Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2014) 38: 110 – 118

doi:10.3906/mat-1303-42

Scattering data in an inverse scattering problem on the semi-axis for a first-order hyperbolic system

Mansur ISGENDEROGLU ISMAILOV*, **İbrahim TEKİN**
Department of Mathematics, Gebze Institute of Technology, Gebze, Kocaeli, Turkey

Received: 20.03.2013 • **Accepted:** 19.06.2013 • **Published Online:** 09.12.2013 • **Printed:** 20.01.2014

Abstract: The inverse scattering problem for the first-order hyperbolic system on the semi-axis in the case of 2 incident and 2 scattered waves under consideration of 2 problems with the same given incident waves and different boundary conditions is considered. The scattering data on the semi-axis are given in terms of the scattering operator on the whole axis for the same system with the coefficients, which are extended in the whole axis by zero.

Key words: Inverse scattering problem, scattering data, first-order hyperbolic system

1. Introduction

The inverse scattering problem (ISP) for differential equations is the problem of finding their unknown coefficients from the scattering operator or scattering data. Such an ISP must satisfy the following requirements: the solution must be unique, the algorithm of the recovering of the coefficients must be given, and the characterization of the scattering data must be determined.

The scattering data are the minimal information from which the ISP can be uniquely solved. Since the number of known functions in the ISP is greater than the number of unknown coefficients of differential equations, selection of the minimal information is important for multidimensional inverse problems.

The ISP on the whole axis for the first-order hyperbolic system was solved in [4] via the Gelfand–Levitan–Marchenko (GLM) equation and the scattering data for the ISP were given in [7,9]. The nonlocal Riemann–Hilbert (RH) approach to the ISP for the first-order hyperbolic system on the whole axis was studied in [8]. A connection of the scattering data between GLM and RH approaches was established in [5].

The present paper is devoted to determination of scattering data for the first-order hyperbolic system on the semi-axis in the case of 2 incident and 2 scattered waves. The uniqueness of the solution of the ISP and the algorithm of the recovering of the coefficients from the scattering operator on the semi-axis were studied in [2,3]. The scattering operator on the semi-axis is closely connected with the scattering operator on the whole axis for the same system with the coefficients that are extended to the whole axis by zero [4]. Thus, the method of determination and the characterization of scattering data in this paper are similar to [9], in which scattering data were made for the first-order hyperbolic system on the whole axis.

The paper is organized as follows:

In Section 2, the ISP on the semi-axis is formulated, the scattering operator on the semi-axis is defined,

*Correspondence: mismailov@gyte.edu.tr
2000 *AMS Mathematics Subject Classification:* 35L50, 35P25.

and the relationship between the scattering operator on the semi-axis and the scattering operator on the whole axis for the same system that is extended by zero in the whole axis are studied. The uniqueness and the algorithm of the solution of the ISP on the semi-axis are established. In Section 3, the scattering data on the semi-axis and their characterization are given.

2. The solution of the ISP on the semi-axis

Consider the first-order hyperbolic system of 4 equations on the semi-axis $\mathbb{R}_+ = [0, +\infty)$:

$$\xi_i \frac{\partial \psi_i(x,t)}{\partial t} - \frac{\partial \psi_i(x,t)}{\partial x} = \sum_{j=1}^{4} u_{ij}(x,t)\psi_j(x,t), \quad i = \overline{1,4}, \tag{1}$$

where $\xi_1 > \xi_2 > 0 > \xi_3 > \xi_4$; $u_{ij}(x,t),\ i,j = \overline{1,4}$ are measurable complex-valued functions and they satisfy the conditions

$$u_{ii}(x,t) = 0, \tag{2}$$

$$u_{ij}(x,t) \in L_2(\mathbb{R}_+ \times \mathbb{R}, \mathbb{C}),\ i,j = \overline{1,4};\ i \neq j.$$

We will consider generalized solutions of the system (1), which are measurable in x and t. Here, these functions belong to space $L_2(\mathbb{R}, \mathbb{C})$ with respect to variable t and their L_2-norms belong to $L_\infty(\mathbb{R}_+, \mathbb{C})$, i.e. the vector-valued functions $\psi(x,t) = \operatorname{col}\{\psi_1, \psi_2, \psi_3, \psi_4\}$ in the space of vector-valued functions with norm

$$\|\psi\| = \max_{k=\overline{1,4}} \operatorname{vrai}\sup_{x} \|\psi_k(x,.)\|_{L_2(\mathbb{R})}.$$

We call such solutions admissible.

