Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2013) 37: 981 – 992

doi:10.3906/mat-1208-23

Orthogonal systems in L^2 spaces of a vector measure

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Received: 04.09.2012 • **Accepted:** 13.12.2012 • **Published Online:** 23.09.2013 • **Printed:** 21.10.2013

Abstract: Let $\mathbf{m} : \Sigma \to X$ be a Banach space valued countably additive vector measure. In this paper we present a procedure to construct an $\mathbf{m}$-orthogonal system in the space $L^2(\mathbf{m})$ of square integrable functions with respect to $\mathbf{m}$. If the vector measure is constructed from a family of indeterminate scalar measures, it is possible to obtain a family of polynomials that is orthogonal with respect to this vector measure. On the other hand, if the vector measure is fixed, then we can obtain sequences of orthogonal functions using the $Kadec - Pelczyński$ disjointification method.

Key words: Orthogonal sequences, vector measures, integration

1. Introduction

Let X be a Banach space and let (Ω, Σ) be a measurable space. Let $\mathbf{m} : \Sigma \to X$ be a countably additive vector measure. We consider a sequence of (nonzero) real functions $\{f_n\}_n$ that are square $\mathbf{m}$-integrable. We say that f_j and f_k are orthogonal with respect to $\mathbf{m}$ if $\int f_j f_k d\mathbf{m} = 0$ for all $j, k \in \mathbb{N}$ $j \neq k$. This notion is defined by imposing *simultaneously* orthogonality with respect to *all* the elements of the family of scalar measures defined by the vector measure. This kind of orthogonality with respect to a vector measure has been studied in a series of papers (see [6, 7, 8, 15]). In these papers we can find several examples (see Examples 4, 5, and 10 [15]). The orthogonality with respect to a vector measure generalizes the usual orthogonality given by the integral with respect to a scalar measure, and provides a natural setting for studying the properties of functions that are orthogonal with respect to a family of measures. Families of polynomials that are orthogonal with respect to a large set of scalar measures were studied at the end of the 19th century. These measures are called indeterminate measures.

We consider the following equation

$$\int_{-\infty}^{\infty} x^n \omega(x) dx = \int_{-\infty}^{\infty} x^n e^{-x^2} dx \quad if \quad n = 0, 1, ...$$

then $\omega(x) = e^{-x^2}$. This problem was presented by Tchebychev in 1885 and it gave rise to the Moment's problem. Let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence of real numbers. The Moment's problem consists of finding necessary and

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Support from the Ministerio de Economía y Competitividad #MTM2012-36740-C02-02 (Spain) is gratefully acknowledged.
2010 *AMS Mathematics Subject Classification:* Primary: 46G10, Secondary: 46C50, 46E30.

sufficient conditions for the existence of a distribution (increasing bounded function) $\alpha(x)$ in $[0,\infty)$, such that

$$\int_0^\infty x^n d\alpha(x) = \mu_n \quad n = 0, 1, ...$$

The first example of such an indeterminate measure was presented by Stieltjes in 1894 (see [16]). He proved that

$$\int_0^\infty x^{n-\log x} \sin[2\pi \log(x)]dx = 0 \quad \textit{for each} \quad n = 0, 1, 2...$$

which implies that all the densities on the half-line

$$d_\lambda(x) = \frac{(1 + \lambda \sin[2\pi \log(x)])}{x^{\log x}}, \quad \lambda \in [-1, 1]$$

have the same Moments. The study of this kind of measures resulted in a beautiful mathematical theory. For instance, the Moment's problem was studied by Nevalinna 1922 using the modern function theory (see [11]). In 1923, Marcel Riesz solved the Moment's problem by obtaining general results on the extension of positive functionals (see [14]). The Stieltjes-Wigert polynomial defines a family of polynomials that are orthogonal with respect to this class of measures. Let $\{d_\lambda\}_{\lambda\in[-1,1]}$ be a family of densities. Then we will see that it is possible to define a vector measure $\mathbf{m}$ using these densities and it is possible to obtain a family of polynomials that is orthogonal with respect to this vector measure. In this paper the existence of orthogonal sequences in the spaces of square integrable functions with respect to a vector measure $\mathbf{m}$ is analyzed. After the preliminary Section 2, we show in Section 3 that we can always build an orthogonal sequence with respect to a vector measure $\mathbf{m}$, where $\mathbf{m}$ has been obtained from a family of indeterminate measures. An example that shows this construction is provided. Finally, in Section 4 a Kadec-Pelczynski disjointification technique that produces sequences of functions that are almost disjoint and then $\mathbf{m}$-orthogonal in $L^2(\mathbf{m})$ is developed and the existence of basic sequences in some subspaces of $L^2(\mathbf{m})$ is shown. Some previous results regarding this technique have been developed in a series of papers (see [5], [9], [10]). In this paper we adapt this technique in the context of spaces of square integrable functions with respect to a suitable vector measure.

