Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2014) 38: 1 – 15

doi:10.3906/mat-1301-60

Coloring hypercomplete and hyperpath graphs

Yusuf CİVAN[1,*], **Demet TAYLAN**[2]
[1]Department of Mathematics, Süleyman Demirel University, Isparta, Turkey
[2]Department of Mathematics, Bozok University, Yozgat, Turkey

Received: 28.01.2013 • **Accepted:** 08.05.2013 • **Published Online:** 09.12.2013 • **Printed:** 20.01.2014

Abstract: Given a graph G with an induced subgraph H and a family $\mathcal{F}$ of graphs, we introduce a (hyper)graph $\mathcal{H}_H(G;\mathcal{F})=(\mathcal{V}_H,\mathcal{E}_H)$, the *hyper-$H$ (hyper)graph of G* with respect to $\mathcal{F}$, whose vertices are induced copies of H in G, and $\{H_1,H_2,\ldots,H_r\}\in\mathcal{E}_H$ if and only if the induced subgraph of G by the set $\cup_{i=1}^{r}H_i$ is isomorphic to a graph F in the family $\mathcal{F}$, and the integer r is the least integer for F with this property. When H is a k-complete or a k-path of G, we abbreviate $\mathcal{H}_{K_k}(G;\mathcal{F})$ and $\mathcal{H}_{P_k}(G;\mathcal{F})$ to $\mathcal{H}_k(G;\mathcal{F})$ and $\mathcal{HP}_k(G;\mathcal{F})$, respectively. Our motivation to introduce this new (hyper)graph operator on graphs comes from the fact that the graph $\mathcal{H}_k(K_n;\{K_{2k}\})$ is isomorphic to the ordinary Kneser graph $K(n;k)$ whenever $2k\leq n$. As a generalization of the Lovász–Kneser theorem, we prove that $\chi(\mathcal{H}_k(G;\{K_{2k}\}))=\chi(G)-2k+2$ for any graph G with $\omega(G)=\chi(G)$ and any integer $k\leq\lfloor\omega(G)/2\rfloor$. We determine the clique and fractional chromatic numbers of $\mathcal{H}_k(G;\{K_{2k}\})$, and we consider the *generalized Johnson graphs* $\mathcal{H}_r(H;\{K_{r+1}\})$ and show that $\chi(\mathcal{H}_r(H;\{K_{r+1}\}))\leq\chi(H)$ for any graph H and any integer $r<\omega(H)$. By way of application, we construct examples of graphs such that the gap between their chromatic and fractional chromatic numbers is arbitrarily large. We further analyze the chromatic number of hyperpath (hyper)graphs $\mathcal{HP}_k(G;P_m)$, and we provide upper bounds when $m=k+1$ and $m=2k$ in terms of the k-distance chromatic number of the source graph.

Key words: Kneser graphs and hypergraphs, chromatic and fractional chromatic numbers, hypercomplete and hyperpath (hyper)graphs

1. Introduction

Most surprised reactions (and equally appreciations) to Lovász's proof of the Kneser conjecture are due to his method of proof that is now counted as the beginning of the discipline: topological combinatorics. Many other proofs with more topological flavors were made afterwards. Mainly, the methods of these proofs concentrate on the famous Borsuk–Ulam theorem or its combinatorial counterparts. However, we here are willing to provide more reasons to celebrate the proof of the Kneser conjecture after more than 30 years. We show that its presence may enable us to determine the chromatic numbers of many other graphs with no additional costs. Our departure begins by understanding the construction of Kneser graphs from complete graphs.

*Correspondence: yusufcivan@sdu.edu.tr
2010 *AMS Mathematics Subject Classification:* 05C15.

The first author is supported by the Turkish Academy of Sciences (TÜBA) through the Young Scientist Award Program (TÜBA-GEBİP/2008), and both authors are partially supported by the Scientific and Technological Research Council of Turkey (TÜBİTAK) via grant TBAG-HD/335-107T879.

To be more explicit, we recall that for given integers n, k with $2k-1 \leq n$, the Kneser graph $K(n;k)$ is the graph whose vertex set consists of all k-element subsets of $[n] := \{1, 2, \ldots, n\}$, where 2 vertices form an edge if and only if they are disjoint. The Kneser conjecture [8] now takes the following form in graph theory language that computes the chromatic number of the Kneser graph.

Theorem 1.1 (Lovász–Kneser Theorem [11]) *If $k > 0$ and $n \geq 2k-1$, then $\chi(K(n;k)) = n - 2k + 2$.*

Once we take $[n]$ as the set of vertices of the complete graph K_n, it is easy to notice that a k-element subset of $[n]$ corresponds to an induced complete graph of order k, and being disjoint for any 2 such sets means exactly that their set union induces a complete graph of order $2k$ in K_n. Such an observation may naturally be generalized to an arbitrary graph by taking it as a source graph instead of K_n. In other words, when we are given a graph G, we may construct a new graph by taking its vertices to be all induced k-complete graphs in G, where $2k \leq \omega(G)$, the clique number of G, and declaring that any such 2 vertices form an edge if and only if the union of their vertices induces a $2k$-complete graph in G. Let us denote the resulting graph by $\mathcal{H}_k(G; K_{2k})$, which we call the k-hypercomplete graph of G with respect to the graph K_{2k} (see Definition 3.2 for a more explicit and general description). The next logical step is to vary the reference family within the above construction. In other words, we consider $\mathcal{H}_k(G; \mathcal{F})$, where $\mathcal{F}$ is a family of graphs. Surprisingly, some of the constructions may end up being hypergraphs depending on the choice of $\mathcal{F}$ as well as the parameter k. As an example, we note that $\mathcal{H}_k(K_n; K_{rk})$ is just the Kneser r-hypergraph of the set system $([n], \binom{[n]}{k})$ with respect to suitably chosen integers n, k, and r (see Section 2.2). The known computation of the chromatic number of $\mathcal{H}_k(K_n; K_{rk})$ can be easily generalized to other graphs, which proves the power of the hypercompletion operator in the case of hypergraphs.

Theorem 4.2 *Let G be a graph with $\omega(G) = \chi(G)$. Then for any integers k and r satisfying $2k+1 \leq n$ and $2 \leq r \leq \lfloor \frac{\omega(G)}{k} \rfloor$, we have*

$$\chi(\mathcal{H}_k(G; K_{rk})) = \lceil \frac{\chi(G) - r(k-1)}{r-1} \rceil.$$

In the particular case where $r = 2$, we state the following result, which can be considered as the generalized Lovász–Kneser theorem:

Corollary 4.3 *If G is a graph with $\chi(G) = \omega(G)$, then $\chi(\mathcal{H}_k(G; K_{2k})) = \chi(G) - 2k + 2$ for any $k \leq \lfloor \frac{\omega(G)}{2} \rfloor$.*

In fact, we show that $\chi(\mathcal{H}_k(G; K_{2k})) \in [\omega(G) - 2k + 2, \chi(G) - 2k + 2]$ for any graph G.

Even if every graph can be realized as an induced subgraph of some Kneser graph, we note that there exist graphs for which $\mathcal{H}_k(G; K_{2k})$ is not an induced subgraph of $K(n;k)$ for any integer n.

Almost all other computations regarding to Kneser (hyper)graphs can be carried out to hypercomplete (hyper)graphs (see Sections 4 and 4.1 for more details). For instance, the clique numbers or the fractional chromatic numbers can be derived similarly. Such results also enable us to provide examples of graphs such that there is a large gap between their chromatic and fractional chromatic numbers that may be of particular interest.

