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Kernel operators on the upper half-space: boundedness and compactness criteria

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Abstract: We establish necessary and sufficient conditions on a weight v governing the trace inequality

$$\|\hat{K}f\|_{L^q_v(\hat{E})} \leq C\|f\|_{L^p(E)},$$

where E is a cone on a homogeneous group, $\hat{E} := E\times\mathbb{R}_+$ and $\hat{K}$ is a positive kernel operator defined on $\hat{E}$. Compactness criteria for this operator are also established.

Key words: Operators with positive kernels, upper half-space, potentials, homogeneous groups, trace inequality, boundedness, compactness, weights

1. Introduction

Our aim is to establish $L^p(E) \to L^q_v(\hat{E})$ boundedness/compactness criteria for the generalized integral operators

$$\hat{K}f(x,t) = \int\limits_{E_{r(x)}} \hat{k}(x,y,t)f(y)dy, \quad (x,t) \in \hat{E}, \tag{1}$$

where $E_{r(x)}$ and E are certain cones in a homogeneous group G, and $\hat{E} := E \times \mathbb{R}_+$. Here $\hat{k} : \{(x,y) \in E \times E : r(y) < r(x)\} \times [0,\infty) \to \mathbb{R}_+$ is a kernel and v is an almost everywhere positive function on $\hat{E}$ (i.e. weight). It should be emphasized that the results are new even for Euclidean case $G = \mathbb{R}^n$.

The problems studied in this paper can be considered as a natural continuation of the investigation carried out in [3] (see also [21], Ch. 3), where the authors derived the similar results for the operator

$$\mathcal{K}f(x) = \int_{E_{r(x)}} k(x,y)f(y)dy, \quad x \in E,$$

defined on cones of homogeneous groups.

Our conditions on the kernel $\hat{k}$ are similar to those introduced in [20] (see also [5], Sec. 2.10) for one-dimensional cases and include kernels of variable parameter fractional integrals on the half-space. In that paper appropriate examples of kernels defined on $\mathbb{R}^2_+$ were also given.

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We point out that the trace inequality

$$||I_\alpha f||_{L^q_v(\Omega\times\mathbb{R}_+)} \le C||f||_{L^p(\Omega)}, \quad 1<p<q<\infty,$$

where $\Omega\subset\mathbb{R}^n$ is a domain and

$$I_\alpha f(x,t)=\int\limits_\Omega (|x-y|+t)^{\alpha-n}f(y)dy, \quad 0<\alpha<n,$$

was characterized by Adams [1] (see also [8] for a more general case).

A complete description of a weight pair (v,w) ensuring the 2-weight inequality for I_α in the case $1<p<q<\infty$ was established in [7]. Sawyer-type necessary and sufficient conditions governing the 2-weight boundedness of I_α and corresponding Hörmander-type maximal operator were obtained in [26]. In [12] necessary and sufficient conditions governing the trace inequality/compactness were derived for truncated potentials defined on $\mathbb{R}^n\times\mathbb{R}_+$.

Such fractional integral operators defined on the half-space arise in the study of boundary value problems in PDEs, particularly in polyharmonic differential equations. Some applications of the operator I_α in weighted estimates for gradients were presented in [30], p. 923.

The $L^p\to L^q_v$ $(p\le q)$ boundedness/compactness criteria for one-sided potentials

$$R_\alpha f(x)=\int\limits_0^x \frac{f(t)}{(x-t)^{1-\alpha}}dt$$

were found in [18] (see also [24]). That result was generalized in [20] for kernel operators involving, for example, Riemann–Liouville, power-logarithmic, Erdelyi-Köber, and Hadamard kernels (see also monograph [5], Ch.2).

In [19] the third author of this paper derived trace inequality criteria for one-sided potential operators defined on the upper half-plane

$$\hat{R}_\alpha f(x,t)=\int\limits_0^x \frac{f(y)}{(x-y+t)^{1-\alpha}}dy, \quad (x,t)\in\mathbb{R}^2_+.$$

We refer also to [5], Chapters 9 and 10, for these and more general results.

The 2-weight problem for higher-dimensional Hardy-type operators defined on cones in $\mathbb{R}^n$ involving the kernels from [4] and [22] was studied in [9] and [29]. A similar problem for Hardy-type transforms on star-shaped regions was investigated in [27]. It should be emphasized that the results of [20] were generalized in [16] for kernel operators defined on star-shaped regions.

Finally, we point out that 2-weight theory for positive kernel operators involving Hardy-type transforms and fractional integrals was delivered in the following well-known monographs: [11], [15], [17], [23], [25], [5], [13], etc.

2. Preliminaries

We begin this section with the definition of a homogeneous group.