2. Basic results

We will use standard Banach and function space notation; our main references are [3, 4, 10, 12]. Let X be a Banach space. We will denote by B_X the unit ball of X, that is $B_X := \{x \in X : \|x\| \leq 1\}$. X' is the topological dual of X and $B_{X'}$ its unit ball. Let (Ω, Σ) be a measurable space. Throughout the paper $\mathbf{m} : \Sigma \to X$ will be a countably additive vector measure, i.e. $\mathbf{m}(\cup_{n=1}^\infty A_n) = \sum_{n=1}^\infty \mathbf{m}(A_n)$ in the norm topology of X for all sequences $\{A_n\}_n$ of pairwise disjoint sets of Σ. We say that a countably vector measure $\mathbf{m} : \Sigma \to X$, where X is a Banach lattice, is positive if $\mathbf{m}(A) \geq 0$ for all $A \in \Sigma$. For each element $x' \in X'$ the formula $\langle \mathbf{m}, x'\rangle(A) := \langle \mathbf{m}(A), x'\rangle$, $A \in \Sigma$, defines a (countably additive) scalar measure. We write $|\langle \mathbf{m}, x'\rangle|$ for its variation, i.e. $|\langle \mathbf{m}, x'\rangle|(A) := \sup \sum_{B\in\Pi} |\langle \mathbf{m}(B), x'\rangle|$ for $A \in \Sigma$, where the supremum is computed over all finite measurable partitions Π of A. We say that an element $x' \in X'$ is $\mathbf{m}$-positive if the scalar measure $\langle \mathbf{m}, x'\rangle$ is positive, i.e. $|\langle \mathbf{m}, x'\rangle| = \langle \mathbf{m}, x'\rangle$. A nonnegative function $\|\mathbf{m}\|$ whose value on a set $A \in \Sigma$ is given by $\|\mathbf{m}\|(A) = \sup\{|\langle \mathbf{m}, x'\rangle|(A) : x' \in X', \|x'\| \leq 1\}$ is called the semivariation of $\mathbf{m}$. The measure $\mathbf{m}$ is absolutely continuous with respect to μ if $\lim_{\mu(A)\to 0} \mathbf{m}(A) = 0$; we say that μ is a control measure for $\mathbf{m}$ and we write

$\mathbf{m} \ll \mu$. It is known that there always exists an element $x' \in X'$ such that $\mathbf{m} \ll |\langle \mathbf{m}, x'\rangle|$ and so $|\langle \mathbf{m}, x'\rangle|$ is a control measure for $\mathbf{m}$. Such measures are called Rybakov measures for $\mathbf{m}$ (see [4, Ch.IX,2]). Note that if $\mathbf{m}$ is positive and x' is a positive element of the Banach lattice X', then $|\langle \mathbf{m}, x'\rangle| = \langle \mathbf{m}, x'\rangle$. The space $L^1(\mathbf{m})$ of integrable functions with respect to $\mathbf{m}$ is a Banach function space over any Rybakov measure μ for $\mathbf{m}$ (see [1, 10]). The elements of this space are (classes of μ-a.e. measurable) functions f that are integrable with respect to each scalar measure $\langle \mathbf{m}, x'\rangle$, and for every $A \in \Sigma$ there is an element $\int_A f d\mathbf{m} \in X$ such that $\langle \int_A f d\mathbf{m}, x'\rangle = \int_A f d\langle \mathbf{m}, x'\rangle$ for every $x' \in X'$. The space $L^1(\mathbf{m})$ of $\mathbf{m}$-a.e. equal $\mathbf{m}$-integrable functions is an order continuous Banach lattice endowed with the norm $\|\cdot\|_{L^1(\mathbf{m})}$ and the $\mathbf{m}$-a.e. order. We consider the spaces $L^2(\mathbf{m})$, which are also order continuous Banach function spaces over the space $(\Omega, \Sigma, |\langle \mathbf{m}, x_0'\rangle|)$ with weak unit where $|\langle \mathbf{m}, x_0'\rangle|$ is a Rybakov measure. We say that a measurable function $f : \Omega \to \mathbf{R}$ is 2-integrable with respect to $\mathbf{m}$ if $|f|^2 \in L^1(\mathbf{m})$ with the norm $\|f\|_{L^2(\mathbf{m})} := \||f|^2\|_{L^1(\mathbf{m})}^{\frac{1}{2}}$, $f \in L^2(\mathbf{m})$.

Let X be a Banach space and let $\{x_n\}_{n=1}^\infty$ be a sequence of X. We say that $\{x_n\}_{n=1}^\infty$ is a *Schauder basis* of X if there exists a unique sequence of scalars $\{\alpha_n\}_{n=1}^\infty$ such that $x = \lim_{n\to\infty} \sum_{k=1}^n \alpha_k x_k$ for all $x \in X$. We say that $\{x_n\}_{n=1}^\infty$ is a *basic sequence* if it is a Schauder basis of its closed span. Let $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ be two bases for the Banach spaces X and Y, respectively. It follows from the closed graph theorem that $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ are equivalent if and only if there exists an isomorphism T from X to Y for which $T(x_n) = y_n$ for all $n \in \mathbb{N}$. The following result can be found in ([10], Proposition 1.a.9.).

Remark 2.1

1. *If $\{x_n\}_{n=1}^\infty$ is a basic sequence of a Banach space X and $\{y_n\}_{n=1}^\infty$ is another sequence in X so that $\|x_n - y_n\| \to 0$ then $\{y_n\}_{n=1}^\infty$ is a basic sequence of a Banach space X.*

2. *If $\{x_n\}_{n=1}^\infty$ is a basic sequence of a Banach space X with basis constant K and $\{y_n\}_{n=1}^\infty$ is another sequence in X such that $\sum_{n=1}^\infty \|x_n - y_n\| < 1/(2K)$. Then $\{y_n\}_{n=1}^\infty$ is a basic sequence which is equivalent to $\{x_n\}_{n=1}^\infty$.*

Now we consider the particular case when X is a space of 2-integrable functions with respect to a vector measure $\mathbf{m}$. In this case, the question of how to recognize a basic sequence arises. The following remark provides a basic test for recognizing a basis in a subspace of $L^2(\mathbf{m})$, (see [3], Theorem 1. Ch.V).