Shortly after Lovász's proof of the Kneser conjecture, Schrijver found a vertex-critical induced subgraph of $K(n;k)$, whose chromatic number is the same as that of $K(n;k)$ [15]. We recall that a set $S \subseteq [n]$ is called *stable* if $i \in S$ implies $i+1 \notin S$, and if $n \in S$ implies $1 \notin S$. The stable Kneser graph (or the Schrijver graph) $SK(n;k)$ is defined to be the graph whose vertices are all stable k-subsets of $[n]$, 2 being adjacent if and only if they are disjoint. We generalize Schrijver's construction to hypercomplete graphs by fixing a coloring of the source graph beforehand. Suppose that κ is a proper n-coloring of G, where $\chi(G) = n$. When $2k \leq \omega(G)$, we call a k-complete S of G κ-stable if $\kappa(S) := \{\kappa(s)\colon s \in S\}$ is a stable set in $[n]$. We define the *stable k-hypercomplete graph* $\mathcal{SH}_k^{\kappa}(G;K_{2k})$ of G with respect to κ to be the graph with vertex set consisting of all κ-stable k-completes of G, where 2 such sets form an edge if and only if their set union induces a $2k$-complete in G. The following is not a surprise:

Corollary 4.12 *Let κ be a proper n-coloring of a graph G with $n = \chi(G) = \omega(G)$. Then, $\chi(\mathcal{SH}_k^{\kappa}(G;K_{2k})) = \chi(G) - 2k + 2$ for any $k \leq \lfloor \frac{n}{2} \rfloor$.*

It should be noted that the subgraphs $\mathcal{SH}_k^{\kappa}(G;K_{2k})$ of $\mathcal{H}_k(G;K_{2k})$ need not be vertex-critical as opposed to the case of complete graphs. Furthermore, for any 2 distinct n-colorings κ_1 and κ_2 of G, the graphs $\mathcal{SH}_k^{\kappa_1}(G;K_{2k})$ and $\mathcal{SH}_k^{\kappa_2}(G;K_{2k})$ are not isomorphic in general.

Our final move is to study the chromatic number of hyperpath (hyper)graphs $\mathcal{HP}_k(G;P_m)$. As opposed to hypercomplete graphs, there seems to be no general relation between $\chi(G)$ and $\chi(\mathcal{HP}_k(G;P_m))$. However, there is a simple upper bound involving the maximum degree of the source graph $\chi(\mathcal{HP}_k(G;P_{2k})) \leq 2\Delta(G) - 1$, which is indeed the best possible. On the other hand, by the use of the hypercompletion operator, we also provide upper bounds on $\chi(\mathcal{HP}_k(G;P_m))$ when $m = k+1$ and $m = 2k$ in terms of the k-distance chromatic numbers of the source graph (see Section 5).

Theorem 5.5 *For any graph G and any integer $k \geq 2$, we have $\chi(\mathcal{HP}_k(G;P_{k+1})) \leq \chi_k(G)$ and $\chi(\mathcal{HP}_k(G;P_{2k})) \leq \chi_{2k-1}(G) - 2k + 2$.*

2. Preliminaries

We first recall some general notions and notations needed throughout the paper, and repeat some of the definitions mentioned in the introduction more formally.

2.1. Graphs

By a simple graph G, we will mean an undirected graph without loops or multiple edges. If G is a graph, $V(G)$ and $E(G)$ (or simply V and E) denote its vertex and edge sets. An edge between u and v is denoted by $e = uv$ or $e = \{u,v\}$ interchangeably. A graph $G = (V,E)$ is called the *null-graph* on V whenever $E = \emptyset$. If $U \subset V$, the graph induced on U is written as $G[U]$. We denote by $|G|$ and $\|G\|$ the order and size of G, while $d(v)$ denotes the degree of a given vertex $v \in V$. A graph $G = (V,E)$ is said to be r-regular if $d(v) = r$ for any $v \in V$.

The *complement* $\overline{G} := (V, \overline{E})$ of a graph $G = (V,E)$ is the graph on V with $uv \in \overline{E}$ if and only if $uv \notin E$. Throughout the paper, K_n, C_n, and P_n will denote the complete, cycle, and path graphs on n vertices, respectively. The distance $d(x,y)$ between 2 vertices x and y is defined to be the length of a shortest path in G between them.

A subset $S \subseteq V$ is called a *complete* of G if $G[S]$ is isomorphic to a complete graph, and a *k-complete* of G is a complete of cardinality k. The set of all k-completes of G will be denoted by $\mathcal{C}_k(G)$. In particular, a complete that is maximal with respect to inclusion is called a *clique* of G, and the largest k such that G has a k-clique is called the *clique number* of G and denoted by $\omega(G)$. An *independent set* of G is a complete of $\overline{G}$, and the *independence number* is defined by $\alpha(G) := \omega(\overline{G})$.

We recall that a mapping $\kappa\colon V \to [m]$ is a (proper vertex) *coloring* of a graph G if $\kappa(u) \neq \kappa(v)$ whenever $uv \in E$, where $[m] := \{1, 2, \ldots, m\}$. The least integer m for which G admits a coloring with m colors is called the *chromatic number* of G and is denoted by $\chi(G)$.

The *line graph* $L(G)$ of a graph G is the graph on the edges, 2 of which form an edge if and only if they share a common vertex.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be 2 nonempty graphs on (disjoint) sets of r and s vertices. The *Cartesian product* $G_1 \square G_2$ is the graph with vertex set $V_1 \times V_2$ and $(x_1, x_2)(y_1, y_2)$ is an edge if and only if either $x_1y_1 \in E_1$ and $x_2 = y_2$ or $x_2y_2 \in E_2$ and $x_1 = y_1$, whereas the *tensor product* (also known as the categorical or weak product) $G_1 \times G_2$ is the graph with vertex set $V_1 \times V_2$, where 2 vertices $(x_1, x_2)(y_1, y_2)$ are adjacent if and only if both $x_1y_1 \in E_1$ and $x_2y_2 \in E_2$. The *strong product* $G_1 \boxtimes G_2$ is the graph with vertex set $V_1 \times V_2$ and edge set $(E(G_1 \square G_2)) \cup (E(G_1 \times G_2))$. Furthermore, their *disjoint union* $G_1 \cup G_2$ is the graph $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$, and their *join* $G_1 \vee G_2$ is the graph on $n = r + s$ vertices obtained from $G_1 \cup G_2$ by inserting new edges from each vertex of G_1 to each vertex of G_2. We write nG to denote the disjoint union of n copies of G. A (graph) homomorphism $\phi\colon G_1 \to G_2$ is a mapping from V_1 to V_2 such that $\phi(x)\phi(y) \in E_2$ whenever $xy \in E_1$.

For a given positive integer n, we denote by $\binom{[n]}{k}$ the set of k-subsets of $[n]$. The *Kneser graph* $K(n; k)$ for $2k - 1 \leq n$ is defined to be the simple graph on $\binom{[n]}{k}$, where 2 vertices form an edge if and only if they are disjoint. Note that $K(n; 1)$ is the n-vertex complete graph, and $K(2k - 1, k)$ is the null-graph.

The *fractional chromatic number* $\chi_f(G)$ of a graph G is defined to be the infimum of the fractions $\frac{n}{k}$ such that there exists a graph homomorphism from G to $K(n; k)$ (see [2, 16]).

2.2. Hypergraphs

A *hypergraph* is a set system $\mathcal{H} = (V, \mathcal{E})$, where V is a set, the set of vertices of $\mathcal{H}$, and $\mathcal{E} \subseteq 2^V$. The hypergraphs that we consider may have loops; that is, $\mathcal{E}$ may contain singletons as edges. However, we do not allow repeated edges in general. A hypergraph $\mathcal{H} = (V, \mathcal{E})$ is said to be *simple* if $A \nsubseteq B$ and $B \nsubseteq A$ for any 2 edges $A, B \in \mathcal{E}$. We call a hypergraph r-*uniform* if all of its edges have the same cardinality r.

When $S \subseteq V$, the *induced subhypergraph* of $\mathcal{H}$ by the set S is defined to be the hypergraph $\mathcal{H}[S] := \{A \in \mathcal{E}\colon A \subseteq S\}$.

A mapping $\kappa\colon V \to [m]$ is a (proper) *coloring* of a hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ if none of the edges of $\mathcal{H}$ are monochromatic under κ. The least integer m for which $\mathcal{H}$ admits a coloring with m colors is called the *chromatic number* of $\mathcal{H}$ and denoted by $\chi(\mathcal{H})$. The r-*colorability defect* $\mathrm{cd}_r(\mathcal{H})$ of a hypergraph $\mathcal{H}$ is defined by

$$\mathrm{cd}_r(\mathcal{H}) := \min\{|S|\colon S \subseteq V \text{ and } \chi(\mathcal{H}[V \backslash S]) \leq r\}.$$

Given a hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$, the *Kneser* r-*hypergraph* of $\mathcal{H}$, denoted by $\mathcal{K}_r(\mathcal{H})$, has vertex set $\mathcal{E}$, and an r-tuple $(A_1, \ldots A_r)$ of edges of $\mathcal{H}$ forms an edge iff $A_i \cap A_j = \emptyset$ for all $i \neq j$. We will simply write $\mathcal{K}_r(n; k)$ when $\mathcal{H} = ([n], \binom{[n]}{k})$.

