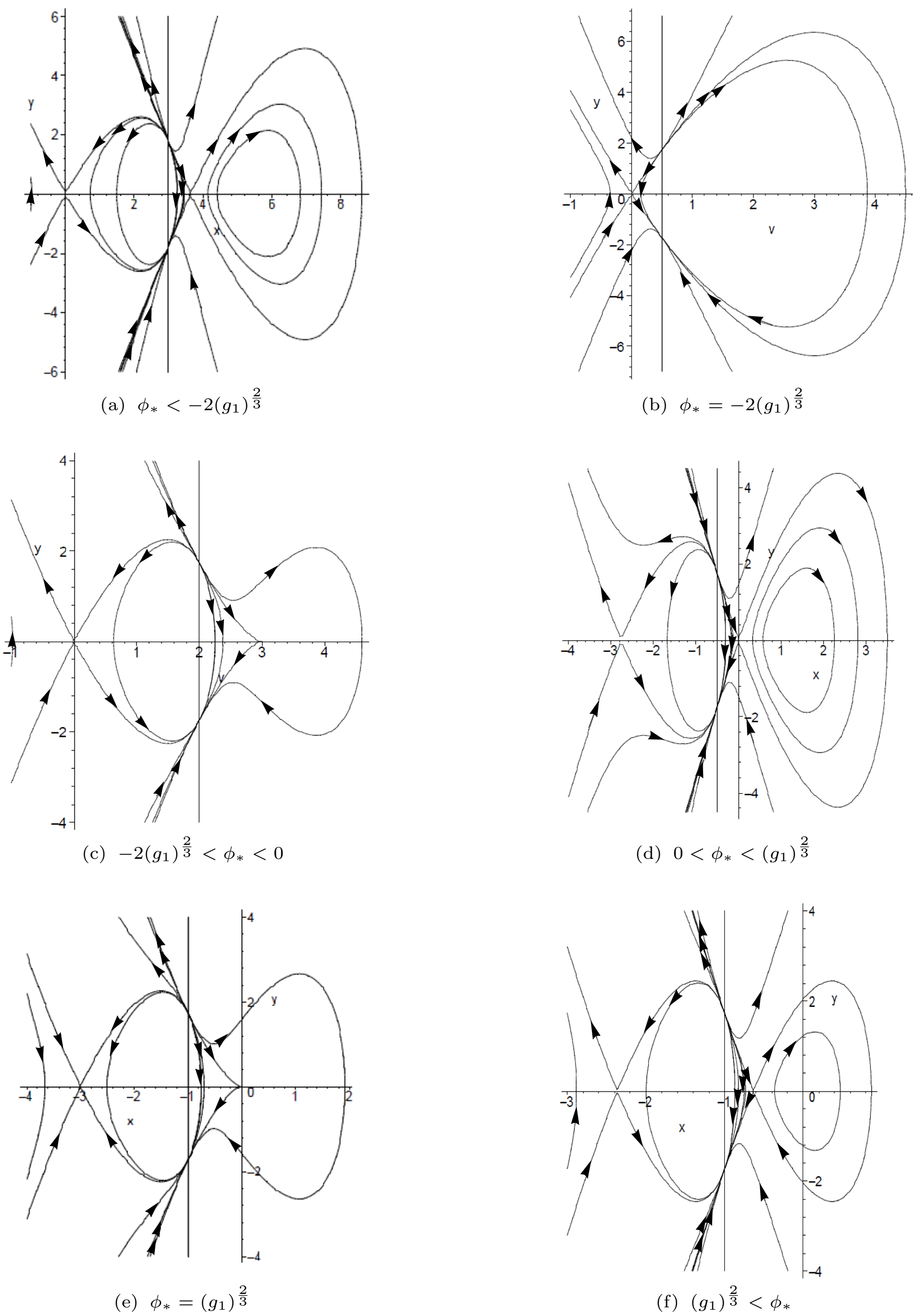
(1) Corresponding to the curves defined by $H(\nu, y) = h_0$, there are 2 heteroclinic orbits connecting the equilibrium points O and $S_{1,2}$, and a heteroclinic orbit connecting the equilibrium points S_1 and S_2, on the left side of the straight line $\nu = -\phi_*$. There is a stable manifold and an unstable manifold of the saddle point O on the left side of the y-axis. Thus, the curves defined by $H(\nu, y) = h_0$ consist of 5 orbits of (1.9) and the equilibrium O.