Remark 2.2 *Let $\{f_n\}_{n=1}^\infty$ be a sequence of nonzero functions in $L^2(\mathbf{m})$, then in order for $\{f_n\}_{n=1}^\infty$ to be a basic sequence, it is both necessary and sufficient that there exists a positive finite constant K so that for any choice of scalars $\{\alpha_i\}_{i=1}^\infty$ and any integers $m < n$ we have*

$$\left\|\sum_{i=1}^m \alpha_i f_i\right\|_{L^2(\mathbf{m})} \le K \left\|\sum_{i=1}^n \alpha_i f_i\right\|_{L^2(\mathbf{m})}. \tag{1}$$

For instance, if $\mathbf{m} : \Sigma \to X$ is a positive vector measure, $\|f\|_{L^2(\mathbf{m})} = \|\int |f|^2\|_X^{1/2}$ for all $f \in L^2(\mathbf{m})$ (see [2]), and so the criterion above can be written as follows. For any finite sequence of scalars $\{\alpha_i\}_{i=1}^\infty$ and any integers $m < n$,

$$\left\|\int (\sum_{i=1}^m \alpha_i f_i)^2 d\mathbf{m}\right\|_X \le K^2 \left\|\int (\sum_{i=1}^n \alpha_i f_i)^2 d\mathbf{m}\right\|_X. \tag{2}$$

If $\{f_i\}_{i=1}^{\infty}$ is an $\mathbf{m}$*-orthogonal sequence then (2) is equivalent to*

$$\left\|\sum_{i=1}^{m} \alpha_i^2 \int f_i^2 d\mathbf{m}\right\|_X \leq K^2 \left\|\sum_{i=1}^{n} \alpha_i^2 \int f_i^2 d\mathbf{m}\right\|_X . \tag{3}$$

3. m-Orthogonal sequences

The notion of $\mathbf{m}$-orthonormal sequence is the natural generalization of the concept of orthonormal sequence in a Hilbert space $L^2(\mu)$ and has been studied in [6, 7, 8, 13, 15]. We consider a sequence of real functions $\{f_i\}_{i=1}^{\infty}$ that are square $\mathbf{m}$-integrable. We say that $\{f_i\}_{i=1}^{\infty}$ is $\mathbf{m}$-orthogonal if

$$\int f_i^2 d\mathbf{m} \neq 0, \quad \textit{for all} \quad i \in \mathbb{N}, \ \textit{and} \int f_i f_j d\mathbf{m} = 0, \quad i \neq j \quad i, j \in \mathbb{N}. \tag{4}$$

and $\{f_n\}_n$ is a $\mathbf{m}$-orthonormal sequence in $L^2(\mathbf{m})$ if for all $n \in \mathbb{N}$,

$$\left\|\int f_n^2 d\mathbf{m}\right\|_X = 1.$$

Example 3.1 *Consider the Lebesgue measure space* $([0,1], \Sigma, \mu)$. *We can define a vector measure* $\mathbf{m} : \Sigma \longrightarrow \ell^2$ *by* $\mathbf{m}(A) := \sum_{n=1}^{\infty}(\int_A \varphi_n d\mu) e_n$ *for all* $A \in \Sigma$, *where* $\varphi_1(x) := \chi_{[\frac{1}{4}, \frac{3}{4}]}$ *and* $\varphi_n(x) := \chi_{[\frac{1}{4^n}, \frac{3}{4^n}]} - \frac{1}{2}\chi_{[\frac{1}{4^{n-1}}, \frac{2}{4^{n-1}}]}$, $\quad n \geq 2$.

For every $A \in \Sigma$

$$\|\mathbf{m}(A)\|^2 = \sum_{n=1}^{\infty}(\int_A \varphi_n d\mu)^2 \leq \sum_{n=1}^{\infty} \frac{2}{4^n} < \infty.$$

It is clearly a countably additive vector measure and then the corresponding space $L^2(\mathbf{m})$ *is well defined. Let us consider the following sequence of functions defined on the unit interval* $[0,1]$ *(see the Figure). For every* $i \in \mathbb{N}$,

$$f_i(x) := \sqrt{2}2^{i-1}(\chi_{[\frac{2}{4^i}, \frac{4}{4^i}]} - \chi_{[0, \frac{2}{4^i}]}).$$

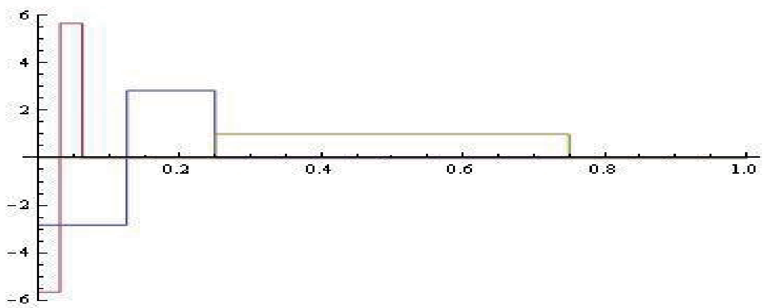

Figure. Functions $f_1(x)$, $f_2(x)$, and $f_3(x)$.

If $j > i$ *then* $f_i f_j = -\sqrt{2}2^{i-1} f_j$, $i, j \in \mathbb{N}$. *A direct calculation shows that*

$$\int_{[0,1]} f_i f_j \varphi_n = \begin{cases} 1 \ \textit{if}\ i = j = n \\ 0 \ \textit{otherwise.} \end{cases}$$

Note that $f_i^2 \leq 1 \in L^1(\mathbf{m})$ *and so* $f_i^2 \in L^2(\mathbf{m})$. *A direct calculation shows that*

$$\int_{[0,1]} f_i f_j d\mathbf{m} = 0 \quad i \neq j$$

and so $\{f_i\}_{i=1}^{\infty}$ *is an* $\mathbf{m}$*-orthogonal sequence.*

Some results on the existence of $\mathbf{m}$-orthonormal sequences in $L^2(\mathbf{m})$ are provided. For any vector measure, it is easy to prove the existence of $\mathbf{m}$-orthonormal sequences of functions in any (nontrivial) space of square integrable functions with respect to a vector measure. To show this fact we present the following lemma:

Lemma 3.2 *Let* $\mathbf{m}$ *be a positive vector measure. Suppose that there is a sequence* $\{A_n\}_{n=1}^{\infty}$ *in* Σ *of disjoint non* $\|\mathbf{m}\|$*-null sets. Then there is a* $\mathbf{m}$*-orthonormal sequence in* $L^2(\mathbf{m})$.