3. Hyper-H (hyper)graphs

In this section, we introduce our main objects of study, hyper-H (hyper)graphs; provide various examples; and construct some known (hyper)graphs as hyper-H (hyper)graphs. We choose to state the definition as broadly as possible.

Definition 3.1 *Let G be a graph and let H be an induced subgraph of G containing at least one edge. Then G is said to be H-*decomposable *if there exists a family $\{H_1,\dots,H_s\}$ consisting of induced subgraphs of G, each being isomorphic to H such that $G[\cup_{i=1}^{t}H_i]\cong G$, and the integer t is called the* size *of this decomposition. When G is H-decomposable, the least integer d for which G admits an H-decomposition of size d is called the H-*width *of G and denoted by $w_H(G)$. In case G is not H-decomposable, we set $w_H(G):=\infty$.*

In particular, when H is a k-complete for some $k\geq 2$, we call $\mathrm{w}_{K_k}(G)$ the *k-clique-width* of G and denote it by $\mathrm{cw}_k(G)$. Note that $\mathrm{w}_H(G)<\infty$ if and only if any vertex of G is contained by at least one induced copy of H in G. Similarly, when H is a k-path for some $k\geq 2$, we call $\mathrm{w}_{P_k}(G)$ the *k-path-width* of G and denote it by $\mathrm{pw}_k(G)$.

Definition 3.2 *Let G be a graph and let H be an induced subgraph of G containing at least one edge. For a given family $\mathcal{F}$ of graphs, we define the* hyper-H hypergraph *$\mathcal{H}_H(G;\mathcal{F})=(\mathcal{V}_H(G),\mathcal{E}_H(G))$ of G with respect to $\mathcal{F}$ by taking $\mathcal{V}_H(G)$ to be the set of induced copies of H in G, and $\{A_1,\dots,A_t\}\in\mathcal{E}_H(G)$ if and only if there exists $F\in\mathcal{F}$ satisfying $w_H(F)=t$ such that $G[\cup_{i=1}^{t}A_i]\cong F$.*

When $\mathcal{H}_H(G;\mathcal{F})$ is just an ordinary graph, we call it as the hyper-H graph *of G with respect to $\mathcal{F}$, which is only possible when $w_H(F)=2$ for all $F\in\mathcal{F}$. We also abbreviate $\mathcal{H}_H(G;\{F\})$ to $\mathcal{H}_H(G;F)$.*

We generally refer to G as the source (or ground) graph of $\mathcal{H}_H(G;\mathcal{F})$, and to $\mathcal{F}$ as the reference family of our operator.

Throughout the paper we only deal with 2 particular cases: $H=K_k$ for some $2\leq k\leq\omega(G)$ and $H=P_k$ for some $k\geq 2$, where we call the resulting hypergraphs the *k-hypercomplete hypergraph* and *k-hyperpath hypergraph* of G and denote them by $\mathcal{H}_k(G;\mathcal{F})$ and $\mathcal{HP}_k(G;\mathcal{F})$, respectively.

Example 3.3 *If we consider the graph G illustrated in Figure 1, then $\mathcal{H}_2(G;K_{1,3})$ is a 3-uniform hypergraph on the set $\{1,2,\dots,8\}$ with edges $\{1,2,4\},\{2,3,5\},\{4,6,7\}$, and $\{5,7,8\}$. We also note that $\mathcal{HP}_3(G;K_{1,3})\cong 4K_3$.*

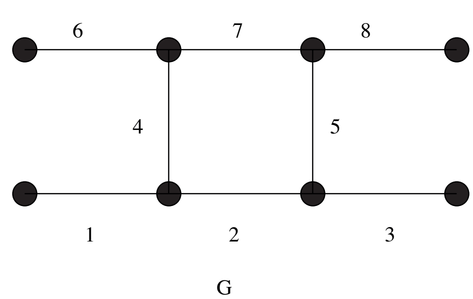

Figure 1. A graph G with the corresponding edge labeling.

We remark that the hypergraph $\mathcal{H}_k(G;\mathcal{F})$ does not have to be a simple hypergraph in general. For instance, $\mathcal{H}_2(G;\{P_3,K_{1,3}\})$ contains both $\{1,2\}$ and $\{1,2,4\}$ as edges for the graph G of Figure 1. This may be overcome by assuming that if $F\in\mathcal{F}$ and F' is an induced subgraph of F with $\mathrm{cw}_k(F')<\infty$, then $F'\notin\mathcal{F}$.

Example 3.4 *For a given graph G, its 2-hypercomplete graph with respect to the family $\{P_3,K_3\}$ is isomorphic to the line graph of G; that is, $\mathcal{H}_2(G;\{P_3,K_3\})\cong L(G)$. This easily follows from the fact that for any 2 intersecting edges in G, the set of vertices incident to these edges induces either a P_3 or a K_3.*

In fact, our construction generalizes many of the known operators on graphs when $k=2$. For instance, $\mathcal{H}_2(G;P_3)$ is the Gallai's graph of G, $\mathcal{H}_2(G;P_4)$ is the wing-graph of G, and $\mathcal{H}_2(G;\{P_4,C_4\})$ is the Chvatal's graph of G (see [3] for details).

It may be useful to note that $\mathcal{H}_2(G;\mathcal{F})$ is a graph if and only if $\mathcal{F}$ is a subset of the set of graphs depicted in Figure 3.

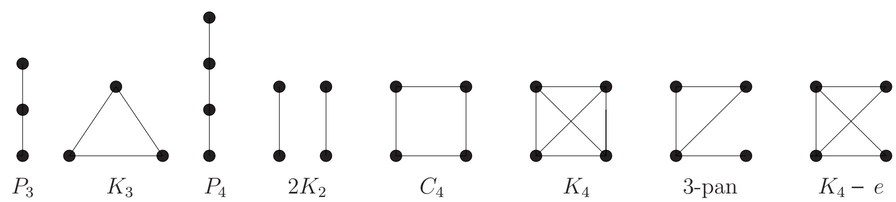

Figure 2. Graphs with $\mathrm{cw}_2(G)=2$.

The power of the hypercompletion operator comes from the fact that many classical results related to Kneser graphs have generalizations to hypercomplete graphs along almost the same lines. We therefore have the chance to expand known results to a wider class of graphs. This section offers such a treatment.

It is easy to verify that if H is an induced subgraph of G, then $\mathcal{H}_k(H;\mathcal{F})$ is an induced sub(hyper)graph of $\mathcal{H}_k(G;\mathcal{F})$ for any family $\mathcal{F}$ and any integer $k\le\omega(H)$. Similarly, if $\mathcal{F}'$ is a subfamily of $\mathcal{F}$, then $\mathcal{H}_k(G;\mathcal{F}')$ is a (not necessarily induced) sub(hyper)graph of $\mathcal{H}_k(G;\mathcal{F})$ for any graph G.

As we mentioned earlier, our starting point to introduce hypercomplete graphs begins with the following observation.

Proposition 3.5 *If $2\le k\le\lfloor\frac{n}{2}\rfloor$, then $\mathcal{H}_k(K_n;K_{2k})\cong K(n;k)$.*

Proof Once we assume that $V(K_n)=[n]$, the claim follows, since 2 k-completes A,B of K_n are disjoint if and only if $K_n[A\cup B]\cong K_{2k}$. □

Remark 3.6 *It is well known that any graph can be obtained as an induced subgraph of some Kneser graph [5]; however, for an arbitrary graph G, it is not true that the graph $\mathcal{H}_k(G;K_{2k})$ is an induced subgraph of $K(n;k)$ for some $n\ge 2k$. For instance, if we consider the graph G illustrated in Figure 3.6, there exists no integer n such that $\mathcal{H}_2(G;K_4)$ is an induced subgraph of $K(n;2)$. This is clearly due to the fact that being disjoint is not enough to form an edge of the hypercomplete graph for any 2 completes.*

We may readily extend the construction of Kneser graphs as hypercomplete graphs to a general graph family obtained from the complete graphs by the same manner.

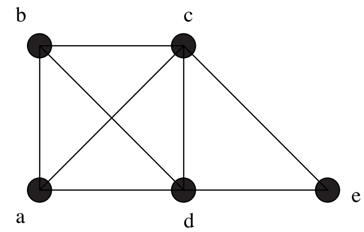

Figure 3. A graph G for which $\mathcal{H}_2(G;K_4)$ is not an induced subgraph of $K(n;2)$ for any n.