Every admissible solution of the system (1) with the potential of satisfying conditions (2) admits the asymptotic representation

$$\psi_i(x,t) = a_i(t + \xi_i x) + o(1), \tag{3_1}$$

$$\psi_{i+2}(x,t) = b_i(t + \xi_{i+2} x) + o(1), \quad i = 1,2,\ x \to +\infty, \tag{3_2}$$

where the functions $a_1(t), a_2(t) \in L_2(\mathbb{R}, \mathbb{C})$ define a profile of the incident waves and $b_1(t), b_2(t) \in L_2(\mathbb{R}, \mathbb{C})$ define a profile of the scattering waves.

The scattering problem on the semi-axis consists of finding the solutions of system (1) in accordance with the given incident waves $a_1(t), a_2(t)$ and specified boundary conditions at $x = 0$.

Consider the 2 scattering problems on the semi-axis for the system (1).

First scattering problem: It is required to find a solution of the system (1) such that the boundary conditions

$$\psi_3(0,t) = \psi_1(0,t), \quad \psi_4(0,t) = \psi_2(0,t) \tag{4}$$

and asymptotic relation (3_1) are satisfied.

Second scattering problem: It is required to find a solution of the system (1) such that the boundary conditions

$$\psi_3(0,t) = \psi_2(0,t), \quad \psi_4(0,t) = \psi_1(0,t) \tag{5}$$

and asymptotic relation (3_1) are satisfied.

We will consider these 2 scattering problems together.

Theorem 1 *(Theorem 2.1, [3]) Let the coefficients of the system (1) satisfy the condition (2). Then for arbitrary* $a(t) = \begin{pmatrix} a_1(t) \\ a_2(t) \end{pmatrix} \in L_2(\mathbb{R}, \mathbb{C}^2)$ *there exists a unique admissible solution of the scattering problems (1)-(4)-(3_1) and (1)-(5)-(3_1).*

According to (3_2), the third and fourth components of the solution of the kth ($k = 1, 2$) scattering problems admits the asymptotic representation

$$\begin{array}{l} \psi_3^k(x,t) = b_1^k(t + \xi_3 x) + o(1), \quad x \to +\infty \\ \psi_4^k(x,t) = b_2^k(t + \xi_4 x) + o(1), \quad x \to +\infty \end{array} \tag{6}$$

where $b_1^k(t), b_2^k(t) \in L_2(\mathbb{R}, \mathbb{C})$. Denote $b^k(t) = \begin{pmatrix} b_1^k(t) \\ b_2^k(t) \end{pmatrix}, k = 1, 2$.

According to Theorem 1 and formula (6), for an arbitrary $a(t) \in L_2(\mathbb{R}, \mathbb{C}^2)$, we can define the operators $\mathbf{S}^1 = (S_{ij}^1)_{i,j=1,2}$ and $\mathbf{S}^2 = (S_{ij}^2)_{i,j=1,2}$ in the space $L_2(\mathbb{R}, \mathbb{C}^2)$, which translate $a(t)$ to $b^1(t)$ and $Ha(t)$ to $b^2(t)$, respectively:

$$b^1(t) = \mathbf{S}^1 a(t), \quad b^2(t) = \mathbf{S}^2 H a(t)$$

where $H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

These operators are invertible and they have forms as $\mathbf{S}^k = I + F^k$, $(\mathbf{S}^k)^{-1} = I + J^k$, where F^k and J^k, $(k = 1, 2)$ are Hilbert–Schmidt integral operators.

We will call the operator $\mathbf{S} = [\mathbf{S}^1, \mathbf{S}^2]$ the scattering operator for the system (1) on the semi-axis with 2 known incident waves.

The problem of finding coefficients of the system (1) from known scattering operators $\mathbf{S}^1$ and $\mathbf{S}^2$ is called the ISP on the semi-axis.