Proof Let $\chi_{A_n} \in L^2(\mathbf{m})$ be the characteristic function of A_n for every $n \in \mathbb{N}$. Moreover, since $\|\mathbf{m}\|(A_n) \neq 0$, there is a subset $B_n \subset A_n$ such that $\|\mathbf{m}(B_n)\|_X > 0$ for every $n \in \mathbb{N}$. Let us define $f_n = \frac{\chi_{B_n}}{\|\mathbf{m}(B_n)\|^{1/2}}, \quad n \in \mathbb{N}$. Then

$$\int f_n^2 d\mathbf{m} = \int \frac{\chi_{B_n}^2}{\|\mathbf{m}(B_n)\|} d\mathbf{m} = \frac{1}{\|\mathbf{m}(B_n)\|} \int \chi_{B_n} d\mathbf{m} = \frac{\mathbf{m}(B_n)}{\|\mathbf{m}(B_n)\|} \neq 0.$$

On the other hand, if $n \neq k$ for $n, k \in \mathbb{N}$, it is clear that $\int f_n f_k d\mathbf{m} = 0$, since $B_n \cap B_k = \emptyset$, and the result is obtained. □

Let $\{f_n\}_n$ be a sequence of $\mathbf{m}$-orthogonal functions and let $\langle \mathbf{m}, x' \rangle$ be a positive Rybakov measure. Then $\int f_l f_j d\langle \mathbf{m}, x' \rangle = \langle \int f_l f_j d\mathbf{m}, x' \rangle = 0 \quad j \neq l$, and so it is clear that an $\mathbf{m}$-orthonormal sequence also is orthogonal for each associated $\mathbf{m}$-positive Rybakov measure $\langle \mathbf{m}, x' \rangle$. The $\mathbf{m}$-orthonormality requirement for a sequence of functions in the nonscalar case introduces a strong restriction, in particular regarding completeness of the orthogonal sequence. An orthonormal set Ψ is said to be complete if there exists no other orthonormal set containing Ψ, that is, Ψ must be a maximal orthonormal set. It is easy to prove that an orthonormal set Ψ is complete if and only if for any f such that f is orthogonal to Ψ, f must be zero. Let us show this fact with an easy construction. Suppose that $(\Omega_0, \Sigma_0, \mu_0)$ is a probability measure space, and consider a complete orthonormal sequence $\{g_i\}_{i=1}^{\infty}$ in $L^2(\mu_0)$ such that $g_1 = \chi_{\Omega_0}$. Suppose that $\{g_i\}_{i=1}^{\infty}$ is also an $\mathbf{m}_0$-orthonormal sequence for a countably additive vector measure $\mathbf{m}_0 : \Sigma_0 \to X$ that is absolutely continuous with respect to μ_0. Then every measure $\langle \mathbf{m}_0, x' \rangle$, $x' \in X'$, is μ_0-continuous, and there is a function $h_{x'} \in L^1(\mu_0)$ such that $d\langle \mathbf{m}_0, x' \rangle = h_{x'} d\mu_0$. For every $k \geq 2$,

$$0 = \langle \int g_1 g_k d\mathbf{m}_0, x' \rangle = \int g_k h_{x'} d\mu_0.$$

If addition $h_{x'} \in L^2(\mu_0)$, since the sequence $\{g_i\}_{i=1}^{\infty}$ is complete, the equalities above imply $h_{x'} = r(x')\chi_{\Omega_0}$ for a real number $r(x')$. Therefore, if we assume that for every $x' \in X'$ the corresponding Radon-Nikodým derivative belongs to $L^2(\mu_0)$, we obtain that

$$(\mathbf{m}_0)_{x'}(A) = r(x')\mu_0(A), \qquad A \in \Sigma_0, \;\; x' \in X'.$$

This relation establishes a strong restriction on $\mathbf{m}_0$. For instance, suppose that X is a Banach space with an unconditional basis $\{e_i\}_{i=1}^{\infty}$. Then $\mathbf{m}_0(A) = \sum_{i=1}^{\infty}\langle \mathbf{m}_0(A), e_i'\rangle e_i$, where $\{e_i'\}_{i=1}^{\infty}$ are the corresponding biorthogonal functionals and $A \in \Sigma_0$. In this case the relation above implies that

$$\mathbf{m}_0(A) = (\sum_{i=1}^{\infty} r(e_i')e_i)\mu_0(A)$$

for every $A \in \Sigma_0$, i.e. $\mathbf{m}_0$ can be in fact considered as a scalar positive measure.

Remark 3.3 *The above argument shows that in general we cannot expect completeness for* $\mathbf{m}$*-orthonormal sequence of functions, although under certain (strong) assumptions it is possible to obtain* $\mathbf{m}$*-orthonormal basis for* $L^2(\mathbf{m})$ *(see [13]). Thus, although the results that we present in what follows can be used to obtain information about standard orthonormal sequences* $\{f_i\}_{i=1}^{\infty}$ *in Hilbert spaces* $L^2(\mu)$*, the procedure of splitting the scalar measure* μ *into a vector measure* $\mathbf{m}$ *preserving orthonormality is essentially limited by a certain noncompleteness assumption for* $\{f_i\}_{i=1}^{\infty}$.