Suppose that $n,r,s \in \mathbb{N}$ are given with $1 \leq s < r \leq \lfloor \frac{n}{2} \rfloor$. We then define $IK(n,r;s)$ to be the graph whose vertices are the r-element subsets of $[n]$, 2 of which form an edge if and only if their intersection is a set with s-element. We omit the proof of the following obvious fact.

Proposition 3.7 *Let n,r,s be integers with $1 \leq s < r \leq \lfloor \frac{n}{2} \rfloor$. Then $\mathcal{H}_r(K_n;K_{2r-s}) \cong IK(n,r;s)$.*

We particularly note that the graph $\mathcal{H}_k(K_n;K_{k+1})$ is the well-known Johnson graph $J(n;k)$, whose vertices are k-element subsets of $[n]$, and 2 such sets form an edge if and only if their intersection has exactly $k-1$ elements (see the discussion after Theorem 4.13).

It may be expected that our construction will always produce simple graphs if we choose the reference family to be the one consisting of a single complete graph. However, this is not the case. As an example, we consider $\mathcal{H}_k(K_n;K_{rk})$ for integers satisfying $2k+1 < n$ and $2 < r \leq \lfloor \frac{n}{k} \rfloor$. It is easily seen that $\mathcal{H}_k(K_n;K_{rk}) \cong \mathcal{K}_r(n;k)$. On the other hand, it should be noted that the hypergraph $\mathcal{H}_k(G;K_{rk})$ is not in general isomorphic to the Kneser r-hypergraph of $(V,\mathcal{C}_k(G))$ for an arbitrary graph $G=(V,E)$ and integers with $2k+1 < \omega(G)$ and $2 < r \leq \lfloor \frac{\omega(G)}{k} \rfloor$. Moreover, $\mathcal{H}_k(G;K_{rk})$ does not have to be an induced subhypergraph of $\mathcal{K}_r(n;k)$ for some n.

One other family of graphs that we will construct as hypercomplete graphs are stable Kneser graphs, also known as Schrijver graphs [15].

In order to construct $SK(n;k)$ as a hypercomplete graph, we first point out that any stable subset S of $[n]$ corresponds to an independent set in the cycle C_n. When $n \geq 4$, we call $SK_n := \overline{C_n}$ the *stable complete graph* of order n.

Proposition 3.8 *For all integers n,k with $1 < k \leq \lfloor \frac{n}{2} \rfloor$, we have*

$$SK(n;k) \cong \begin{cases} \mathcal{H}_k(SK_{2k};SK_{2k}), & \textit{if } n = 2k, \\ \mathcal{H}_k(SK_n;\{\overline{P_{r_1} \cup \ldots \cup P_{r_s}} : r_1 + \ldots + r_s = 2k, r_i \geq 1\}), & \textit{otherwise.} \end{cases}$$

Proof It is enough to show that if A and B are two stable k-subset of $[n] = V(C_n)$, then $A \cap B = \emptyset$ if and only if $C_n[A \cup B] \cong P_{r_1} \cup \ldots \cup P_{r_s}$ for some partition of $2k$, which trivially holds. □

4. Generalized Lovász–Kneser theorem

The generalization of the Lovász–Kneser theorem to Kneser hypergraphs was the main subject of many recent papers [1, 9, 10, 13, 17]. There are 2 main approaches that differ on whether to allow edges consisting of multisubsets of the ground set together with intersection multiplicities. We here only consider Kneser hypergraphs of hypergraphs whose edges consist of ordinary subsets of the vertex set without multiplicities.

In this restricted context, there is a well-known lower bound due to Křiž [9, 12].

Theorem 4.1 ([9]) *For any (finite) hypergraph* $\mathcal{H} = (V, \mathcal{E})$ *and any* $r \geq 2$*, we have* $\chi(\mathcal{K}_r(\mathcal{H})) \geq \frac{1}{r-1} \cdot cd_r(\mathcal{H}))$*.*

We should also note that the upper bound in the specific case of $\chi(\mathcal{H}_k(K_n; K_{rk}))$ was first constructed by Erdös [4] (see also Ziegler [17]), which is just a modification of Kneser's upper bound for $K(n;k)$, and the lower bound is the simple extension of Dol'nikov's 2-colorability defect [12].

Theorem 4.2 *Let* G *be a graph with* $\omega(G) = \chi(G) = n$*. Then for any integers* k *and* r *satisfying* $2k+1 \leq n$ *and* $2 \leq r \leq \lfloor \frac{n}{k} \rfloor$*, we have*

$$\chi(\mathcal{H}_k(G; K_{rk})) = \lceil \frac{n - r(k-1)}{r-1} \rceil.$$

Proof The lower bound follows from Theorem 4.1, since $\omega(G) = n$, implying that $\mathcal{H}_k(K_n; K_{rk})$ is an induced subhypergraph of $\mathcal{H}_k(G; K_{rk})$. For the upper bound, we simply follow Erdös. We let $\lambda\colon V(G) \to [n]$ be a proper coloring of G and define $\Lambda\colon V(\mathcal{H}_k(G; K_{rk})) \to [m]$ by

$$\Lambda(A) := \min\{\lceil \frac{1}{r-1} \min\{\lambda(A)\}\rceil, m\},$$

where $\lambda(A) := \{\lambda(a)\colon a \in A\}$ and $m := \lceil \frac{n-r(k-1)}{r-1} \rceil$. Suppose that $\{A_1, \ldots, A_r\}$ is a monochromatic edge of $\mathcal{H}_k(G; K_{rk})$ under Λ. If $\Lambda(A_j) = i < m$ for all $1 \leq j \leq r$, then there must exist a vertex $a_j \in A_j$ with $\lambda(a_j) = i$ for each $j \in [r]$. However, this is impossible since $G[\cup_{j=1}^{r} A_j] \cong K_{rk}$. On the other hand, if $\Lambda(A_j) = m$ for each $j \in [r]$, it follows that $\cup_{j=1}^{r} \lambda(A_j) \subseteq \{(r-1)m, (r-1)m+1, \ldots, n\}$. This is also impossible, since the set on the left-hand side contains exactly rk elements by $G[\cup_{j=1}^{r} A_j] \cong K_{rk}$, while on the right-hand side there are at most $rk - r + 1$ elements. □

Corollary 4.3 *If* G *is a graph with* $\chi(G) = \omega(G)$*, then* $\chi(\mathcal{H}_k(G; K_{2k})) = \chi(G) - 2k + 2$ *for any* $k \leq \lfloor \frac{\omega(G)}{2} \rfloor$*.*

In general, the condition $\chi(G) = \omega(G)$ in the statement of Theorem 4.2 cannot be dropped. For this, we consider the graph G depicted in Figure 4. Note that $\chi(G) = 5$ and $\omega(G) = 4$, while $\chi(\mathcal{H}_2(G; K_4)) = 2$.

We may, however, obtain a lower and an upper bound in the general case in the view of Theorem 4.2.

Corollary 4.4 *For any graph* G *and for any integers* k *and* r *satisfying* $2k+1 \leq n$ *and* $2 \leq r \leq \lfloor \frac{n}{k} \rfloor$*, we have*

$$\lceil \frac{\omega(G) - r(k-1)}{r-1} \rceil \leq \chi(\mathcal{H}_k(G; K_{rk})) \leq \lceil \frac{\chi(G) - r(k-1)}{r-1} \rceil.$$

Proof Assume that G is a graph with $\chi(G) = m > n = \omega(G)$. We then have $\chi(\mathcal{H}_k(G; K_{rk})) \geq \chi(\mathcal{H}_k(K_n; K_{rk}))$, since $\omega(G) = n$. On the other hand, let $H := G \cup K_m$; that is, H is the disjoint union of G and K_m. We note that $\chi(H) = \omega(H) = m$; hence, $\chi(\mathcal{H}_k(H; K_{rk})) \geq \chi(\mathcal{H}_k(G; K_{rk}))$, since G is an induced subgraph of H. □

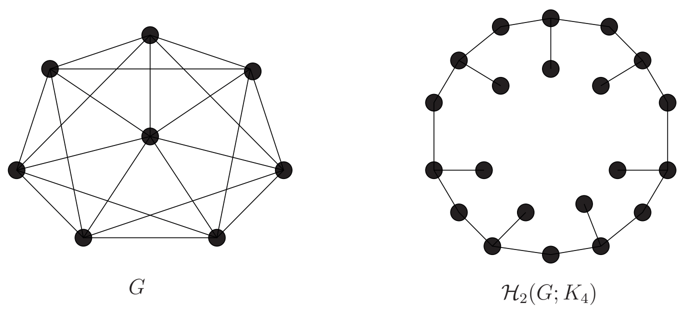

Figure 4. A graph G with $\chi(G) \neq \omega(G)$ and its 2-hypercomplete graph $\mathcal{H}_2(G;K_4)$.