A homogeneous group G is a simply connected nilpotent Lie group G on which Lie algebra g is given a one-parameter group of transformations $\delta_t = \exp(A \log t)$, $t > 0$, where A is a diagonalized linear operator on G with positive eigenvalues. For G the mappings $\exp \circ \delta_t \circ \exp^{-1}$, $t > 0$, are automorphisms on G, which will be denoted by δ_t. The number $Q = \mathrm{tr}A$ is called homogeneous dimension of G. The symbol e will stand for the neutral element in G.

It is possible to equip G with a homogeneous norm $r : G \to [0, \infty)$, which is a continuous function on G and smooth on $G \setminus \{e\}$, satisfying the following conditions:

(i) $r(x) = r(x^{-1})$ for every $x \in G$;

(ii) $r(\delta_t x) = t \cdot r(x)$ for every $x \in G$ and $t > 0$;

(iii) $r(x) = 0$ if and only if $x = e$;

(iv) there exists $c_0 \geq 1$ such that

$$r(xy) \leq c_0(r(x) + r(y)), \quad x, y \in G.$$

A ball in G, centered at x and of radius ρ, is defined as

$$B(x, \rho) = \{y \in G : r(xy^{-1}) < \rho\}.$$

It can be observed that $\delta_\rho B(e, 1) = B(e, \rho)$.

Let us fix a Haar measure $|\cdot|$ in G so that $|B(e, 1)| = 1$. Then $|\delta_t E| = t^Q |E|$; in particular, $|B(x, s)| = s^Q$ for $x \in G$, $s > 0$.

Examples of homogeneous groups are Euclidean n-dimensional space, Heisenberg groups, upper triangular groups, etc (see [6] for the definition and basic properties of homogeneous groups).

Let S be the unit sphere in G, i.e. $S := \{x \in G : r(x) = 1\}$. The next statement is useful for us.

Proposition A ([6], p. 14) *Let G be a homogeneous group. There is a (unique) Radan measure σ on S such that for all $u \in L^1(G)$,*

$$\int_G u(x)dx = \int_0^\infty \int_S u(\delta_s \bar{y}) s^{Q-1} d\sigma(\bar{y}) ds.$$

Furthermore, let A be a measurable subset of S with positive measure. We denote by E a measurable cone in G:

$$E := \{x \in G : x = \delta_s \bar{x}, 0 < s < \infty, \bar{x} \in A\}.$$

We denote

$$E_t := \{y \in E : r(y) < t\}.$$

Now we define the kernel operator given by (1), where $\hat{k}(x, y, t)$ is a nonnegative function defined on

$$\tilde{E} := \{(x, y) \in E \times E : r(y) < r(x)\} \times \mathbb{R}_+.$$

In the sequel we will also use the notation:

$$S_x := E_{r(x)/2c_0}, \quad F_x := E_{r(x)} \setminus S_x,$$

$$\hat{F} := F \times [0, \infty), \quad \lambda' := \frac{\lambda}{\lambda - 1},$$

where the constant c_0 is from the triangle inequality for the homogeneous norm r, F is a measurable subset of G, and λ is a number satisfying the condition $\lambda \in (1,\infty)$.

Let Ω be a measurable subset of G and let w be an almost everywhere positive function (i.e. weight) on Ω. Denote by $L^p_w(\Omega)$ $(0<p<\infty)$ the weighted Lebesgue space, which is the space of all measurable functions $f:\Omega\to\mathbb{C}$ with the finite norm (quasi-norm if $0<p<1$):

$$\|f\|_{L^p_w(\Omega)} = \left(\int\limits_{\Omega} |f(x)|^p w(x)dx\right)^{1/p}.$$

If $w\equiv 1$, then we denote $L^p_w(\Omega)$ by $L^p(\Omega)$.

Now we introduce a class of kernels defined on $\hat{E}$.

Definition 1 *We say that the kernel $\hat{k}\in \hat{V}_\lambda$, $1<\lambda<\infty$, if*
(i) there are positive constant c_1 and c_2 such that

$$\hat{k}(x,y,t) \le c_1 \hat{k}(x,\delta_{1/(2c_0)}x,t) \tag{2}$$

for all $x,y\in E$ with $0<r(y)\le r(x)/(2c_0)$ and $t>0$;

$$\hat{k}(x,y,t) \ge c_2 \hat{k}(x,\delta_{1/(2c_0)}x,t) \tag{3}$$

for all $x,y\in E$ with $0<r(x)/(2c_0)\le r(y)\le r(x)$ and $t>0$;
(ii) there exists a positive constant c_3 such that for all $x\in E$ and $t>0$

$$\int\limits_{F_x} \hat{k}^{\lambda'}(x,y,t)dy \le c_3\big(r(x)\big)^Q \hat{k}^{\lambda'}(x,\delta_{1/(2c_0)}x,t). \tag{4}$$

Such conditions for kernel operators defined on the semi-axis first appeared in [20].