4. A constructive procedure for finding m-orthogonal sequences: Indeterminate measures

In this section we show a procedure to build families of orthogonal polynomials on a vector measure that is made of indeterminate measures. The following procedure shows that an orthogonal sequence with respect to a vector measure $\mathbf{m}$ can be built, where $\mathbf{m}$ has been obtained from a family of indeterminate measures. Consider the Lebesgue measure space (Ω, Σ, dx). Let $\{S_n\}_n$ be a sequence of real numbers and let Δ_n be the $(n+1) \times (n+1)$ Hankel matrix where

$$\Delta_n = \begin{bmatrix} S_0 & S_1 & \dots & S_n \\ S_1 & \ddots & & \vdots \\ \vdots & & & \\ S_n & \dots & & S_{2n} \end{bmatrix} \tag{5}$$

It is well known that if $\det(\Delta_n) > 0$ for all $n \in \mathbb{N}$ then the sequence $\{S_n\}_{n=0}^{\infty}$ is defined positive. In this case we define the linear operator $\mathfrak{L} : \mathbf{P}[x] \to \mathbb{R}$ such that $\mathfrak{L}(Q_n(x)) = \sum_{k=0}^{\infty} a_k S_k$, where $Q_n(x) = \sum_{k=0}^{n} a_k x^k$. The Moment's Problem is determined if $\{S_n\}_n$ is a defined positive sequence, i.e. there exists a nondecreasing distribution, $F(x)$, with support into Ω such that for all n, $\int_\Omega x^n F(x)dx = S_n$ is satisfied.

We call indeterminate measure to a family $F_\lambda(x)$ of nondecreasing distributions with support into Ω which have the same Moments, i.e. the Moments $\int_\Omega x^n F_\lambda(x)dx = S_n$ do not depend on the parameter λ. We define the following sequence:

$$P_n(x) = \begin{vmatrix} S_0 & \dots & & S_n \\ \vdots & \ddots & & \vdots \\ S_{n-1} & & & S_{2n-1} \\ 1 & x & \dots & x^n \end{vmatrix}$$

Then it is easy to prove that

$$\mathfrak{L}(x^k P_n(x)) = \begin{cases} 0 & \text{if} \quad k < n \\ \det(\Delta_n) & \text{if} \quad k = n \end{cases} \tag{6}$$

Let X be a Banach space. We can define a countably additive vector measure $\nu : \Sigma \to X$ as $\nu(A) = \int_A F_\lambda(x)dx$. Then the construction presented above shows that

$$\int_\Omega P_j(x)P_l(x)d\nu = 0 \text{ } if \text{ } j \neq l.$$

Example 4.1 *In this case we consider the first example of an indeterminate measure that was presented by Stieltjes in 1894 (see [16]). In this example Stieljes shows that for each* $n \in \mathbb{N}$

$$\int_0^\infty x^n e^{-\ln^2(x)}[1 + \lambda \sin(2\pi \ln(x))]dx = \sqrt{\pi}e^{(n+1)^2/4}.$$

If we take $|\lambda| < 1$ *then* $F_\lambda(x) = e^{-\ln^2(x)}[1 + \lambda \sin(2\pi \ln(x))] > 0$ *is a positive function for all* $x \in [0, \infty[$*; thus* $\mu_\lambda(y) = \int_0^y e^{-\ln^2(x)} x^n [1 + \lambda \sin(2\pi \ln(x))]dx$ *is a family of nondecreasing distributions with support into* $[0, \infty[$*, which has the same moments* $S_n = \sqrt{\pi}e^{(n+1)^2/4}$*. It is immediate to prove that the sequence* $\{S_n = \sqrt{\pi}e^{(n+1)^2/4}\}_{n=0}^\infty$ *is positive defined; therefore,*

$$\int_0^\infty x^k P_n(x) e^{-\ln^2(x)}[1 + \lambda \sin(2\pi \ln(x))]dx = \begin{cases} 0 & if \quad k < n \\ \det(\Delta_n) & if \quad k = n \end{cases}$$

for every $\lambda \in \mathbb{R}, |\lambda| < 1$*. The above construction provides a sequence of* ν*-orthogonal polynomials* $P_n(x)$ *for a suitable vector measure.*

$$\begin{aligned} P_0(x) &= S_0 = 2.27588, \\ P_1(x) &= -4.81803 + 2.27588x, \\ P_2(x) &= 183.457 - 139.22x + 15.059x^2, \\ P_4(x) &= -655344. + 611203.x - 106211.x^2 + 3438.93x^3, \\ &\vdots \end{aligned}$$

Now we take $\Omega = (0, \infty)$ *and* Σ *the* σ*-algebra of the Lebesgue subsets of* Ω*. We can define* $\nu : \Omega \to c_0$ *by*

$$\nu(A) = \left\{ \int_A \frac{e^{-\ln^2(x)}}{m}[1 + \frac{1}{m+1} \sin(2\pi \ln(x))]dx \right\}_{m=1}^\infty,$$

where dx *is the Lebesgue measure and* $A \in \Sigma$*. Using elementary integral calculus, it is easy to prove that for every* $A \in \Sigma$

$$\lim_{m \to \infty} \int_A \frac{e^{-\ln^2(x)}}{m}[1 + \frac{1}{m+1} \sin(2\pi \ln(x))]dx = 0.$$

This shows that ν *is well defined and so countably additive. Moreover, it is also clear that the functions* $p_l(x) \in L^2(\mathbf{m})$ *and they satisfy that for all* $j < l$*,*

$$\int_0^\infty p_j(x)p_l(x)d\nu = 0$$

and

$$\left\| \int_0^\infty p_j(x)p_l(x)d\nu \right\|_{c_0} = B_l$$

if $j = l$ *where* B_l *is a nonnull constant for all* $l \in \mathbb{N}$*.*

5. Kadec-Pełczyński decomposition

The aim of this section is to give a canonical procedure to obtain disjoint sequences in the space $L^2(\mathbf{m})$. This will be the first step for finding $\mathbf{m}$-orthogonal sequences and providing the corresponding existence theorems. In what follows, a well-known result of *Kadec* and *Pelczyński* will be applied to the context of sequences of functions on spaces of integrable functions with respect to a vector measure (see [9]). Throughout this section we will consider a positive vector measure $\mathbf{m}$.