(2) Corresponding to the curves defined by $H(\nu, y) = h_2$, there is a homoclinic orbit connecting the equilibrium point A_2, and 2 heteroclinic orbits connecting the equilibrium points A_2 and $S_{1,2}$, respectively, on the right side of the straight line $\nu = -\phi_*$. There is a stable manifold and an unstable manifold of the saddle point $A_2(\nu_2, 0)$ on the right side of the straight line $\nu = -\phi_*$. Hence, the curves defined by $H(\nu, y) = h_2$ consist of 5 orbits of (1.9) and the equilibrium point A_2.

(3) Corresponding to the curves defined by $H(\nu, y) = h, h \in (-\infty, h_0)$, there are 2 heteroclinic orbits connecting the equilibrium points S_1 and S_2, which lie on the left and right sides of the straight line $\nu = -\phi_*$, respectively; there is an open orbit on the left side of y-axis. Namely, the curves defined by $H(\nu, y) = h$ consist of 3 orbits of (1.9) and 2 equilibrium points $S_{1,2}$.

(4) Corresponding to the curves defined by $H(\nu, y) = h, h \in (h_0, h_1)$ or $H(\nu, y) = h, h \in (h_2, \infty)$, there is a heteroclinic orbit connecting the equilibrium points S_1 and S_2, which lies on the right side of the straight line $\nu = -\phi_*$; there is an unstable manifold (a stable manifold) to the equilibrium point S_1 (S_2). The curves defined by $H(\nu, y) = h$ consist of 3 orbits of (1.9) and the equilibria $S_{1,2}$.

(5) Corresponding to the curves defined by $H(\nu, y) = h, h \in (h_1, h_2)$, there is a heteroclinic orbit connecting the equilibrium points S_1 and S_2, which lies on the right side of the straight line $\nu = -\phi_*$; there is a periodic orbit enclosing the center A_1 and an unstable manifold (a stable manifold) to the equilibrium point S_1 (S_2). The curves defined by $H(\nu, y) = h$ consist of 3 orbits of (1.9) and the equilibria A_1 and $S_{1,2}$.



(a) $\phi_* < -2(g_1)^{\frac{2}{3}}$ (b) $\phi_* = -2(g_1)^{\frac{2}{3}}$

(c) $-2(g_1)^{\frac{2}{3}} < \phi_* < 0$ (d) $0 < \phi_* < (g_1)^{\frac{2}{3}}$

(e) $\phi_* = (g_1)^{\frac{2}{3}}$ (f) $(g_1)^{\frac{2}{3}} < \phi_*$

Figure 2. The phase portraits of (1.9) for $g_1 > 0$.

2. Case II: $\phi_* = -2(g_1)^{\frac{2}{3}}$ (see Figure 2b).

(6) Corresponding to the curves defined by $H(\nu, y) = h_0$, there are 2 heteroclinic orbits connecting the equilibria O and $S_{1,2}$ on the left side of the straight line $\phi = -\phi_*$, and a heteroclinic orbit connecting the equilibria S_1 and S_2 on the right side of the straight line $\nu = -\phi_*$. There are a stable manifold and an unstable

manifold of the saddle point $O(0,0)$ on the left side of the y-axis. The curves defined by $H(\nu,y)=h_0$ consist of 5 orbits of (1.9) and the equilibrium point O.