Remark 1 It can be checked easily that if $\hat{k}\in\hat{V}_\lambda$, then $v\hat{k}\in\hat{V}_\lambda$, where v is a weight on $\hat{E}$.

Example 1 *Let $G=\mathbb{R}^n$ and let λ be a number greater than 1. Suppose that $r(x)=|x|$, $\delta_t x = tx$, $\hat{k}(x,y,t) = \big(|x-y|+t\big)^{\alpha(x)-n}$, where $\alpha(\cdot)$ is a measurable function satisfying the condition $n/\lambda<\alpha(x)<n$. Then $\hat{k}\in\hat{V}_\lambda$.*

Indeed, first observe that in this case $c_0=1$. It is easy to check that (2) and (3) are satisfied for $\hat{k}$. Let us verify that (4) holds. Denote

$$I(x) := \int\limits_{E_{|x|}\setminus E_{|x|/2}} \big(|x-y|+t\big)^{(\alpha(x)-n)\lambda'}dy.$$

(i) Let $t>|x|$. Then we have

$$I(x)\le ct^{(\alpha(x)-n)\lambda'}|x|^n \le c\big(t+|x|\big)^{(\alpha(x)-n)\lambda'}|x|^n \le c\hat{k}^{\lambda'}(x,x/2,t)|x|^n.$$

(ii) Let now $t \le |x|$. Then

$$\begin{aligned} I(x) &\le \int\limits_{E_{|x|}} |x-y|^{(\alpha(x)-n)\lambda'} dy \le c|x|^{(\alpha(x)-n)\lambda'+n} \\ &\le c\big(t+|x|\big)^{(\alpha(x)-n)\lambda'+n} \le c\hat{k}^{\lambda'}(x, x/2, t)|x|^n. \end{aligned}$$

Finally we see that (4) holds. □

Let

$$Hf(x) = \int\limits_{E_{r(x)}} f(y)dy, \quad x \in E,$$

be the Hardy–type transform defined on a cone E.

Proposition B ([3]) *Let $1 < p \le q < \infty$. Suppose that E is a cone in a homogeneous group G. Then the operator H is bounded from $L^p(E)$ to $L^q_u(E)$ if and only if*

$$A := \sup_{s>0} \left(\int\limits_{E \setminus E_s} u(x)dx \right)^{1/q} s^{Q/p'} < \infty.$$

For the next statements we refer to [17] (see Sec. 1.3.2) in the case of $1 \le q < p < \infty$, and [28] for $0 < q < 1 < p < \infty$.

Proposition C *Let $0 < q < p < \infty$ and let $p > 1$. Suppose that $w^{1-p'}$ is locally integrable on $\mathbb{R}_+$. Then the inequality*

$$\left(\int\limits_0^\infty v(x) \left(\int\limits_0^x f(t)dt \right)^q dx \right)^{1/q} \le c \left(\int\limits_0^\infty f^p(x)w(x)dx \right)^{1/p}, \quad f \ge 0$$

holds if and only if

$$\left(\int\limits_0^\infty \left[\left(\int\limits_t^\infty v(x)dx \right) \left(\int\limits_0^t w^{1-p'}(x)dx \right)^{q-1} \right]^{p/(p-q)} w^{1-p'}(t)dt \right)^{(p-q)/(pq)} < \infty.$$

The next lemma is well known (see [2] and [14], Sections 5.3 and 5.4), which is formulated here for the special case.

Proposition D *Let $0 < q < \infty$, $1 < p < \infty$, and $q < p$. Suppose that v and w are almost everywhere positive functions defined on $\hat{E}$ and E, respectively. If the kernel operator*

$$A_E f(x,t) = \int\limits_E a(x,y,t)f(y)dy, \quad (x,t) \in \hat{E}$$

is bounded from $L^p_w(E)$ *to* $L^q_v(\hat{E})$, *then* A_E *is compact.*

Now we prove the next statement.

Lemma 1 *Let* $1 < p \le q < \infty$, v *be a weight on* $\hat{E}$. *Then the 2-weight inequality*

$$\int\limits_{\hat{E}} v(x,t)\Big(\int\limits_{E_{r(x)}} f(y)dy\Big)^q dxdt\Big)^{1/q} \le c\Big(\int\limits_E w(f(x))^p dx\Big)^{1/p}, \quad f \ge 0,$$

holds if and only if

$$\sup_{s>0}\Big(\int\limits_{E\setminus E_s}\int\limits_0^\infty v(x,t)dtdx\Big)^{1/q} s^{Q/p'} < \infty. \tag{5}$$

Proof Necessity follows immediately by taking test functions $f(y) = \chi_{E_s}(y)$ in the weighted inequality.