Let us define the transmission operator $\widetilde{S}$ transforming the boundary data $\mathrm{col}\{\psi_1(0,t), \psi_2(0,t), \psi_3(0,t), \psi_4(0,t)\}$ of the admissible solution to the asymptotic $\mathrm{col}\{a_1(t), a_2(t), b_1(t), b_2(t)\}$.

The following theorem holds for the relationship between operators $\mathbf{S}$ and $\widetilde{S}$.

Theorem 2 *(Theorem 2, [2]) The transmission operator* $\widetilde{S}$ *is uniquely determined by the scattering operator* $\mathbf{S} = [\mathbf{S}^1, \mathbf{S}^2]$ *on the semi-axis.*

Consider the system of differential equations (1) in the whole axis with the coefficients $v_{ij}(x,t)$, which are extended to the whole axis $-\infty < x < +\infty$ by zero, i.e. $v_{ij}(x,t) = \begin{cases} u_{ij}(x,t), & x \geq 0, \\ 0, & x < 0. \end{cases}$ The system of equation (1) becomes the system

$$\frac{\partial \tilde{\psi}_i(\varkappa, \tau)}{\partial \tau} - \xi_i \frac{\partial \tilde{\psi}_i(\varkappa, \tau)}{\partial \varkappa} = \sum_{j=1}^{4} \tilde{v}_{ij}(\varkappa, \tau) \tilde{\psi}_j(\varkappa, \tau), \quad i = \overline{1,4},$$

by applying the change of variables $x = -\tau$, $t = \varkappa$. Here $\tilde{\phi}_j(\varkappa, \tau) \equiv \phi_j(-\tau, \varkappa)$. For this system, when the condition (2) is satisfied, there exist (p. 40, [6]) unique matrix functions $\tilde{w}^{(\pm)}(\varkappa, \tau; y) = (\tilde{w}_{ij}^{(\pm)}(\varkappa, \tau; y))_{i,j=1}^4$, which are solutions of the matrix integral equation

$$\tilde{w}(\varkappa,\tau;y)=h(\varkappa,\tau;y)+(K\tilde{w})(\varkappa,\tau;y), \tag{7}$$

in the region $\pm\varkappa\geq\pm y$. Here

$$\begin{aligned} h(\varkappa,\tau;y) &= (h_{ij}(\varkappa,\tau;y))_{i,j=1}^{4} \\ h_{ij}(\varkappa,\tau;y) &= \frac{1}{\xi_i-\xi_j}\tilde{v}_{ij}(\varkappa+\frac{\xi_i}{\xi_i-\xi_j}(y-\varkappa),\tau-\frac{y-\varkappa}{\xi_i-\xi_j}), \\ h_{ii} &= 0,\ i=\overline{1,4} \end{aligned}$$

and $(Kw)(\varkappa,\tau;y)$ is the matrix integral operator as follows:

$$\begin{aligned} (K\tilde{w})_{ij}(\varkappa,\tau;y) &= -\frac{1}{\xi_i-\xi_j}\sum_{k=1}^{4}\int\limits_{y}^{\varkappa}\tilde{v}_{ik}(\varkappa+\frac{\xi_i}{\xi_i-\xi_j}(s-\varkappa),\tau-\frac{s-\varkappa}{\xi_i-\xi_j}) \\ &\quad \tilde{w}_{kj}(\varkappa+\frac{\xi_i}{\xi_i-\xi_j}(s-\varkappa),\tau-\frac{s-\varkappa}{\xi_i-\xi_j};y+\frac{\xi_j}{\xi_i-\xi_j}(s-\varkappa))ds, \\ (K\tilde{w})_{ii}(\varkappa,\tau;y) &= -\sum_{k=1}^{4}\int\limits_{-\infty}^{0}\tilde{v}_{ik}(\varkappa+\xi_i s,\tau-s)\tilde{w}_{kj}(\varkappa+\xi_i s,\tau-s;y+\xi_j s)ds,\ i,j=\overline{1,4}. \end{aligned}$$

The solution of (7) admits the following estimates:

$$\left\|vrai\max_{x}\left|\tilde{w}^{(\pm)}(\varkappa,\tau;y)\right|\right\|_{L_2(\mathbb{R}^2)}<+\infty.$$

For fixed y and j, the columns of the solution $\operatorname{col}\left\{\tilde{w}_{1j}^{(\pm)}(\varkappa,\tau;y),...,\tilde{w}_{nj}^{(\pm)}(\varkappa,\tau;y)\right\}$ of (7) are the solution of the system (1), which are extended in the whole axis by zero.