When $r=2$, the Corollary 4.3 simply translates that $\chi(\mathcal{H}_k(G;K_{2k})) \in [w(G)-2k+2, \chi(G)-2k+2]$ for any $k \leq \lfloor \frac{\omega(G)}{2} \rfloor$.

Example 4.5 *We define H_m^s to be the join of s-copies of the $2m$-cycle. If we assume that $m, s \geq 3$, then we have $\chi(\mathcal{H}_2(H_m^s;K_{2r})) = \lceil \frac{2s-r}{r-1} \rceil$ for any $s \geq r > 2$ by Theorem 4.2.*

Example 4.6 *Assume that $G = C_5 \vee C_5$, the join of the 5-cycle with itself, so $\omega(G) = 4$ and $\chi(G) = 6$; hence, $2 \leq \chi(\mathcal{H}_2(G;K_4)) \leq 4$ by Corollary 4.3. However, the chromatic number of $\mathcal{H}_2(G;K_4)$ is indeed equal to 3. To ensure that, we let $\{a,b,c,d,e\}$ and $\{A,B,C,D,E\}$ denote the set of vertices of two 5-cycles respectively with edges in cyclic fashion. We then note that the graph $\mathcal{H}_2(G;K_4)$ is disconnected with 2 connected components; one is isomorphic to the complete bipartite graph $K_{5,5}$, induced by the set of vertices $U := \{ab, bc, cd, de, ae, AB, BC, CD, DE, AE\}$, and the other component is isomorphic to $C_5 \times C_5$, the tensor product of C_5 with itself, induced by the rest of the vertices of $\mathcal{H}_2(G;K_4)$.*

The above analysis can naturally be generalized to higher order odd cycles. In other words, if we set $G_r := C_{2r+1} \vee C_{2r+1}$ for $r \geq 2$, then the vertices of $\mathcal{H}_2(G_r;K_4)$ corresponding to the initial edges of the copies of C_{2r+1} in G_r form a complete bipartite graph $K_{2r+1,2r+1}$, and the rest of the vertices induce the component being isomorphic to the tensor product $C_{2r+1} \times C_{2r+1}$. Therefore, $\chi(\mathcal{H}_2(G_r;K_4)) = 3$.

Proposition 4.7 *If G is a graph and $2k \leq \omega(G)$, then $\omega(\mathcal{H}_k(G;K_{2k})) = \lfloor \frac{\omega(G)}{k} \rfloor$.*

Proof Let $\mathcal{W} = \{A_1, \ldots, A_s\}$ be a complete of $\mathcal{H}_k(G;K_{2k})$. Then $G[A_i \cup A_j] \cong K_{2k}$ for all $i,j \in [s]$ with $i \neq j$. Therefore, we must have $G[\cup_{i=1}^s A_i] \cong K_{sk}$, which in turn implies that $\omega(\mathcal{H}_k(G;K_{2k})) = \lfloor \frac{\omega(G)}{k} \rfloor$ as claimed. □

Our next computation concerns the independence numbers of hypercomplete graphs $\mathcal{H}_k(G;K_{2k})$. We first point out that the classical result of Erdös–Ko–Rado [7] takes the following simple form in the language of hypercomplete graphs.

Theorem 4.8 (Erdös–Ko–Rado Theorem) *If $n \geq 2k$, then*

$$\omega(\mathcal{H}_k(K_n;\{K_{k+1}, K_{k+2}, \ldots, K_{2k-1}\})) = \binom{n-1}{k-1}.$$

Therefore, we have $\alpha(\mathcal{H}_k(K_n;K_{2k})) = \binom{n-1}{k-1}$ whenever $2k \leq n$. It seems that we are not that lucky for the general case. We are only able to propose the computation with respect to particularly chosen graphs. We remark that what really differs from the specific case of complete graphs is that the consideration of the maximal intersecting families of k-completes is not enough to deduce the result for an arbitrary graph. Furthermore, the maximal independent sets do not have to be uniquely determined, as opposed to the cases of many generalizations of the Erdös–Ko–Rado theorem [6].

Let $G = (V,E)$ be a graph. We denote the strong product graph $G \boxtimes K_m$ by $G^{[m]}$ for any $m \geq 2$. As an example, the graph $P_3^{[2]}$ is depicted in Figure 5.

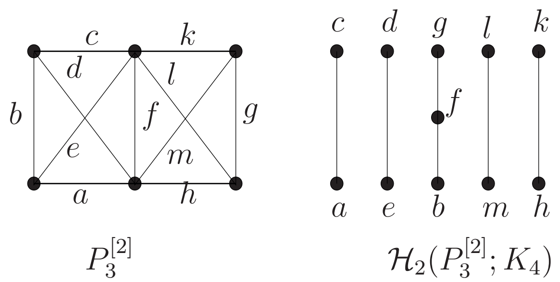

Figure 5. The graph $P_3^{[2]}$ and its 2-hypercompletion.

The reason why we consider the strong product of a graph with complete graphs is that $\omega(G^{[m]}) = m \cdot \omega(G)$, while $\alpha(G^{[m]}) = \alpha(G)$ for any graph G and $m \geq 2$.

Proposition 4.9 *For any $2 \leq k \leq m$ and $n \geq 2$, we have*

$$\alpha(\mathcal{H}_k(P_n^{[m]};K_{2k})) = \begin{cases} (n-1)\cdot\binom{2m-1}{k-1} - \lfloor\frac{n-1}{2}\rfloor \cdot \binom{m-1}{k-1}, & \text{if } n \text{ is even,} \\ (n-1)\cdot\binom{2m-1}{k-1} - \lfloor\frac{n-2}{2}\rfloor \cdot \binom{m-1}{k-1}, & \text{if } n \text{ is odd.} \end{cases} \tag{4.10}$$

Proof We verify the claim by the help of Theorem 4.8 together with an analysis of $\mathcal{H}_k(P_n^{[m]};K_{2k})$. We write $V(P_n^{[m]}) = \{(i,j) \colon i \in [n] \text{ and } j \in [m]\}$, and we define $V_i := \{(i,r),(i+1,r) \colon r \in [m]\}$ and $R_i := P_n^{[m]}[V_i]$ for $i \in [n-1]$. Similarly, we let $U_j := \{(j,r) \colon r \in [m]\}$ and $T_j := P_n^{[m]}[U_j]$ for $2 \leq j \leq n-1$. We note that $R_i \cong K_{2m}$ and $T_j \cong K_m$. We then consider the families of k-completes of $P_n^{[m]}$ defined by

$$\mathcal{A}_{2r-1} := \{A \in \mathcal{C}_k(R_{2r-1}) | (2r-1,1) \in A\} \text{ and } \mathcal{A}_{2r} := \{A \in \mathcal{C}_k(R_{2r}) \backslash \mathcal{C}_k(T_{2r}) | (2r,1) \in A\}$$

whenever $2r-1, 2r \in [n-1]$. When n is even, it is straightforward to show that $\cup_{i=1}^{n-1}\mathcal{A}_i$ is an independent set in $\mathcal{H}_k(P_n^{[m]};K_{2k})$ with the requested cardinality. When n is odd, we consider the independent set $\cup_{i=1}^{n-2}\mathcal{A}_i \cup \mathcal{A}_n$, where $W_n := \{(n-1,r),(n,r) \colon r \in [m]\}$, $S_n := P_n^{[m]}[W_n]$ and $\mathcal{A}_n := \{A \in C_k(S_n) \colon (n,1) \in A\}$. This proves the lower bound.

For the upper bound, we simply note that the contribution of k-completes contained by any 2 consecutive $2m$-cliques R_i and R_{i+1} to an independent set of $\mathcal{H}_k(P_n^{[m]};K_{2k})$ can not be greater than $[2\binom{2m-1}{k-1} - \binom{m-1}{k-1}]$. □

Our next aim is to prove an analogue of Schrijver's result [15] for hypercomplete graphs.