Let H be a Hilbert space and let us consider a positive countably additive vector measure $\mathbf{m} : \Sigma \to H$. We suppose that $\|\chi_\Omega\|_{L^2(\mathbf{m})} = \|\chi_\Omega\|^{1/2}_{L^1(\mathbf{m})} = \|\mathbf{m}\|^{1/2}(\Omega) = 1$ and $\{f_n\}_{n=1}^\infty \in L^2(\mathbf{m})$. We define the subsets of Ω

$$\sigma(f,\varepsilon) = \{t \in \Omega : |f(t)| \geq \varepsilon \|f\|_{L^2(\mathbf{m})}\}$$

and the subsets of $L^2(\mathbf{m})$

$$M_{L^2(\mathbf{m})}(\varepsilon) = \{f \in L^2(\mathbf{m}) : \|\mathbf{m}\|(\sigma(f,\varepsilon)) \geq \varepsilon\}.$$

By normalizing if necessary, we assume that $\|f_n\|_{L^2(\mathbf{m})} = 1$ for all $n \in \mathbb{N}$.

Remark 5.1 *Note that the classes* $M_{L^2(\mathbf{m})}(\varepsilon)$ *have the following properties:*

(1) *If* $\varepsilon_1 < \varepsilon_2$ *, then* $M_{L^2(\mathbf{m})}(\varepsilon_1) \supset M_{L^2(\mathbf{m})}(\varepsilon_2)$*.*

(2) $\bigcup_{\varepsilon > 0} M_{L^2(\mathbf{m})}(\varepsilon) = L^2(\mathbf{m})$*.*

(3) *If* $f \neq 0$ *does not belong to* $M_{L^2(\mathbf{m})}(\varepsilon)$*, then there exists a set* A *such that* $\|\mathbf{m}\|(A) < \varepsilon$ *and*

$$\left\| \int_A \left| \frac{f(t)}{\|f\|_{L^2(\mathbf{m})}} \right|^2 d\mathbf{m} \right\|_H \geq 1 - \varepsilon^2.$$

The first property is obvious. To prove the second, we suppose that there exists a square $\mathbf{m}$-integrable function g so that it is not in $\bigcup_{\varepsilon>0} M_{L^2(\mathbf{m})}(\varepsilon)$ for all $\varepsilon > 0$, in particular $g \neq 0$, that is, $\|\mathbf{m}\|(Supp\ g) > 0$. Since $Supp\ g = \cup_{n \geq 1} \sigma(g, \frac{\varepsilon}{2^n})$ for every $\varepsilon > 0$, then

$$\|\mathbf{m}\|(Supp\ g) \leq \sum_{n \geq 1} \|\mathbf{m}\|(\sigma(g, \frac{\varepsilon}{2^n})) \leq \sum_{n \geq 1} \frac{\varepsilon}{2^n} = \varepsilon.$$

So $\|\mathbf{m}\|(Supp\ g) = 0$, which is a contradiction. To prove the third we denote by A the set $\sigma(f,\varepsilon)$. Then

$$1 = \left\| \int_\Omega \left| \frac{f(t)}{\|f\|_{L^2(\mathbf{m})}} \right|^2 d\mathbf{m} \right\|_H \leq \left\| \int_A \left| \frac{f(t)}{\|f\|_{L^2(\mathbf{m})}} \right|^2 d\mathbf{m} \right\|_H + \varepsilon^2 \|\mathbf{m}(\Omega/A)\|_H$$

$$= \left\| \int_A \left| \frac{f(t)}{\|f\|_{L^2(\mathbf{m})}} \right|^2 d\mathbf{m} \right\|_H + \varepsilon^2 \|\mathbf{m}\|(\Omega/A) \leq \left\| \int_A \left| \frac{f(t)}{\|f\|_{L^2(\mathbf{m})}} \right|^2 d\mathbf{m} \right\|_H + \varepsilon^2.$$

This finishes the proof of (3).

Lemma 5.2 *Let (Ω, Σ, μ) be a measure space and let $X(\mu)$ be an order continuous Banach function space; then for all $f \in X(\mu)$*

$$\lim_{\mu(A)\to 0} \|f\chi_A\|_{X(\mu)} = 0.$$

The proof can be found in (Lemma 2.37, [12]).

The following result shows 2 mutually excluding possibilities for a sequence $\{f_n\}_n$ of functions in $L^2(\mathbf{m})$. On the one hand, when $\{f_n\}_n$ is included in the set $M_{L^2(\mathbf{m})}(\varepsilon)$ for some $\varepsilon > 0$, the norms $\|\cdot\|_{L^2(\mathbf{m})}$ and $\|\cdot\|_{L^1(\mu)}$ are equivalent, where μ is a Rybakov control measure for $\mathbf{m}$. On the other hand, when $\{f_n\}_n \not\subseteq M_{L^2(\mathbf{m})}(\varepsilon)$ for every $\varepsilon > 0$, we can build another sequence $\{h_k\}_k$ of disjoint functions of $L^2(\mathbf{m})$, in such a way that $\{f_n\}_n$ and $\{h_k\}_k$ are equivalent ([5], Chapter 1.9.). This procedure gives us a tool for building disjoint sequences in subspaces of $L^2(\mathbf{m})$ that in fact are unconditional basic sequences. The order continuity of the space is the key point for the construction.