Let us introduce the matrices

$$\mathbf{W}_-(x)f(t)=\int\limits_{-\infty}^{t}w^{(+)}(x,t;y)f(y)dy,\ \mathbf{W}_+(x)f(t)=\int\limits_{t}^{+\infty}w^{(-)}(x,t,y)f(y)dy,$$

where $w^{(\pm)}(x,t;y)=\tilde{w}^{(\pm)}(t,-x;y)$.

From (7), the Hilbert–Schmidt kernels $w_{ij}^{(\pm)}(x,t,\tau)$ are uniquely determined by the coefficient of the system (1) and they are related with the coefficient as the following formulas for $x\geq 0$:

$$(\xi_i-\xi_j)w_{ij}^{(\pm)}(x,t,t)=\pm u_{ij}(x,t),\ i,j=\overline{1,4}. \tag{8}$$

There exists another transmission operator (p. 51, [3]) $\widehat{S}=I+\hat{F}$ in the space $L_2(\mathbb{R},\mathbb{C}^4)$ that is invertible and admits the left factorization

$$\mathcal{F}_x\widehat{S}\mathcal{F}_{-x}=(I+\mathbf{W}_-(x))^{-1}(I+\mathbf{W}_+(x)) \tag{9}$$

for every $x \geq 0$, where $\mathcal{F}_x = diag\{T_{\xi_1 x}, T_{\xi_2 x}, T_{\xi_3 x}, T_{\xi_4 x}\}$ is diagonal shift operator with $T_{\xi_i x} f(t) = f(t+\xi_i x)$, $i = \overline{1,4}$.

The factorization (9) becomes the following matrix factorizations at $x = 0$ and $x = +\infty$ (p. 56, [3]):

$$\widehat{S} = (I + D_-)(I + S_-^{rz})(I + S_+^{lz})^{-1} = (I + S^{lz})^{-1}(I + S^{rz})(I + D_+), \tag{10}$$

where I is identity matrix operator, $S^{rz}, S_-^{rz} (S^{lz}, S_+^{lz})$ are strictly lower (upper) triangular Hilbert–Schmidt matrix operators, and D_+, D_- are diagonal matrix operators. In addition, $S_-^{rz}, D_- (S_+^{lz}, D_+)$ are upper (lower) Volterra integral operators.

Two-sided matrix factorization of the operator $\widetilde{S}$ are obtained as follows (p. 56, [3]):

$$\widetilde{S} = (I + S^{rz})(I + D_+)(I + S_+^{lz}) = (I + S^{lz})(I + D_-)(I + S_-^{rz}). \tag{11}$$

In contrast to the general case, if $u_{ij}(x,t) = 0,\ x < 0,\ i,j = \overline{1,4}$, the transmission operator $\widehat{S}$ admits left matrix factorization on the triangle and Volterra factors as follows:

$$\widehat{S} = (I + D_-)(I + S_-^{rz})(I + S_+^{lz})^{-1}.$$

Therefore, we obtain the following theorem by using Theorem 2 and the uniqueness of the representations (10) and (11).

Theorem 3 *The transmission operators* $\widetilde{S}$*,* $\widehat{S}$ *and the scattering operator* $\mathbf{S} = [\mathbf{S}^1, \mathbf{S}^2]$ *on the semi-axis uniquely determine each other.*

The following theorem holds for the uniqueness of the ISP on the semi-axis.

Theorem 4 *Let* $\widetilde{S}$ *be a transmission operator for the system (1) with the coefficients satisfying condition (2). Then the coefficients of the system (1) are uniquely determined from the operator* $\widetilde{S}$.