(7) Corresponding to the curves defined by $H(\nu,y)=h, h\in(-\infty,h_0)$, there are 2 heteroclinic orbits connecting the equilibrium points S_1 and S_2, which lie on the left and right sides of the straight line $\nu=-\phi_*$, respectively; there is an open orbit on the left side of the y-axis. Namely, the curves defined by $H(\nu,y)=h$ consist of 3 orbits of (1.9) and 2 equilibrium points $S_{1,2}$.

(8) Corresponding to the curves defined by $H(\nu,y)=h, h\in(h_0,\infty)$, there is a heteroclinic orbit connecting the equilibrium points S_1 and S_2, which lies on the right side of the straight line $\nu=-\phi_*$; there is an unstable manifold (a stable manifold) to the equilibrium point S_1 (S_2). The curves defined by $H(\nu,y)=h$ consist of 3 orbits of (1.9) and 2 equilibrium points $S_{1,2}$.

Similarly, we can discuss the cases where $-2(g_1)^{\frac{2}{3}}<\phi_*<0,\ 0<\phi_*<(g_1)^{\frac{2}{3}},\ \phi_*=(g_1)^{\frac{2}{3}}$, and $(g_1)^{\frac{2}{3}}<\phi_*$ (see Figures 2c, 2d, 2e, and 2f). We omit them to save space.

Now we study the dynamical behavior of orbits of (1.6). We notice again that the singular system (1.6) has the same orbits as the regular system (1.9) except on the singular straight line $\nu=-\phi_*$. When $\nu\neq-\phi_*$, the transformation of variables $\mathrm{d}\xi=(\nu+\phi_*)\mathrm{d}\zeta$ only derives the difference between the directions of orbits of (1.6) and the counterparts of (1.9). The orbits of (1.6) and (1.9) have the same directions on the right side of the straight line $\nu=-\phi_*$, and have different directions on the left side of the line $\nu=-\phi_*$.

To discuss the existence of the solitary wave and periodic wave, we need to use the following lemma of the existence of finite time interval(s) of solutions with respect to ξ in the positive or (and) negative direction(s) and the vector field defined by (1.6).

Lemma 1

Let $(\nu, y=\nu')$ be the parametric representation of an orbit γ of system (1.6) and $S_{1,2}(-\phi_*,\pm\sqrt{3})$ be 2 equilibrium points on the singular straight line $\nu=-\phi_*$. Suppose that the phase point (ϕ,y) tends to the points $S_{1,2}$ along the orbit γ, respectively, as ξ increases or (and) decreases; then, there exists a constant $\widetilde{\xi}$ such that $\lim_{\xi\to\widetilde{\xi}}\nu(\xi)=-\phi_*$.

Proof Consider the case where there is an orbit γ, defined by $H(\nu,y)=h_a$, connecting the points S_1 and S_2 (see Figure 2a) and let $(\nu_0(\xi_0),y_0(\xi_0))$ be the initial point on the orbit γ. As ξ increases from ξ_0 to ξ, it follows from (1.8) and the first equation of (1.6) that

$$\xi-\xi_0=\int_{\nu_0}^{\nu}\frac{d\nu}{y}=\int_{\nu_0}^{\nu}\frac{d\nu}{|\nu+\phi_*|\sqrt{-\frac{3}{g_1^2}\left(-\frac{g_1^2}{(\nu+\phi_*)^2}+\frac{2g_1^2+\phi_*^3}{\phi_*(\nu+\phi_*)}+\nu\right)+h_a}}. \tag{2.3}$$

Notice that

$$\begin{aligned}&\lim_{\nu\longrightarrow-\phi_*}\left(\frac{1}{|\nu+\phi_*|\sqrt{-\frac{3}{g_1^2}\left(-\frac{g_1^2}{(\nu+\phi_*)^2}+\frac{2g_1^2+\phi_*^3}{\phi_*(\nu+\phi_*)}+\nu\right)+h_a}}\right)^2\\ =&\lim_{\nu\longrightarrow-\phi_*}\frac{1}{3+(\nu+\phi_*)\left(-3\frac{2g_1^2+\phi_*^3}{\phi_*}-\frac{3\nu(\nu+\phi_*)}{g_1^2}+h_a(\nu+\phi_*)\right)}=\frac{1}{3}\neq0.\end{aligned}$$