Theorem 5.3 *Let $\mu = |\langle \mathbf{m}, x_0' \rangle|$ be a Rybakov measure for a vector measure $\mathbf{m}$ and let (Ω, Σ, μ) be a probability measure space. Let $\{f_n\}_n$ be a sequence of functions into $L^2(\mathbf{m})$.*

(1) *If $\{f_n\}_{n=1}^\infty \subset M_{L^2(\mathbf{m})}(\varepsilon)$ for some $\varepsilon > 0$ then $\{f_n\}_{n=1}^\infty$ converges to zero in $L^2(\mathbf{m})$ if and only if $\{f_n\}_{n=1}^\infty$ converges to zero in $L^1(\mu)$.*

(2) *If $\{f_n\}_{n=1}^\infty \not\subseteq M_{L^2(\mathbf{m})}(\varepsilon)$ for all $\varepsilon > 0$ then there exists a subsequence $\{n_k\}_{k=1}^\infty$ and a disjointly supported function $\{h_k\}_{k=1}^\infty \subset L^2(\mathbf{m})$ such that $|h_k| \leq |f_{n_k}|$ for all k and $\{h_k\}_{k=1}^\infty$ and $\{f_{n_k}\}_{k=1}^\infty$ are equivalent unconditional basic sequences that satisfy $\lim_{k\to\infty} \|f_{n_k} - h_k\|_{L^2(\mathbf{m})} = 0$.*

Proof It is well known that $L^2(\mathbf{m})$ is continuously embedded into $L^1(\mu)$ and it is an order continuous Banach lattice with weak unit. There are 2 excluding cases.

(1) We suppose that $\{f_n\}_n \subset M_{L^2(\mathbf{m})}(\varepsilon)$ for some $\varepsilon > 0$ then

$$\|f_n\|_{L^2(\mathbf{m})} \geq \|f_n\|_{L^1(\mu)} = \int_\Omega |f_n(t)| d\mu \geq \int_{\sigma(f_n,\varepsilon)} |f_n(t)| d\mu$$

$$\geq \varepsilon \|f_n\|_{L^2(\mathbf{m})} \mu(\sigma(f_n, \varepsilon)).$$

The direct implication is obtained from the inclusion $L^2(\mathbf{m}) \hookrightarrow L^1(\mu)$. Conversely, we suppose that $\mu(\sigma(f_n, \varepsilon))$ converges to 0. Since μ is a Rybakov measure and thus it is a control measure $\|\mathbf{m}\|(\sigma(f_n, \varepsilon))$ converges to 0, but it gives a contradiction because $M_{L^2(\mathbf{m})}(\varepsilon) = \{f \in L^2(\mathbf{m}) : \|\mathbf{m}\|(\sigma(f, \varepsilon)) \geq \varepsilon\}$. Therefore, $\{f_n\}_n$ converges to zero in $L^2(\mathbf{m})$ if and only if $\{f_n\}_n$ converges to zero in $L^1(\mu)$.

(2) If the above supposition does not hold, then $\{f_n\}_n \nsubseteq M_{L^2(\mathbf{m})}(\varepsilon)$ for all ε. In order to simplify the notation we consider $\|f_n\|_{L^2(\mathbf{m})} = 1$. Thus, there exists an index $n_1 \in \mathbb{N}$ such that f_{n_1} is not in $M_{L^2(\mathbf{m})}(\varepsilon)$ where $j_1 = 2$. We take $\varepsilon = 4^{-j_1}$. Then $\|\mathbf{m}\|(\sigma(f_{n_1}, 4^{-j_1})) < 4^{-j_1}$ and

$$\|\chi_{\sigma(f_{n_1},4^{-j_1})^c} f_{n_1}\|_{L^2(\mathbf{m})} = \left\| \int |\chi_{\sigma(f_{n_1},4^{-j_1})^c} f_{n_1}|^2 d\mathbf{m} \right\|_H^{1/2}$$

$$= \left\| \int_{\chi_{\sigma(f_{n_1},4^{-j_1})^c}} |f_{n_1}|^2 d\mathbf{m} \right\|_H^{1/2} \leq 4^{-j_1} \left\| \mathbf{m}(\sigma(f_{n_1}, 4^{-j_1})^c) \right\|_H^{1/2}$$

$$\leq 4^{-j_1} \|\mathbf{m}\|(\Omega)^{1/2} = 4^{-j_1}.$$

Now we apply Lemmma 5.2, so there exists $\delta_1 > 0$ such that for all $A \in \Sigma$ with $\|\mathbf{m}\|(A) < \delta_1$ it follows $\|\chi_A f_{n_1}\| < 4^{-(j_1+1)}$. We take $j_2 > j_1$ such that $4^{-j_2} < \delta_1$. Following the same argument, there exists $n_2 > n_1$ such that f_{n_2} is not in $M_{L^2(\mathbf{m})}(4^{-j_2})$; thus $\|\mathbf{m}\|(\sigma(f_{n_2}, 4^{-j_2})) < 4^{-j_2} < \delta_1$

$$\|\chi_{\sigma(f_{n_2},4^{-j_2})} f_{n_1}\| \leq 4^{-(j_1+1)},$$

$$\|\chi_{\sigma(f_{n_2},4^{-j_2})^c} f_{n_2}\| \leq \|4^{-j_2} \chi_{\sigma(f_{n_2},4^{-j_2})^c}\| \leq 4^{-j_2}.$$