Proof Let $(u_{ij}^1(x,t))_{i,j=1}^4$ and $(u_{ij}^2(x,t))_{i,j=1}^4$ be 2 coefficients corresponding to transmission operators $\widetilde{S}_1$ and $\widetilde{S}_2$, respectively. Let us show that if $\widetilde{S}_1 = \widetilde{S}_2$, then $u_{ij}^1(x,t) = u_{ij}^2(x,t)$ for $i,j = \overline{1,4}$. Since the matrix representation (11) is unique and $\widetilde{S}_1 = \widetilde{S}_2$, its factors coincide. Then $\widehat{S}_1$ and $\widehat{S}_2$ also coincide by (10). Because the Volterra factorization (9) is unique, the Volterra factors are equal for the operators $\mathcal{F}_x \widehat{S_1} \mathcal{F}_{-x}$ and $\mathcal{F}_x \widehat{S_2} \mathcal{F}_{-x}$. Then $u_{ij}^1(x,t) = u_{ij}^2(x,t)$ by formula (8). □

The algorithm of finding the coefficients of the system (1) by the scattering operator $\mathbf{S} = [\mathbf{S}^1, \mathbf{S}^2]$ consists of the following steps:

If $\mathbf{S} = [\mathbf{S}^1, \mathbf{S}^2] = \begin{pmatrix} \mathbf{S}_{11}^1 & \mathbf{S}_{12}^1 & \mathbf{S}_{11}^2 & \mathbf{S}_{12}^2 \\ \mathbf{S}_{21}^1 & \mathbf{S}_{22}^1 & \mathbf{S}_{21}^2 & \mathbf{S}_{22}^2 \end{pmatrix}$ is the known scattering operator for the system (1) on the semi-axis with the coefficients satisfying the conditions (2), then:

1. The matrix operator $\widetilde{S}$ can be determined with respect to $\mathbf{S}$ by Theorem 2 (see [2] for clarification).

2. The triangle factors of the operator $\widetilde{S}$ can be found as a linear system of the operator equation from the formula (11).

3. The operator $\widehat{S}$ can be defined from the formula (10).

4. The operator $F_x\widehat{S}F_{-x}$ admits the factorization (9). The Volterra factors $\mathbf{W}_+(x,t,\tau)$ and $\mathbf{W}_-(x,t,\tau)$ can be found using Krein's theory of factorization of the second kind of Fredholm integral operator (p. 183, [1]).

5. The coefficients $u_{ij}(x,t),\ (i,j=1,4;\ i\neq j)$ are defined with respect to the kernels of $\overset{(\pm)}{W}(x,t,\tau)$ by the formula (8).

3. Scattering data and their properties

The ISP on the semi-axis means the recovering of the coefficients $u_{ij}(x,t),\ x\geq 0,\ t\in\mathbb{R},\ \ (i,j=\overline{1,4}; i\neq j)$ of the system (1) from its scattering operator $\mathbf{S}=[\mathbf{S}^1,\mathbf{S}^2]$, where $\mathbf{S}^k=I+F_k,\ F_k=(F_{ij}^k)_{i,j=1}^4,\ \ k=1,2$ and

$$F_{ij}^k f(x)=\int\limits_{-\infty}^{+\infty}F_{ij}^k(x,t)f(t)dt,\ x\in\mathbb{R}.$$

As is seen, the scattering operator $\mathbf{S}$ is a 2×4 matrix integral operator with Hilbert–Schmidt kernels, which are defined in all planes $\mathbb{R}^2$. However, the coefficients of the system (1) are known in half-plane $\mathbb{R}_+\times\mathbb{R}$. By the way, we have 16 functions in the half-plane for which it is needed to find 12 functions (coefficients). This means that there is overdetermination in finding coefficients of the system (1). Therefore, it is important to minimize the data of the considered inverse problem.

It is suitable to select the minimal information in terms of operator $\widehat{S}$, which is in a one-to-one relation with $\mathbf{S}$ by Theorem 4.

Let

$$S_-^{rz}=(S_{ij-})_{i,j=1,i>j}^4,\ S_+^{lz}=(S_{ij+})_{i,j=1,i<j}^4,$$
$$D_\pm=diag\,(D_{11\pm},D_{22\pm},D_{33\pm},D_{44\pm})$$

in (11). From the representation (10) we obtain that

$$(I+S_-^{rz})(I+S_+^{lz})^{-1}=(I+D_-)^{-1}(I+S^{lz})^{-1}(I+S^{rz})(I+D_+). \tag{12}$$

From (12), the matrix D_- is uniquely determined by the matrix elements of S_-^{rz} and S_+^{lz}. As is known, the matrix $I+S_+^{lz}$ is invertible and its inverse is the same matrix structure as $(I+S_+^{lz})^{-1}=I+M_+^{lz}$, $M_+^{lz}=(M_{ij+})_{i,j=1,i<j}^4$.