Let us denote

$$V(x) := \int\limits_0^\infty v(x,t)dt.$$

For sufficiency, observe that (5) together with Proposition B implies

$$\begin{aligned}
\|Hf\|_{L^q_v(\hat{E})} &= \Big[\int\limits_E \Big(\int\limits_0^\infty v(x,t)dt\Big)\Big(\int\limits_{E_{r(x)}} f(y)dy\Big)^q dx\Big]^{1/q} \\
&= \Big[\int\limits_E V(x)\Big(\int\limits_{E_{r(x)}} f(y)dy\Big)^q dx\Big]^{1/q} \\
&\le c\Big(\int\limits_E f^p(x)dx\Big)^{1/p}.
\end{aligned}$$

□

The next statement can be found, for example, in [10] (see Ch. 11, Section 4).

Lemma 2 *Let* $1 < p, q < \infty$ *and let* (X,μ) *and* (Y,ν) *be* σ*-finite measure spaces. If*

$$\|\|a(x,y)\|_{L^{p'}_\mu(X)}\|_{L^q_\nu(Y)} < \infty,$$

then the operator

$$Af(x) = \int\limits_X a(x,y)f(y)d\mu$$

is compact from $L^p_\mu(X)$ *to* $L^q_\nu(Y)$.

3. The main results

We begin this section with the boundedness result.

Theorem 1 *Let* $1 < p \leq q < \infty$ *and let* $\hat{k} \in \hat{V}_p$. *The following statements are then equivalent:*

(i) $\hat{K}$ *is bounded from* $L^p(E)$ *to* $L^q_v(\hat{E})$;

(ii) $$B := \sup_{s>0} \left(\int_{E \setminus E_s} \int_0^{\infty} v(x,t) \hat{k}^q(x, \delta_{1/(2c_0)}x, t) dt dx \right)^{1/q} s^{Q/p'} < \infty;$$

(iii) $$B_1 := \sup_{k \in \mathbb{Z}} \left(\int_{E_{2^{k+1}} \setminus E_{2^k}} \int_0^{\infty} v(x,t) \hat{k}^q(x, \delta_{1/(2c_0)}x, t) dt dx \right)^{1/q} 2^{kQ/p'} < \infty.$$

Proof Taking Remark 1 into account, without loss of generality we can assume that $v \equiv 1$. First we show that (ii) $\Rightarrow$ (i). Let $f \geq 0$. We have

$$\begin{aligned} \|\hat{K}f\|^q_{L^q(\hat{E})} &\leq c \int_{\hat{E}} \left(\int_{S_x} \hat{k}(x,y,t) f(y) dy \right)^q dx dt \\ &\quad + c \int_{\hat{E}} \left(\int_{F_x} \hat{k}(x,y,t) f(y) dy \right)^q dx dt \\ &=: cI_1 + cI_2. \end{aligned}$$

Lemma 1 and the condition $\hat{k} \in \hat{V}_p$ yield that

$$\begin{aligned} I_1 &\leq c \int_{\hat{E}} \hat{k}^q(x, \delta_{1/(2c_0)}x, t) \left(\int_{E_{r(x)}} f(y) dy \right)^q dx dt \\ &\leq cB^q \left(\int_E f^p(y) dy \right)^{q/p}. \end{aligned}$$

Applying Hölder's inequality and the condition $\hat{k} \in \hat{V}_p$, we find that

$$\begin{aligned} I_2 &\leq \int_{\hat{E}} \left(\int_{F_x} f^p(y) dy \right)^{q/p} \left(\int_{F_x} \hat{k}^{p'}(x,y,t) dy \right)^{q/p'} dx dt \\ &\leq c \int_{\hat{E}} \hat{k}^q(x, \delta_{1/(2c_0)}x, t) \big(r(x)\big)^{Qq/p'} \left(\int_{E_{r(x)}} f^p(y) dy \right)^{q/p} dx dt \end{aligned} \tag{6}$$

$$\leq \quad c\sum_{k\in\mathbb{Z}}\Big(\int\limits_{E_{2^{k+1}}\setminus E_{2^k}}\int\limits_0^\infty \hat{k}^q(x,\delta_{1/(2c_0)}x,t)dxdt\Big)$$

$$\times\Big(\int\limits_{E_{2^{k+1}}\setminus E_{2^k}} f^p(y)dy\Big)^{q/p}2^{kQq/p'}$$

$$\leq \quad cB^q\|f\|^q_{L^p(E)}.$$

Now we prove that (i) $\Rightarrow$ (iii). Let $f_k(x)=\chi_{E_{2^{k+1}}}(x)$. Then $\|f_k\|_{L^p(E)}=c2^{kQ/p}$, where c does not depend on k. Furthermore, by the condition $k\in\hat{V}_p$ (in particular, by (3)), we have