We take $\varepsilon = 4^{-(j_2+1)}$. Again, we apply Lemma 5.2 and there exists $\delta_2 > 0$ such that for all $A \in \Sigma$ with $\|\mathbf{m}\|(A) < \delta_2$ it follows $\|\chi_A f_{n_1}\|$, $\|\chi_A f_{n_2}\| < 4^{-(j_2+1)}$. Let $j_3 > j_2$ be an integer satisfying that $4^{-j_3} < \delta_2$. Again there exists a integer $n_3 > n_2 > n_1$ such that f_{n_3} is not in $M_{L^2(\mathbf{m})}(4^{-j_3})$, as in the above case we have $\|\mathbf{m}\|(\sigma(f_{n_3}, 4^{-j_3})) < 4^{-j_3} < \delta_2$, and therefore

$$\|\chi_{\sigma(f_{n_3},4^{-j_3})} f_{n_1}\|, \quad \|\chi_{\sigma(f_{n_3},4^{-j_3})} f_{n_2}\| \leq 4^{-(j_2+1)},$$

$$\|\chi_{\sigma(f_{n_3},4^{-j_3})^c} f_{n_3}\| \leq 4^{-j_3}.$$

In the same way, it is possible to find 2 subsequences $\{f_{n_k}\}_{k=1}^{\infty}$ and $\sigma(f_{n_k}, 4^{-j_k})_k$ that satisfy the following inequalities:

$$\|\mathbf{m}\|(\sigma(f_{n_k}, 4^{-j_k})) < 4^{-j_k},$$

$$\|\chi_{\sigma(f_{n_k},4^{-j_k})^c} f_{n_k}\| \leq 4^{-j_k},$$

$$\|\chi_{\sigma(f_{n_k},4^{-j_k})} f_{n_i}\| \leq 4^{-(j_{k-1}+1)}, \quad i = 1, ..., k-1.$$

Now, we define the following disjoint sequence of sets:

$$\varphi_k = \sigma(f_{n_k}, 4^{-j_k}) \setminus \bigcup_{i=k+1}^{\infty} \sigma(f_{n_i}, 4^{-j_i}).$$

$$\varphi_k^c = \sigma(f_{n_k}, 4^{-j_k})^c \bigcup \left(\bigcup_{i=k+1}^{\infty} \sigma(f_{n_i}, 4^{-j_i}) \right).$$

Thus $\varphi_k \cap \varphi_l = \emptyset$ for $k \neq l$. This allows us to construct the sequence of disjoint functions $h_k = \chi_{\varphi_k} f_{n_k}$. Due to the lattice properties of $L^2(\mathbf{m})$ and to Remark 2.2 we obtain that the sequence $\{h_k\}_{k=1}^{\infty}$ is a basic sequence. On the other hand, we check that $\lim_{k\to\infty} \|f_{n_k} - h_k\| = 0$. Indeed

$$\|f_{n_k} - h_k\| = \|\chi_{\varphi_k^c} f_{n_k}\| \leq \|\chi_{\sigma(f_{n_k},4^{-j_k})^c} f_{n_k}\| + \|\chi_{\bigcup_{i=k+1}^{\infty} \sigma(f_{n_i},4^{-j_i})} f_{n_k}\|$$

$$\leq 4^{-j_k} + \sum_{i=k+1}^{\infty} \|\chi_{\sigma(f_{n_i},4^{-j_i})} f_{n_k}\| \leq 4^{-j_k} + \sum_{i=k+1}^{\infty} 4^{-(j_{i-1}+1)}$$

$$\leq 4^{-j_k} + \frac{4^{-(j_k+1)}}{1-1/4} = 4^{-j_k} + \frac{1}{3}4^{-j_k} = \frac{1}{3}4^{-(j_k-1)}.$$

So if we apply Remark 2.1 we obtain that $\{f_{n_k}\}_{k=1}^{\infty}$ is a basic sequence. Let K be the basic constant of the sequence $\{h_k\}_k$. Note that —by passing to a subsequence if necessary— we can obtain that $\sum_{k=1}^{\infty} \|f_{n_k} - h_k\| < 1/(2K)$. Remark 2.1 enables us to give that $\{h_k\}_k$ and $\{f_{n_k}\}_k$ are equivalent unconditional basic sequences.

□

The next result is a direct consequence of the *Kadec* and *Pelczynski* process for obtaining disjoint sequences. It provides a method for, given a convenient sequence $\{f_n\}_n$, finding a disjoint (and then $\mathbf{m}$-orthogonal) subsequence of $\{f_n\}_n$.

Corollary 5.4 *Let H be Hilbert space that is also a Banach lattice and $\mathbf{m}\colon \Sigma \to H$ a positive vector measure such that $\|\mathbf{m}\|(\Omega) = 1$. Take a Rybakov control measure $\mu = |\langle \mathbf{m}, x_0' \rangle|$ for $\mathbf{m}$ with $\|x_0'\| = 1$. If $\{f_n\}_n \subset L^2(\mathbf{m})$ is such that $\|f_n\|_{L^2(m)} = 1$ for all n and $\{f_n\}_n \nsubseteq M_{L^2(\mathbf{m})}(\varepsilon)$ for all $\varepsilon > 0$ then there exists a subsequence $\{h_k\}_{k=1}^{\infty} \subset L^2(\mathbf{m})$ such that $|h_k| \leq |f_{n_k}|$ for all k and $\{h_k\}_{k=1}^{\infty}$ is a $\mathbf{m}$-orthogonal sequence in $L^2(\mathbf{m})$. Moreover, $\{f_{n_k}\}_{k=1}^{\infty}$ and $\{h_k\}_{k=1}^{\infty}$ are equivalent unconditional basic sequences that satisfy $\lim_{k\to\infty} \|f_{n_k} - h_k\|_{L^2(\mathbf{m})} = 0$.*

The proof is a direct application of Theorem 5.3. Note that $\lim_{k\to\infty} \|f_{n_k} - h_k\|_{L^2(\mathbf{m})} = 0$ and $\|f_{n_k}\|_{L^2(\mathbf{m})} = 1$ for all k implies that $\int h_k^2 d\mathbf{m} \neq 0$ for large enough k.

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