From (10), we obtain the following left factorizations:

$$\begin{array}{r}(I+D_{44-})^{-1}(I+D_{44+})=S_{41-}M_{14+}+S_{42-}M_{24+}+S_{43-}M_{34+}+I,\\(I+D_{33-})^{-1}(I+D_{33+})=S_{31-}M_{13+}+S_{32-}M_{23+}+I-AB,\\(I+D_{22-})^{-1}(I+D_{22+})=S_{21-}M_{12+}+I-EF-CD,\\(I+D_{11-})^{-1}(I+D_{11+})=I-LM-JK-GH,\end{array} \tag{13}$$

where

$$A=(S_{31-}M_{14+}+S_{32-}M_{24+}+M_{34+})(I+D_{44+})^{-1},\quad G=M_{14+}(I+D_{44+})^{-1},$$
$$B=(I+D_{44-})(S_{41-}M_{13+}+S_{42-}M_{23+}+S_{43-}),\quad H=(I+D_{44-})S_{41-},$$
$$C=(S_{21-}M_{14+}+M_{24+})(I+D_{44+})^{-1},\quad J=(M_{13+}-GB)(I+D_{33+})^{-1},$$
$$D=(I+D_{44-})(S_{41-}M_{12+}+S_{42-}),\quad K=(I+D_{33-})(S_{31-}-AH),$$
$$E=(S_{21-}M_{13+}+M_{23+}-CB)(I+D_{33+})^{-1},\quad L=(M_{12+}-JF-GD)(I+D_{22+})^{-1},$$
$$F=(I+D_{33-})(S_{31-}M_{12+}+S_{32-}-AD),\quad M=(I+D_{22-})(S_{21-}-EK-CH).$$

It is known from (12) that the factors are uniquely determined by the factorization relations (13). Therefore, the matrix D_- is uniquely determined by S_-^{rz} and S_+^{lz}. This means that the transmission matrix $\widehat{S}$ is uniquely determined by S_-^{rz} and S_+^{lz}.

Let

$$S_{ij-}f(t) = \int_t^{+\infty} s_{ij}(t,\tau)f(\tau)d\tau,\ i,j=\overline{1,4};\ i>j,$$

$$S_{ij+}f(t) = \int_{-\infty}^{t} s_{ij}(t,\tau)f(\tau)d\tau,\ i,j=\overline{1,4};\ i<j.$$

Definition 1 *The collection of functions* $\{s_{ij}(t,\tau),\ i,j=\overline{1,4},\ i>j;\ s_{ij}(t,\tau),\ i,j=\overline{1,4},\ i<j\}$, *which are the kernels of Volterra integral operators* $\{S_{ij-},\ i,j=\overline{1,4},\ i>j;\ S_{ij+},\ i<j\}$, *is called the scattering data for the system (1) on the semi-axis.*

From the uniqueness of the matrix factorizations (10) and (11), it follows that $\widehat{S}$ is uniquely determined by scattering data. Therefore, the scattering operator $\mathbf{S}=[\mathbf{S}^1,\mathbf{S}^2]$ is also uniquely determined from the scattering data by Theorem 3.

Theorem 5 *Let* $(s_{ij}(t,\tau))_{i,j=1,i>j}^4$ *and* $(s_{ij}(t,\tau))_{i,j=1,i<j}^4$ *be 2 strictly upper and strictly lower triangular matrix functions, which are zero when* $\tau>t$ *and* $\tau<t$*, respectively. Suppose that they are matrices with complex valued square summable entries. The collection* $\{s_{ij}(t,\tau),\ i,j=\overline{1,4},\ i>j;\ s_{ij}(t,\tau),\ i,j=\overline{1,4},\ i<j\}$ *is the scattering data for the system (1) on the semi-axis with the coefficients satisfying the condition (2) if and only if the matrix* $\mathcal{F}_x\widehat{S}\mathcal{F}_{-x}$ *admits the left factorization as in (9), for every* $x>0$. *Here*