$$\begin{aligned}\|\hat{K}f\|^q_{L^q(\hat{E})} &\geq \int\limits_{E_{2^{k+1}}\setminus E_{2^k}}\int\limits_0^\infty\Big(\int\limits_{F_x}k(x,y,t)dy\Big)^q dtdx\\ &\geq c\int\limits_{E_{2^{k+1}}\setminus E_{2^k}}\int\limits_0^\infty k^q(x,\delta_{1/(2c_0)}x,t)\big(r(x)\big)^{Qq}dtdx\\ &\leq c\Big(\int\limits_{E_{2^{k+1}}\setminus E_{2^k}}\int\limits_0^\infty k^q(x,\delta_{1/(2c_0)}x,t)dtdx\Big)2^{kQq}.\end{aligned}$$

Hence, we conclude that (i) implies (iii).

To prove the implication (iii) $\Rightarrow$ (ii), we take $s>0$. Then $s\in[2^m,2^{m+1})$ for some integer m. Then

$$\begin{aligned}&\Big(\int\limits_{E\setminus E_s}\int\limits_0^\infty k^q(x,\delta_{1/(2c_0)}x,t)dtdx\Big)s^{Qq/p'}\\ &\leq c\Big(\int\limits_{E\setminus E_{2^m}}\int\limits_0^\infty k^q(x,\delta_{1/(2c_0)}x,t)dtdx\Big)2^{mQq/p'}\\ &= c\sum_{k=m}^\infty\Big(\int\limits_{E_{2^{k+1}}\setminus E_{2^k}}\int\limits_0^\infty k^q(x,\delta_{1/(2c_0)}x,t)dtdx\Big)2^{mQq/p'}\\ &\leq cB_1^q2^{mQq/p'}\sum_{k=m}^\infty 2^{-kQq/p'}\leq cB_1^q.\end{aligned}$$

Hence, $B\leq B_1$. □

The compactness result reads as follows:

Theorem 2 *Let $1 < p \le q < \infty$ and let $\hat{k} \in \hat{V}_p$. Then the following statements are equivalent.*

(i) $\hat{K}$ *is compact from $L^p(E)$ to $L^q_v(\hat{E})$;*

(ii) $B < \infty$ *and* $\lim\limits_{s\to 0} B(s) = \lim\limits_{s\to\infty} B(s) = 0,$ *where B is defined in Theorem* 1 *and*

$$B(s) := \left(\int\limits_{E\setminus E_s} \int\limits_0^\infty v(x,t)\hat{k}^q(x,\delta_{1/(2c_0)}x,t)dtdx \right)^{1/q} s^{Q/p'};$$

(iii) $B_1 < \infty$ *and* $\lim\limits_{k\to -\infty} B_1(k) = \lim\limits_{k\to +\infty} B_1(k) = 0,$ *where B_1 is defined in Theorem* 1 *and*

$$B_1(k) = \left(\int\limits_{E_{2^{k+1}}\setminus E_{2^k}} \int\limits_0^\infty v(x,t)\hat{k}^q(x,\delta_{1/(2c_0)}x,t)dtdx \right)^{1/q} 2^{kQ/p'}.$$

Proof Due to Remark 1 we assume that $v \equiv 1$. Let us first we show that (ii) $\Rightarrow$ (i). Denoting $\hat{E}_t = E_t \times \mathbb{R}_+$ we have that

$$\begin{aligned}
\hat{K}f(x,t) &= \chi_{\hat{E}_a}(x,t)\hat{K}f(x,t) + \chi_{\hat{E}_b\setminus\hat{E}_a}(x,t)\hat{K}f(x,t) \\
&\quad + \chi_{\hat{E}\setminus\hat{E}_b}(x,t)\hat{K}(f\chi_{\hat{E}_{b/(2c_0)}})(x,t) + \chi_{\hat{E}\setminus\hat{E}_b}(x,t)\hat{K}(f\chi_{\hat{E}\setminus\hat{E}_{b/(2c_0)}})(x,t) \\
&=: \hat{K}_1f(x,t) + \hat{K}_2f(x,t) + \hat{K}_3f(x,t) + \hat{K}_4f(x,t),
\end{aligned}$$

where $0 < a < b < \infty$. It is obvious that

$$\hat{K}_2f(x,t) = \int\limits_E k^*(x,y,t)f(y)dy,$$

where $k^*(x,y,t) = \chi_{\hat{E}_b\setminus\hat{E}_a}(x,t)\chi_{E_{r(x)}}(y)k(x,y,t)$. Now observe that the condition $\hat{k} \in \hat{V}_p$ yields