It then follows that

$$\lim_{\nu\longrightarrow -\phi_*}\int_{\nu_0}^{\nu}\frac{d\nu}{|\nu+\phi_*|\sqrt{-\frac{3}{g_1^2}\left(-\frac{g_1^2}{(\nu+\phi_*)^2}+\frac{2g_1^2+\phi_*^3}{\phi_*(\nu+\phi_*)}+\nu\right)+h_a}}=B_{1s}=constant.$$

Thus, there is a constant $\widetilde{\xi}=\xi_0+B_{1s}$ such that $\lim_{\xi\to\widetilde{\xi}}\nu(\xi)=-\phi_*$.

For the regular system (1.9), $S_{1,2}$ are 2 equilibrium points, so one can apply standard existence and uniqueness results to prove that the heteroclinic orbit γ does not intersect with $S_{1,2}$. The point (ν_0,y_0) travels along the orbit γ and can never reach $S_{1,2}$ in a finite time. However, for the singular system (1.6), (ν_0,y_0) travels along the orbit γ and can maybe reach $S_{1,2}$ in a finite time. In this case $S_{1,2}$ are no longer the equilibrium points but are 2 regular points of (1.6), and the solution defined by γ is a solitary wave or a periodic wave solution of (1.6). □

From Lemma 1 and the vector fields defined by (1.6),we have the following conclusions.

Theorem 2

(i) The 2 curves in Figures 2a–2f, which connect S_1 and S_2 on the left and right sides of the straight line $\nu=-\phi_*$ and have the same level set $H(\nu,y)=h_a$, can be seen as a periodic orbit of (1.6). This case generates a periodic wave solution of (1.6).

(ii) The 3 curves in Figures 2a–2c, which connect the 3 points O, S_1, and S_2 on the left and right sides of the straight line $\nu=-\phi_*$ and have the same level set $H(\nu,y)=h_0$, can be seen as a homoclinic orbit of (1.6) to the origin O or $R(\nu,0)$. This case generates a solitary wave solution of (1.6).

(iii) The 3 curves in Figures 2d–2f, which connect the 3 points A_2, S_1, and S_2 on the left and right sides of the straight line $\nu=-\phi_*$ and have the same level set $H(\nu,y)=h_2$, can be seen as a homoclinic orbit of (1.6) to A_2, S_1, S_2. This case generates a solitary wave solution of (1.6).

It is easy to see that the orbits of (1.6) and the regular system (1.9) have the same level sets given by (1.8). However, one can show from Theorem 2 that, in contrast to the regular system (1.9), the orbits of (1.6) passing through the straight line $\nu=-\phi_*$ can be seen as a periodic solution or a homoclinic orbit. In the next section, we will use the theorem to study the exact solitary wave solutions and periodic wave solutions for system (1.1).

3. Exact and explicit traveling wave solutions of (1.6)

In this section, by using the above results obtained in Section 2, we consider the dynamical behavior of the traveling wave solutions of (1.1) and compute their exact parametric representations for these traveling wave solutions.

1. (1) Corresponding to the heteroclinic orbits of (1.9) defined by $H(\nu,y)=h_0$, connecting the saddle point O and the nodes $S_{1,2}$ and connecting the nods $S_{1,2}$, respectively (see Figures 2a–2c), by Theorem 1, we can obtain a solitary wave solution of peak type of (1.1) with respect to ν. In fact, when $h=h_0$, we see from (1.7) that $y^2=\frac{3}{g_1^2\phi_*^2}(g_1^2-\phi_*^3-\phi_*^2\nu)\nu^2$. By using the first equation of (1.6), we obtain the parametric representations of the solitary wave solution as follows:

$$\nu_1(\xi)=(g_1^2-\phi_*^3)\left(1-\tanh^2\frac{\sqrt{3(g_1^2-\phi_*^3)}}{2g_1\phi_*}\xi\right),\ \xi\in(-\infty,\infty). \tag{3.1}$$