$$\widehat{S} = (I+D_-)(I+S_-^{rz})(I+S_+^{lz})^{-1},$$

$$S_-^{rz}f(t) = \int_t^{+\infty}(s_{ij}(t,\tau))_{i,j=1,i>j}^4 f(\tau)d\tau,$$

$$S_{+}^{lz} f(t) = \int_{-\infty}^{t} (s_{ij}(t,\tau))_{i,j=1,i<j}^{4} f(\tau) d\tau,$$

$$D_{-} = diag(D_{11-}, D_{22-}, D_{33-}, D_{44-}),$$

where $D_{11-}, D_{22-}, D_{33-}$ and D_{44-} are defined as in formula (13).

Proof Since the operator $\mathcal{F}_x \widehat{S} \mathcal{F}_{-x}$ admits left factorization $\mathcal{F}_x \widehat{S} \mathcal{F}_{-x} = (I + \mathbf{W}_{-}(x))^{-1}(I + \mathbf{W}_{+}(x))$, we obtain the following operator equation with respect to $\mathbf{W}_{-}(x)$ and $\mathbf{W}_{+}(x)$:

$$\mathbf{W}_{\pm}(x) + \overset{(\pm)}{\Phi}(x) + \mathbf{W}_{\pm}(x) \overset{(\pm)}{\Phi}(x) = \mathbf{W}_{\pm}(x),$$

where $\overset{(-)}{\Phi}(x) = \mathcal{F}_x \widehat{S} \mathcal{F}_{-x} - I$, $\overset{(+)}{\Phi}(x) = \mathcal{F}\left(\widehat{S}\right)^{-1} \mathcal{F}_{-x} - I$. These equations can be rewriten with respect to kernels of its integral operators as below:

$$\omega(x,t;\tau) \mp \int_{t}^{\mp\infty} \omega(x,t;\tau) \overset{(\pm)}{\Phi}(x,s,\tau) ds = \mp \overset{(\pm)}{\Phi}(x,t,\tau), \tag{14}$$

where $\omega(x,t;\tau) = \begin{cases} \omega^{(+)}(x,t;\tau), & \tau \le t, \\ \omega^{(-)}(x,t;\tau), & \tau \ge t. \end{cases}$

These systems of equations are the GLM type of integral equations. Because the factorization (9) is admitted, these systems are uniquely solvable in the space $L_2(\mathbb{R}, \mathbb{C}^4)$ for every $0 < x < +\infty$; $-\infty < t < +\infty$ and for the arbitrary right hand side function in space L_2 (p. 183, [1]). Therefore, we obtain the functions

$$\pm u_{ij}(x,t) = (\xi_i - \xi_j) \omega_{ij}^{(\pm)}(x,t;t), \ i,j = \overline{1,4} \tag{15}$$

in the space $L_2(\mathbb{R}_{+} \times \mathbb{R}, \mathbb{C})$.

Now, consider the system (1) with the coefficients (15). It is needed to show that the scattering data corresponding to the system with the coefficients (15) coincide with the data mentioned in Theorem 5.

By using the equality (14), it can be shown that the vector function

$$v(x,t;\tau) = \omega(x,t;\tau) - h(x,t,\tau) - (K\omega)(x,t;\tau)$$

is the solution of the homogeneous case of the equation (14). Since these homogeneous equations have only a trivial solution, then $v(x,t;\tau) = 0$ and $\omega(x,t;\tau)$ is a solution of (7) [9]. Therefore, $\omega(x,t;\tau) = w(x,t;\tau)$. This fact implies that the scattering data related with the kernel $w(x,t;\tau)$ coincide with the data $\{s_{ij}(t,\tau), \ i,j = \overline{1,4}, \ i > j; \ s_{ij}(t,\tau), \ i,j = \overline{1,4}, \ i < j\}$. □