$$\begin{aligned}
S &:= \int\limits_{\hat{E}} \left(\int\limits_E \big(k^*(x,y,t)\big)^{p'} dy \right)^{q/p'} dxdt \\
&= \int\limits_{\hat{E}_b\setminus\hat{E}_a} \left(\int\limits_{E_{r(x)}} \big(\hat{k}(x,y,t)\big)^{p'} dy \right)^{q/p'} dxdt \\
&\le c \int\limits_{\hat{E}_b\setminus\hat{E}_a} \left(\int\limits_{E_{r(x)/2}} \big(\hat{k}(x,y,t)\big)^{p'} dy \right)^{q/p'} dxdt \\
&\quad + c \int\limits_{\hat{E}_b\setminus\hat{E}_a} \left(\int\limits_{E_{r(x)}\setminus E_{r(x)/2}} \big(\hat{k}(x,y,t)\big)^{p'} dy \right)^{q/p'} dxdt \\
&\le c \int\limits_{\hat{E}_b\setminus\hat{E}_a} \hat{k}^q(x,\delta_{1/(2c_0)}x,t)\big(r(x)\big)^{Qq/p'} dxdt \\
&\le cb^{Qq/p'} \int\limits_{\hat{E}_b\setminus\hat{E}_a} \hat{k}^q(x,\delta_{1/(2c_0)}x,t)dxdt \ < \ \infty.
\end{aligned}$$

Hence $S < \infty$ and, consequently, by Lemma 2 we have that $\hat{K}_2$ is compact for every a and b. In a similar manner we conclude that $\hat{K}_3$ is also compact. Furthermore, taking into account arguments used in the proof of Theorem 1, we find that

$$\begin{aligned} \|\hat{K}_2\| &\leq cB^{(a)} := c\sup_{s\leq a}\Bigg(\int\limits_{\hat{E}_a\setminus\hat{E}_s} \hat{k}^q(x,\delta_{1/(2c_0)}x,t)dxdt\Bigg)^{1/q} s^{Q/p'}; \\ \|\hat{K}_3\| &\leq cB_{(b)} := c\sup_{s\geq b}\Bigg(\int\limits_{\hat{E}\setminus\hat{E}_s} \hat{k}^q(x,\delta_{1/(2c_0)}x,t)dxdt\Bigg)^{1/q} \big(s^Q - b^Q\big)^{1/p'}. \end{aligned}$$

Hence,

$$\|\hat{K} - \hat{K}_1 - \hat{K}_4\| \leq \|\hat{K}_2\| + \|\hat{K}_3\| \leq c\big(B^{(a)} + B_{(b)}\big) \to 0$$

as $a \to 0$ and $b \to \infty$ because $\lim\limits_{t\to 0} B(t) = \lim\limits_{t\to\infty} B(t) = 0$.

The implication (iii) $\Rightarrow$ (ii) follows in the same way as in the case of the implication (iii) $\Rightarrow$ (ii) in the proof of Theorem 1; therefore, we omit the details.

Now we prove that (i) $\Rightarrow$ (iii). Let us take $f_j(y) = \chi_{\hat{E}_{2^{j+1}}\setminus\hat{E}_{2^{j-1}/c_0}}(y)2^{-jQ/p}$. Then for $\phi \in L^p(E)$, we have

$$\Big|\int\limits_E f_j(y)\phi(y)dy\Big| \leq \Bigg(\int\limits_{E_{2^{j+1}}\setminus E_{2^{j-1}/c_0}} |\phi(y)|^{p'}dy\Bigg)^{1/p'} \longrightarrow 0$$

as $j \to -\infty$ or $j \to +\infty$. On the other hand, condition (3) implies

$$\begin{aligned} \|\hat{K}f_j\|_{L^q(\hat{E})} &\geq \Bigg(\int\limits_{\hat{E}_{2^{j+1}}\setminus\hat{E}_{2^j}} \big(\hat{K}f_j(x,t)\big)^q dxdt\Bigg)^{1/q} \\ &\geq c\Bigg[\int\limits_{\hat{E}_{2^{j+1}}\setminus\hat{E}_{2^j}} \hat{k}^q(x,\delta_{1/(2c_0)}x,t)\Bigg(\int\limits_{F_x} f_j(y)dy\Bigg)^q dxdt\Bigg]^{1/q} \\ &\geq c\Bigg(\int\limits_{\hat{E}_{2^{j+1}}\setminus\hat{E}_{2^j}} \hat{k}^q(x,\delta_{1/(2c_0)}x,t)2^{jQq/p'}\big(r(x)\big)^{Qq}dxdt\Bigg)^{1/q} \\ &\geq c\Bigg(\int\limits_{\hat{E}_{2^{j+1}}\setminus\hat{E}_{2^j}} \hat{k}^q(x,\delta_{1/(2c_0)}x,t)\Bigg)^{1/q} 2^{jQq/p} = cB(j). \end{aligned}$$

By virtue of the fact that the compact operator maps weakly convergent sequence into a strongly convergent one, we conclude that (i) implies (iii). □

Let us now consider the case $q < p$.