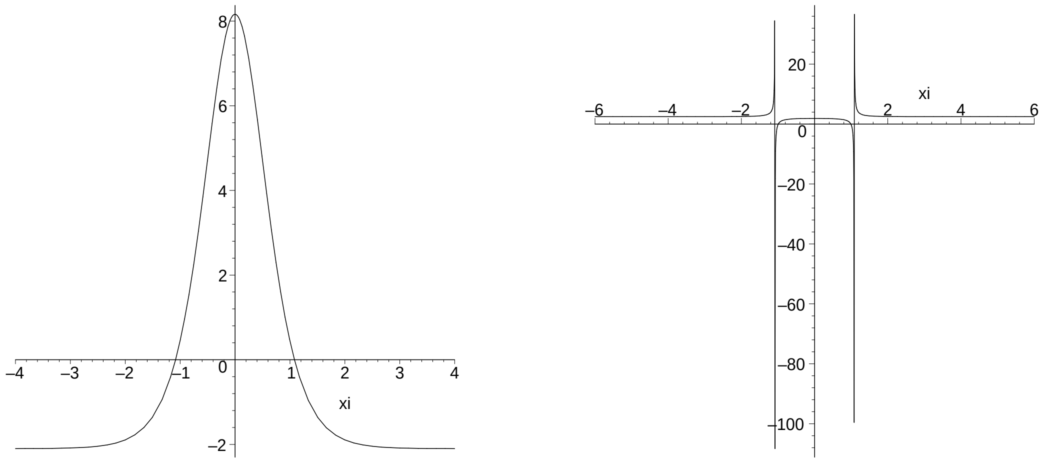
Thus, we obtain the corresponding exact traveling wave solutions of (1.1):

$$\eta_1(x,t) = \phi_* + (g_1^2 - \phi_*^3)\left(1 - \tanh^2 \frac{\sqrt{3(g_1^2-\phi_*^3)}}{2g_1\phi_*}(x-ct)\right), \tag{3.2a}$$

$$u_1(x,t) = c - \frac{g_1}{\phi_* + (g_1^2-\phi_*^3)\left(1-\tanh^2\frac{\sqrt{3(g_1^2-\phi_*^3)}}{2g_1\phi_*}(x-ct)\right)}. \tag{3.2b}$$

It is easy to see that $lim_{\xi\to\pm\infty}u(\xi) = c - \frac{g_1}{\phi_*}$. In addition, when $\tanh\frac{\sqrt{3(g_1^2-\phi_*^3)}}{2g_1\phi_*}\xi = \sqrt{\frac{g_1^2-\phi_*^3+\phi_*}{g_1^2-\phi_*^3}}$, the right side of $(3.2b)$ becomes ∞, which means that $(3.2b)$ gives rise to a discontinuous wave solution with respect to the variable u of (1.1). In other words, the discontinuity happens when the orbit in the phase space hits the singular line $\nu = \phi_*$.

The wave profiles of $\eta_1(\xi)$ and $u_1(\xi)$ are shown in Figure 3 for $c = 2, g_1 = 1, \phi_* = -2.1, \nu_2 = 2.762548006, and \nu_1 = 3.537451994$.



(a) (b)

Figure 3. The wave profiles of $\eta_1(\xi)$ and $u_1(\xi)$ in (a) and (b), respectively.