Definition 4.11 *Suppose that κ is a proper n-coloring of G, where $\chi(G) = n$. When $2k \leq \omega(G)$, we call a k-complete S of G κ-*stable *if $\kappa(S) := \{\kappa(s) \colon s \in S\}$ is a stable set in $[n]$. We define the* stable k-hypercomplete graph $\mathcal{SH}_k^{\kappa}(G; K_{2k})$ *of* G *with respect to* κ *to be the graph with vertex set consisting of all* κ*-stable* k*-completes of* G*, where 2 such sets form an edge if and only if their set union induces a* $2k$*-complete in* G*.*

Corollary 4.12 *Let* κ *be a proper* n*-coloring of a graph* G *with* $n = \chi(G) = \omega(G)$*. Then* $\chi(\mathcal{SH}_k^{\kappa}(G; K_{2k})) = \chi(G) - 2k + 2$ *for any* $k \leq \lfloor \frac{n}{2} \rfloor$*.*

Proof The upper bound is provided by the fact that $\mathcal{SH}_k^{\kappa}(G; K_{2k})$ is an induced subgraph of $\mathcal{H}_k(G; K_{2k})$, and the lower bound is a consequence of Schrijver's theorem, since $\omega(G) = n$. □

As noted earlier, the graphs $\mathcal{SH}_k^{\kappa_1}(G; K_{2k})$ and $\mathcal{SH}_k^{\kappa_2}(G; K_{2k})$ are not in general isomorphic for any distinct 2 n-colorings κ_1 and κ_2 of G. Furthermore, these subgraphs need not be always vertex-critical. For instance, we consider two 4-colorings $\kappa_i \colon V(P_3^{[2]}) \to [4]$ of $P_3^{[2]}$ depicted in Figure 6. Note that neither $\mathcal{SH}_2^{\kappa_1}(P_3^{[2]}; K_4)$ nor $\mathcal{SH}_2^{\kappa_2}(P_3^{[2]}; K_4)$ is vertex-critical.

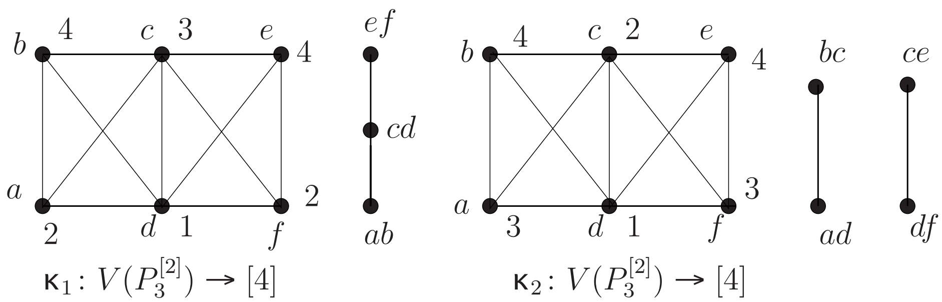

Figure 6. Two 4-colorings of $P_3^{[2]}$, and the corresponding stable 2-hypercompletions $\mathcal{SH}_2^{\kappa_1}(P_3^{[2]}; K_4)$ and $\mathcal{SH}_2^{\kappa_2}(P_3^{[2]}; K_4)$ of $P_3^{[2]}$, in which the isolated vertices bd and de are deleted.

As we have seen so far, our computations are only valid with respect to an even order complete graph taken as the reference family. There is one more exceptional case that we examine next.

Theorem 4.13 $\omega(G) - k + 1 \leq \chi(\mathcal{H}_k(G; K_{k+1})) \leq \chi(G)$ *for any graph* G *and any integer* $k < \omega(G)$*.*

Proof Suppose that G is a graph with $\omega(G) = n \leq m = \chi(G)$. Since G contains an induced K_n, the graph $\mathcal{H}_k(G; K_{k+1})$ must contain an induced K_{n-k+1}; thus, $\chi(\mathcal{H}_k(G; K_{k+1})) \geq n - k + 1$. Let $\lambda \colon V(G) \to \{0, 1, \ldots, m-1\}$ be a proper m-coloring of G. We define a coloring $\Lambda \colon V(\mathcal{H}_k(G; K_{k+1})) \to \{0, 1, \ldots, m-1\}$ of $\mathcal{H}_k(G; K_{k+1})$ by $\Lambda(\{x_1, \ldots, x_k\}) := \sum_{i=1}^{k} \lambda(x_i) \pmod m$ for any k-complete $\{x_1, \ldots, x_k\}$ of G. Suppose that $\Lambda(A) = \Lambda(B)$ for 2 k-completes of G. If $G[A \cup B] \cong K_{k+1}$, then we must have $|A \cap B| = k - 1$. We may therefore let $A = \{x_1, \ldots, x_{k-1}, y\}$ and $B = \{x_1, \ldots, x_{k-1}, z\}$. However, the equality $\Lambda(A) = \Lambda(B)$ implies that $\lambda(y) = \lambda(z)$; that is, $yz \notin E(G)$. Therefore, $G[A \cup B] \ncong K_{k+1}$, a contradiction. So, Λ is a proper coloring of $\mathcal{H}_k(G; K_{k+1})$ as claimed. □

In the particular case where $k = 2$, Theorem 4.13 forces $n - 1 \leq \chi(\mathcal{H}_2(G; K_3)) \leq n$ whenever $\chi(G) = \omega(G) = n \geq 2$ for a given graph G. We also remark that both bounds are attainable in this special case. For example, if $G = K_{2r}$, then $\chi(\mathcal{H}_2(K_{2r}; K_3)) = 2r - 1$, and similarly, $\chi(\mathcal{H}_2(K_{2r-1}; K_3)) = 2r - 1$ for $r \geq 2$, as the resulting hypercomplete graphs simply correspond to the line graphs of complete graphs. Furthermore, the graphs $\mathcal{H}_k(G; K_{k+1})$ can be considered as the *generalized Johnson graphs*, since $\mathcal{H}_k(G; K_{k+1}) \cong J(n; k)$, when $G = K_n$ as noted earlier.

Example 4.14 *We let* $H_r := C_{2r} \vee C_{2r}$ *for* $r \geq 2$*, and we consider the graph* $\mathcal{H}_2(H_r; K_3)$*. By Theorem 4.13, we must have* $3 \leq \chi(\mathcal{H}_2(H_r; K_3)) \leq 4$*. In fact, we claim that* $\chi(\mathcal{H}_2(H_r; K_3)) = 3$*. To verify that, we first note that the graph* $\mathcal{H}_2(H_r; K_3)$ *is a connected graph with* $4r(r+1)$ *vertices. If* U *is the set of vertices of* $\mathcal{H}_2(H_r; K_3)$ *corresponding to the initial edges of copies of* C_{2r}*, then* U *is an independent set. On the other hand, the subgraph of* $\mathcal{H}_2(H_r; K_3)$ *induced by the set* $V(\mathcal{H}_2(H_r; K_3))\backslash U$ *is isomorphic to the Cartesian product* $C_{2r} \square C_{2r}$ *from which the claim follows.*

4.1. Fractional chromatic number and examples

In this subsection, we prove the fractional analogue of the generalized Lovász–Kneser theorem that turns out to have no difference from the specific case of complete graphs.

Theorem 4.15 *If* G *is a graph with* $\chi(G) = \omega(G) = n$*, then* $\chi_f(\mathcal{H}_k(G; K_{2k})) = \frac{n}{k}$ *for any* $k \leq \lfloor \frac{\omega(G)}{2} \rfloor$*.*

Proof For the upper bound, we construct an explicit graph homomorphism from $\mathcal{H}_k(G; K_{2k})$ to $\mathcal{H}_k(K_n; K_{2k})$. Since $\omega(G) = n$, a copy of K_n is contained in G as an induced subgraph. We let $\kappa\colon V(G) \to [n]$ be a proper n-coloring of G, and we define $\psi\colon V(G) \to V(K_n) \subseteq V(G)$ by $\psi(v) = x$ if and only if $\kappa(v) = \kappa(x)$.