Theorem 3 *Let $0 < q < p < \infty$ and let $p > 1$. Suppose that $k \in \hat{V}_p$. Then the following statements are equivalent.*

(i) $\hat{K}$ *is bounded from* $L^p(E)$ *to* $L^q_v(\hat{E})$*;*

(ii) $\hat{K}$ *is compact from* $L^p(E)$ *to* $L^q_v(\hat{E})$*;*

(iii)

$$D := \left[\int\limits_E \left(\int\limits_{\hat{E}\setminus\hat{E}_{r(x)}} v(y,t) k^q(y, \delta_{1/(2c_0)} y, t) dy dt \right)^{\frac{p}{p-q}} \big(r(x)\big)^{\frac{Qp(q-1)}{p-q}} dx \right]^{\frac{p-q}{pq}} < \infty.$$

Proof Due to Remark 1, without loss of generality we assume that $v \equiv 1$. Let us prove that the implication (iii) $\Rightarrow$ (i) holds. Let $f \geq 0$. Keeping the notation of the proof of Theorem 1 and taking Proposition A into account, we see that

$$\begin{aligned} I_1 &\leq c \int\limits_{\hat{E}} \hat{k}^q(x, \delta_{1/(2c_0)} x, t) \Big(\int\limits_{S_x} f(y) dy \Big)^q dx \\ &= c \int\limits_E \Big(\int\limits_0^\infty \hat{k}^q(x, \delta_{1/(2c_0)} x, t) dt \Big) \Big(\int\limits_{S_x} f(y) dy \Big)^q dx \\ &= c \int\limits_E \overline{v}(x) \Big(\int\limits_{S_x} f(y) dy \Big)^q dx \\ &= c \int\limits_0^\infty s^{Q-1} \Big[\int\limits_A \overline{v}(\delta_s \overline{x}) d\sigma(\overline{x}) \Big] \Big[\int\limits_0^{s/2c_0} \tau^{Q-1} \Big(\int\limits_A f(\delta_\tau \overline{y}) d\sigma(\overline{y}) \Big) d\tau \Big]^q ds \\ &\leq c \int\limits_0^\infty \tilde{v}(s) \Big(\int_0^s F(\tau) d\tau \Big)^q ds, \end{aligned}$$

where

$$\begin{aligned} \overline{v}(x) &:= \int\limits_0^\infty \hat{k}^q(x, \delta_{1/(2c_0)} x, t) dt; \\ \tilde{v}(s) &:= s^{Q-1} \int\limits_A \overline{v}(\delta_s \overline{x}) d\sigma(\overline{x}); \\ F(\tau) &:= \tau^{Q-1} \int\limits_A f(\delta_\tau \overline{y}) d\sigma(\overline{y}). \end{aligned}$$

Now observe that

$$\begin{aligned} D &= \Big[\int\limits_0^\infty s^{Q-1} \Big(\int\limits_{E\setminus E_s} \int\limits_0^\infty \hat{k}^q(y, \delta_{1/(2c_0)} y, t) dt dy \Big)^{p/(p-q)} s^{Qp(q-1)/(p-q)} ds \Big]^{(p-q)/(pq)} \\ &= \Big[\int\limits_0^\infty s^{Qp(q-1)/(p-q)+Q-1} \Big(\int\limits_{E\setminus E_s} \overline{v}(y) dy \Big)^{p/(p-q)} ds \Big]^{(p-q)/(pq)} \end{aligned} \tag{7}$$

$$
\begin{aligned}
&= \Big[\int_0^\infty s^{Qp(q-1)/(p-q)+Q-1}\Big(\int_s^\infty \tilde{v}(s)ds\Big)^{p/(p-q)} ds\Big]^{(p-q)/(pq)} \\
&= c\Big[\int_0^\infty \Big(\int_s^\infty \tilde{v}(s)ds\Big)^{p/(p-q)} \Big(\int_0^s \tau^{(Q-1)(1-p)(1-p')}d\tau\Big)^{p(q-1)/(p-q)} \\
&\qquad \times s^{(Q-1)(1-p)(1-p')}ds\Big]^{(p-q)/(pq)}.
\end{aligned}
$$