(2) Corresponding to the homoclinic orbits of (1.9) defined by $H(\nu,y) = h_2$, connecting the saddle point $(\nu_2, 0)$ (see Figure 2a), we can gain a solitary wave solution of peak type of (1.1) with respect to ν. In fact, when $h = h_2$, we see from (1.7) that $y^2 = \frac{3}{g_1^2}(\nu-\nu_2)^2(\nu_M-\nu)$, $\nu_M = \frac{1}{3}g_1^2h_2 - 2\phi_* - 2\nu_2 > \nu_1 > \nu_2 > -\phi_*$. By using the first equation of (1.6), we obtain the parametric representations of the solitary wave solution as follows:

$$\nu_2(\xi) = \nu_M - (\nu_M - \nu_2)\tanh^2\frac{\sqrt{3(\nu_M-\nu_2)}}{2g_1}\xi,\ \xi \in (-\infty,\infty). \tag{3.3}$$

Thus, we obtain the corresponding exact traveling wave solutions of (1.1):

$$\eta_2(x,t) = \phi_* + \nu_M - (\nu_M-\nu_2)\tanh^2\frac{\sqrt{3(\nu_M-\nu_2)}}{2g_1}(x-ct), \tag{3.4a}$$

$$u_2(x,t) = c - \frac{g_1}{\phi_* + \nu_M - (\nu_M - \nu_2)\tanh^2 \frac{\sqrt{3(\nu_M-\nu_2)}}{2g_1}(x-ct)}. \tag{3.4b}$$

The wave profiles of $\eta_2(\xi)$ and $u_2(\xi)$ are shown in Figure 4 for $c = 2, g_1 = 1, \phi_* = -2.1, \nu_2 = 2.762548006, \nu_1 = 3.537451994, \nu_M = 4.37806073, and h_2 = 17.10947023$.

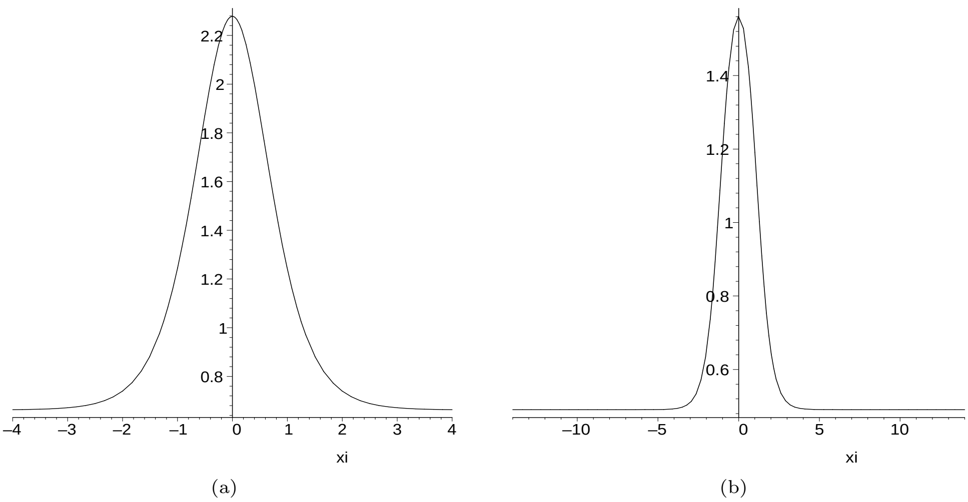


Figure 4. The wave profiles of $\eta_2(\xi)$ and $u_2(\xi)$ in (a) and (b), respectively.

2. Similarly, we can construct the parametric representations of the solitary wave solution of (1.1) as follows:

$$\eta(x,t) = \frac{1}{4}\phi_* \left(1 - 9\tanh^2 \frac{3\sqrt{3}}{4\phi_*}\xi\right), \tag{3.5a}$$

$$u(x,t) = c - \frac{4g_1}{\phi_* \left(1 - 9\tanh^2 \frac{3\sqrt{3}}{4\phi_*}\xi\right)}. \tag{3.5b}$$

Remark. To the best of our knowledge, solutions (3.4) and (3.5) obtained for Eq. (1.1) have not been reported before in the literature.

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