If we define $\Psi\colon V(\mathcal{H}_k(G; K_{2k})) \to V(\mathcal{H}_k(K_n; K_{2k}))$ by $\Psi(A) := \{\psi(a)\colon a \in A\}$, we claim that Ψ is a graph homomorphism. We first note that if $a, b \in A$, then we necessarily have $\psi(a) \neq \psi(b)$ so that $\Psi(A)$ is a k-complete in K_n. On the other hand, if $G[A \cup B] \cong K_{2k}$ for some $A, B \in V(\mathcal{H}_k(G; K_{2k}))$, then $\Psi(A) \cap \Psi(B) = \emptyset$. We therefore conclude that $\chi_f(\mathcal{H}_k(G; K_{2k})) \leq \frac{n}{k}$.

For the other direction, we simply note that $\chi_f(\mathcal{H}_k(G; K_{2k})) \geq \chi_f(\mathcal{H}_k(K_n; K_{2k})) = \frac{n}{k}$, since G contains an induced K_n. □

Following the same steps in the proof of Corollary 4.3, we can easily obtain the following by Theorem 4.15.

Corollary 4.16 $\chi_f(\mathcal{H}_k(G; K_{2k})) \in [\frac{\omega(G)}{k}, \frac{\chi(G)}{k}]$ *for any graph* G *and any integer* $k \leq \lfloor \frac{\omega(G)}{2} \rfloor$*.*

Before providing an example, we recall that $\chi_f(G \cup H) = \max\{\chi_f(G), \chi_f(H)\}$ for any graphs G and H [14], and $\chi_f(G \times C_s) = \min\{\chi_f(G), \chi_f(C_s)\}$ for any graph G and any integer $s \geq 3$ by a recent result of Alon and Lubetzky [2].

Example 4.17 *Consider the graph* $G_r := C_{2r+1} \vee C_{2r+1}$ *for* $r \geq 2$*. We know from Example 4.6 that* $\mathcal{H}_2(G_r; K_4) \cong K_{2r+1,2r+1} \cup (C_{2r+1} \times C_{2r+1})$*. By Corollary 4.16, we have* $\chi_f(\mathcal{H}_2(G_r; K_4)) \in [2, 3]$*. On the other hand, we note that* $\chi_f(K_{2r+1,2r+1}) = 2$ *and* $\chi_f(C_{2r+1}) = 2 + \frac{1}{r}$*, since both* $K_{2r+1,2r+1}$ *and* C_{2r+1} *are vertex-transitive ([14]). By the same reason together with Alon and Lubetzky's result, we conclude that* $\chi_f(\mathcal{H}_2(G_r; K_4)) = 2 + \frac{1}{r}$ *for any* $r \geq 2$*.*

We next provide examples of graphs such that there is a large gap between their chromatic and fractional chromatic numbers, as we promised. Let us first mention an already existing example. The Kneser graph $\mathcal{H}_k(K_{3k-1}; K_{2k})$ contains no triangle while having chromatic number $k+1$ for $k \geq 2$. By Corollary 4.3 and Theorem 4.15, any graph (other than complete graphs) whose clique and chromatic numbers are equal to $3k-1$ will produce a new example of a graph with the desired property.

Example 4.18 *We may consider the complete* t*-partite graphs* $K_{n_1,\ldots,n_t}$ *with* $n_i \geq 2$*. By choosing* $t = 3k-1$*, we conclude that* $\chi(\mathcal{H}_k(K_{n_1,\ldots,n_{3k-1}}; K_{2k})) = k+1$*, while it contains no triangle and* $\chi_f(\mathcal{H}_k(K_{n_1,\ldots,n_{3k-1}}; K_{2k})) = 3 - \frac{1}{k}$*.*

Example 4.19 *Let* $m, s \geq 2$ *be given. Recall that* H_m^s *is the join of* s*-copies of the* $2m$*-cycle. If we choose* $k > 2$ *and* $s = 2k+1$*, then* $\chi(\mathcal{H}_k(H_m^{2k+1}; K_{2k})) = 2k+4$ *by Corollary 4.3,* $\omega(\mathcal{H}_k(H_m^{2k+1}; K_{2k})) = 4$ *by Proposition 4.7, and* $\chi_f(\mathcal{H}_k(H_m^{2k+1}; K_{2k})) = 4 + \frac{2}{k}$ *by Theorem 4.15.*

5. Hyperpath hypergraphs

In this section, we provide a study of the chromatic number of hyperpath (hyper)graphs $\mathcal{HP}_k(G; P_m)$, where we implicitly assume that the source graph G is always P_m*-dense* in the sense that any vertex of G is contained in an induced m-path of G, since otherwise we have $\mathcal{HP}_k(G; P_m) \cong \mathcal{HP}_k(G - x; P_m)$, if $x \in V(G)$ is not contained by any m-path of G. We note that $\mathcal{HP}_k(G; P_m)$ is a simple graph whenever $k < m \leq 2k$. For notational purposes, if $A = \{x_1, \ldots, x_k\}$ is a vertex of $\mathcal{HP}_k(G; P_m)$, we then assume that $x_i x_{i+1} \in E(G)$ for any $1 \leq i < k$, and we call x_1 and x_k the ends of A.

We first show that every graph can be obtained as an induced subgraph of a hyperpath graph, for which we first need a definition.

Definition 5.1 *Let* $G = (V, E)$ *be a graph and let* $S = \{x_1^1, x_2^1, \ldots, x_n^1\} \subseteq V$ *be given. For any given* $k \geq 2$*, we define the* whisker $W_k^S(G)$ *of* G *of length* k *with respect to* S *to be the graph whose vertex set is* $V_k^S(G) := V \cup \{x_i^j : 1 < j \leq k \text{ and } i \in [n]\}$ *and whose edge set is* $E_k^S(G) := E \cup \{x_i^j x_i^{j+1} : 1 \leq j < k \text{ and } i \in [n]\}$*. In particular, we write* $W_k(G)$ *when* $S = V$*.*

Lemma 5.2 *For any given graph* G*, the hyperpath graph* $\mathcal{HP}_k(W_k(G); P_{2k})$ *contains* G *as an induced subgraph for any* $k \geq 2$*.*

Proof Denote by A_i the vertex of $\mathcal{HP}_k(W_k(G); P_{2k})$ induced by the set of vertices $\{x_i^1, \ldots, x_i^k\}$; then $A_i A_j \in E(\mathcal{HP}_k(W_k(G); P_{2k}))$ if and only if $x_i^1 x_j^1 \in E(G)$, from which the claim follows. □

As opposed to the case of hypercompletion $\mathcal{H}_k(G; K_{2k})$, the chromatic number of $\mathcal{HP}_k(G; P_{2k})$ may well exceed that of G. This is easily verified when k is even, since, for example, the graph $\mathcal{HP}_2(C_6; P_4)$ has chromatic number 3. When k is odd, it is rather illusive to construct such examples. However, we provide one such example as follows.

Example 5.3 *Let* G *be the graph depicted in Figure 7a. It is straightforward to check that* $\chi(G) = 3$*, while the graph* $\mathcal{HP}_3(G; P_6)$ *contains a* 4*-complete illustrated in Figure 7b; hence, it has a chromatic number of at least* 4*.*

There is still a chance that $\chi(G)$ and $\chi(\mathcal{HP}_k(G; P_{2k}))$ may well be equal.

Proposition 5.4 *If* G *is bipartite and* $k \geq 3$ *is an odd integer, then* $\chi(\mathcal{HP}_k(G; P_m)) = 2$ *for any* $m \geq 2k$*.*

Proof Let $V(G) = U_1 \cup U_2$ be the bipartition. Since k is odd, any k-path in G has its ends in either U_1 or U_2. Therefore, if we color any such k-path by i if its ends are contained by U_i for $i = 1, 2$, this is definitely a proper coloring of $\mathcal{HP}_k(G; P_m)$. □

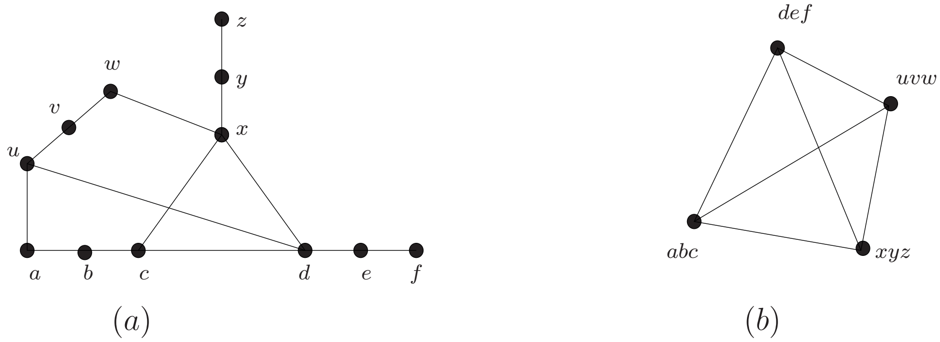

Figure 7. A graph G for which the chromatic number of $\mathcal{HP}_3(G;P_6)$ exceeds that of G.