Example 1 *Let us recover the coefficients of the system (1) from the scattering data $\{s_{ij}(t,\tau), \ i,j = \overline{1,4}, \ i > j; \ s_{ij}(t,\tau), \ i,j = \overline{1,4}, \ i < j\}$ with $s_{3k}(t,\tau) = 0, \ k = 1,2$; $s_{4k}(t,\tau) = 0, \ k = \overline{1,3}$; $s_{1k}(t,\tau) = 0, \ k = \overline{2,4}$; $s_{2k}(t,\tau) = 0, \ k = 3,4$ and $s_{21}(t,\tau) = 0, \ \tau > t$; $s_{34}(t,\tau) = 0, \ \tau > t$ with $s_{21}(t,\tau), \ s_{34}(t,\tau) \in L_2(\mathbb{R}^2, \mathbb{C})$. In* *this case the factorization (9) easily becomes the following expressions with respect to* $\omega_{ij}(x,t;\tau)$*:*

$$\begin{aligned}
\omega_{ij}^{(-)}(x,t;\tau) &= 0,\ i,j=\overline{1,4};(i,j)\neq(2,1),(3,4),\ \tau\geq t,\\
\omega_{21}^{(-)}(x,t;\tau) &= -s_{21}(t+\xi_2 x,\tau+\xi_1 x),\ \tau\geq t,\\
\omega_{34}^{(-)}(x,t;\tau) &= \begin{cases} s_{34}(t+\xi_3 x,\tau+\xi_4 x),\ t\leq\tau\leq t+(\xi_3-\xi_4)x,\\ 0,\ \tau\geq t+(\xi_3-\xi_4)x,\end{cases}
\end{aligned}$$

$$\begin{aligned}
\omega_{ij}^{(+)}(x,t;\tau) &= 0,\ i,j=\overline{1,4};\ (i,j)\neq(2,1),(3,4),\ \tau\leq t,\\
\omega_{34}^{(+)}(x,t;\tau) &= -s_{34}(t+\xi_3 x,\tau+\xi_4 x),\ \tau\leq t,\\
\omega_{21}^{(+)}(x,t;\tau) &= \begin{cases} s_{21}(t+\xi_2 x,\tau+\xi_1 x),\ t+(\xi_2-\xi_1)x\leq\tau\leq t,\\ 0,\ \tau\leq t+(\xi_2-\xi_1)x.\end{cases}
\end{aligned}$$

By using the formula (15), the coefficients are determined as below:

$$\begin{aligned}
u_{ij}(x,t) &= 0,\ i,j=\overline{1,4};\ (i,j)\neq(2,1),(3,4),\\
u_{21}(x,t) &= (\xi_2-\xi_1)s_{21}(t+\xi_2 x,t+\xi_1 x),\\
u_{34}(x,t) &= (\xi_4-\xi_3)s_{34}(t+\xi_3 x,t+\xi_4 x).
\end{aligned}$$

References

[1] Gohberg I, Krein M. Theory and Applications of Volterra Operators in Hilbert Space. Providence, RI, USA: American Mathematical Society, 1970.

[2] Iskenderov NSh, Ismailov MI. Inverse non-stationary scattering problem for hyperbolic system of four equations of the first order on a semi-axis. Proc Inst Math Mech AN Azerb 1996; 4: 161–168 (article in Russian with an abstract in English).

[3] Ismailov MI. Scattering problem for the first order hyperbolic systems in the semi-axis. PhD, Institute of Mathematics and Mechanics of Academy Science of Azerbaijan, Baku, Azerbaijan, 1996.

[4] Ismailov MI. Inverse scattering problem for hyperbolic system on a semi-axis in the case of equal number of incident and scattered waves. Inverse Problems 2006; 22: 955–974.

[5] Kulish PP, Lipovskii VD, Shirokov AV. Scattering data for the nonstationary Dirac equation. J Soviet Math 1989; 47, 2: 2488–2492.

[6] Nizhnik LP. Inverse Scattering Problems for Hyperbolic Equations. Kiev: Nauk Dumka, 1991 (in Russian).

[7] Nizhnik LP, Tarasov VG. The inverse nonstationary scattering problem for a hyperbolic system of equations. Sov Math Dokl 1977; 18: 397–401.

[8] Sung LY, Fokas AS. Inverse problems for N × N hyperbolic systems on the plane and N-wave interactions. Commun Pure Appl Math 1991; 64: 535–571.

[9] Tarasov VG. Scattering data in an inverse scattering problem for a hyperbolic system of first-order differential equations. Dopov Nats Akad Nauk Ukr Mat. Prirodozn Tekh Nauki 1997; 7: 60–67 (article in Russian with an abstract in English).