Even if there seems to be no general relation between $\chi(G)$ and $\chi(\mathcal{HP}_k(G;P_m))$, we can bound the latter by the k-distance chromatic number of the source graph for a particularly chosen m. We recall that a *k-distance coloring* of a graph G is a vertex coloring of it, in which 2 vertices lying at a distance of less than or equal to k must be assigned different colors. The minimum number of colors needed in such a coloring is called the *k-distance chromatic number* of G and is denoted by $\chi_k(G)$. In terms of graph operations, $\chi_k(G)$ is equal to the chromatic number of G^k, the kth-power graph of G, which is the graph on the same set of vertices such that any 2 vertices whose distance is at most k are adjacent.

Theorem 5.5 *For any graph G and any integer $k \geq 2$, we have $\chi(\mathcal{HP}_k(G;P_{k+1})) \leq \chi_k(G)$ and $\chi(\mathcal{HP}_k(G;P_{2k})) \leq \chi_{2k-1}(G) - 2k + 2$.*

Proof If A is a k-path in G, then it is a k-complete of G^r when $k-1 \leq r \leq 2k$. Therefore, the inclusion maps $\mathcal{HP}_k(G;P_{k+1}) \hookrightarrow \mathcal{H}_k(G^k;K_{k+1})$ and $\mathcal{HP}_k(G;P_{2k}) \hookrightarrow \mathcal{H}_k(G^{2k-1};K_{2k})$ are graph homomorphisms; hence, claims follow from Theorem 4.13 and Corollary 4.3. □

Theorem 5.6 *For any graph G and any integer $k \geq 2$, we have $\chi(\mathcal{HP}_k(G;P_{2k})) \leq 2\Delta(G)-1$, and this bound is tight.*

Proof Suppose $\Delta = \Delta(G)$ and let $A = \{x_1,\ldots,x_k\}$ be a vertex of $\mathcal{HP}_k(G;P_{2k})$. If B is any other vertex of $\mathcal{HP}_k(G;P_{2k})$ satisfying $AB \in E(\mathcal{HP}_k(G;P_{2k}))$, then we must have either $N_G(x_1) \cap B \neq \emptyset$ or $N_G(x_k) \cap B \neq \emptyset$. When $N_G(x_1) \cap B \neq \emptyset$, we say that B is a neighbor of A at x_1, and similarly B is a neighbor of A at x_k if $N_G(x_k) \cap B \neq \emptyset$. Now, if $B_1,\ldots,B_r$ are neighbors of A at x_1, we must have $B_i \cap B_j = \emptyset$ for any distinct $i,j \in [r]$ in order to maximize the degree of A in $\mathcal{HP}_k(G;P_{2k})$; thus, $r \leq \Delta - 1$, which in turn implies the claim by Brooks' theorem.

For tightness, let $D(r,k)$ be the graph defined for $r > 1$ on the set $V(D(r,k)) = V_1 \cup V_2 \cup V_3$, where $V_1 = \{x_1,\ldots,x_k\}$, $V_2 = \{y_1^1,\ldots,y_k^1,\ldots,y_1^{(r-1)},\ldots,y_k^{(r-1)}\}$ and $V_3 = \{z_1^1,\ldots,z_k^1,\ldots,z_1^{(r-1)},\ldots,z_k^{(r-1)}\}$ with the set of edges:

$$\begin{aligned} E(D(r,k)) = &\{x_i x_{i+1}\colon 1 \leq i < k\} \cup \{y_i^s y_{i+1}^s\colon 1 \leq i < k \text{ and } 1 \leq s \leq r-1\} \\ &\cup \{z_i^s z_{i+1}^s\colon 1 \leq i < k \text{ and } 1 \leq s \leq r-1\} \cup \{x_1 y_1^s\colon 1 \leq s \leq r-1\} \\ &\cup \{x_k z_1^s\colon 1 \leq s \leq r-1\} \cup \{y_k^s z_k^s\colon 1 \leq s \leq r-1\} \\ &\cup \{y_1^p y_1^q\colon 1 \leq p < q \leq r-1\} \cup \{z_1^p z_1^q\colon 1 \leq p < q \leq r-1\}. \end{aligned}$$

Note that $\Delta(D(r,k)) = r$, and it is easy to check that the hyperpath graph $\mathcal{HP}_k(D(r,k);P_{2k})$ contains a $(2r-1)$-complete. □

There are other ways to bound $\chi(\mathcal{HP}_k(G;P_{2k}))$ that we state next, while noting that the computation of $\chi(\mathcal{HP}_2(G^{k-1};P_4))$ or $\chi(\mathcal{H}_2(\overline{G};K_4-e))$ seems to be as difficult as that of $\chi(\mathcal{HP}_k(G;P_{2k}))$, where K_4-e denotes the complete graph on 4 vertices with an edge removed (see Figure 3).

Proposition 5.7 *Let G be a graph. Then $\chi(\mathcal{HP}_k(G;P_{2k})) \leq \chi(\mathcal{HP}_2(G^{k-1};P_4))$ for any $k \geq 2$ and $\chi(\mathcal{HP}_k(G;P_{2k})) \leq \chi(\mathcal{H}_2(\overline{G};K_4-e))$ for any $k > 2$.*

Proof If $A = \{x_1,\ldots,x_k\}$ is a vertex of $\mathcal{HP}_k(G;P_{2k})$, then the mapping $\Psi(A) := \{x_1,x_k\}$ defines a graph homomorphism in either case. □

Acknowledgments

We would like to thank the anonymous referee for the very useful comments and detailed suggestions, which we found very constructive and helpful in improvement of our manuscript.

References

[1] Alon N, Frankl P, Lovász L. The chromatic number of Kneser hypergraphs. Trans Amer Math Soc 1986; 298: 359–370.

[2] Alon N, Lubetzky E. Independent sets in tensor graph powers. J Graph Theory 2007; 54: 73–87.

[3] Brandstadt A, Le VB, Spinrad JP. Graph Classes: A Survey. Philadelphia, PA, USA: SIAM Monographs on Discrete Mathematics and Applications, 1999.

[4] Erdös P. Problems and results in combinatorial analysis. In: International Colloquium on Combinatorial Theory 1973. Rome: Acad Naz Lincei, 1976; 3–17.

[5] Hamburger P, Por A, Walsh M. Kneser representations of graphs. SIAM J Discrete Math 2009; 23: 1071–1081.

[6] Hilton AJW, Spencer CL. A graph-theoretical generalizations of Berge's analogue of the Erdös-Ko-Rado theorem. Trends in Math 2006; 225–242.

[7] Katona GOH. A simple proof of the Erdös-Chao Ko-Rado theorem. J Combin Theory Ser B 1972; 13: 183–184.

[8] Kneser M. Jahresbericht der Deuctschen Mathematiker-Vereinigung. Aufgabe 360, Ableilung 1978; 58:2: S 27.

[9] Kriz I. Equivariant cohomology and lower bounds for chromatic numbers. Trans Amer Math Soc 1992; 333: 567–577.

[10] Lange C, Ziegler G. On generalized Kneser hypergraph colorings. J Combin Theory Ser A 2007; 114: 159–166.

[11] Lovász L. Kneser's conjecture, chromatic number, and homotopy. J Combin Theory Ser A 1978; 25: 319–324.

[12] Matoušek J. Using the Borsuk-Ulam Theorem: Lectures on Topological Methods in Combinatorics and Geometry. Heidelberg: Springer-Verlag, 2003.

[13] Sarkaria K. A generalized Kneser conjecture. J Combin Theory Ser B 1990; 49: 236–240.

[14] Scheinerman E, Ullman D. Fractional Graph Theory. New York: Wiley-Interscience, 1997.

[15] Schrijver A. Vertex-critical subgraphs of Kneser graphs. Nieuw Arch Wiskd III Ser 1978; 26: 454–461.

[16] Stahl S. n-tuple colorings and associated graphs. J Combin Theory Ser B 1976; 20: 185–203.

[17] Ziegler G. Generalized Kneser coloring theorems with combinatorial proofs. Inventiones Math 2002; 147: 671–691.