Consequently, Proposition C, Hölder's inequality, and Proposition A imply

$$
\begin{aligned}
I_1 &\leq c\Big(\int_0^\infty s^{(Q-1)(1-p)}\big(F(s)\big)^p ds\Big)^{q/p} \\
&= c\Big[\int_0^\infty s^{(Q-1)(1-p)+(Q-1)p}\Big(\int_A f(\delta_s\overline{x})d\sigma(\overline{x})\Big)^p ds\Big]^{q/p} \\
&\leq c\Big[\int_0^\infty s^{Q-1}\Big(\int_A f^p(\delta_s\overline{x})d\sigma(\overline{x})\Big)ds\Big]^{q/p} \\
&= c\|f\|^q_{L^p(E)}.
\end{aligned}
$$

Furthermore, due to Hölder's inequality and the condition $\hat{k} \in \hat{V}_p$ we find that

$$
\begin{aligned}
I_2 &\leq \int_{\hat{E}} \Big(\int_{F_x} f^p(y)dy\Big)^{q/p}\Big(\int_{F_x} \hat{k}^{p'}(x,y,t)dy\Big)^{q/p'} dxdt \\
&\leq c\int_{\hat{E}} \Big(\int_{F_x} f^p(y)dy\Big)^{q/p} \hat{k}^q(x,\delta_{1/(2c_0)}x,t)\big(r(x)\big)^{Qq/p'}dxdt \\
&\leq c\sum_{k\in\mathbb{Z}}\Big(\int_{E_{2^{k+1}}\setminus E_{2^k}} \int_0^\infty \hat{k}^q(x,\delta_{1/(2c_0)}x,t)\big(r(x)\big)^{Qq/p'}dtdx\Big) \\
&\qquad \times\Big(\int_{E_{2^{k+1}}\setminus E_{2^{k-1}/c_0}} f^p(y)dy\Big)^{q/p} \\
&\leq c\Big[\sum_{k\in\mathbb{Z}}\int_{E_{2^{k+1}}\setminus E_{2^{k-1}/c_0}} f^p(y)dy\Big]^{q/p} \\
&\qquad \times\Big[\sum_{k\in\mathbb{Z}}\Big(\int_{E_{2^{k+1}}\setminus E_{2^k}} \overline{v}(x)\big(r(x)\big)^{Qq/p'}dx\Big)^{p/(p-q)}\Big]^{(p-q)/p} \\
&=: c\|f\|^q_{L^p(E)}\big(\overline{D}\big)^q,
\end{aligned}
$$

where

$$\overline{D} := \left[\sum_{k\in\mathbb{Z}}\left(\int\limits_{E_{2^{k+1}}\setminus E_{2^k}} \overline{v}(x)\big(r(x)\big)^{Qq/p'}dx\right)^{p/(p-q)}\right]^{(p-q)/pq}.$$

Furthermore, it is clear that

$$\begin{aligned}
\big(\overline{D}\big)^{pq/(p-q)} &\leq c\sum_{k\in\mathbb{Z}} 2^{kQq(p-1)/(p-q)}\left(\int\limits_{E_{2^{k+1}}\setminus E_{2^k}} \overline{v}(x)dx\right)^{p/(p-q)} \\
&\leq c\sum_{k\in\mathbb{Z}}\int\limits_{E_{2^k}\setminus E_{2^{k-1}}} \big(r(y)\big)^{kQp(q-1)/(p-q)}\left(\int\limits_{E\setminus E_{r(y)}} \overline{v}(x)dx\right)^{p/(p-q)} dy \\
&= \int\limits_{E}\big(r(y)\big)^{kQp(q-1)/(p-q)} \\
&\quad\times\left(\int\limits_{E\setminus E_{r(y)}}\int\limits_0^\infty k^q(x,\delta_{1/(2c_0)}x,t)dtdx\right)^{p/(p-q)}dy \\
&= cD^{pq/(p-q)} < \infty.
\end{aligned}$$

Now we show that (i) $\Rightarrow$ (iii). Let $n\in\mathbb{Z},\ n\geq 2$, and let

$$\overline{v}_n(x) := \left(\int\limits_0^\infty \hat{k}^q(x,\delta_{1/(2c_0)}x,t)dt\right)\chi_{E_n\setminus E_{1/n}}(x).$$

Suppose that

$$f_n(x) := \left(\int\limits_{E\setminus E_{r(x)}} \overline{v}_n(y)dy\right)^{1/(p-q)}\big(r(x)\big)^{Q(p-1)/(p-q)}.$$

Then

$$\begin{aligned}
\|f_n\|_{L^p(E)} &= \left[\int\limits_E\left(\int\limits_{E\setminus E_{r(x)}}\overline{v}_n(y)dy\right)^{p/(p-q)}\big(r(x)\big)^{Qp(q-1)/(p-q)}dx\right]^{1/p} \\
&= \left[\int\limits_E \chi_{E_n\setminus E_{1/n}}(x)\left(\int\limits_{E\setminus E_{r(x)}}\int\limits_0^\infty \hat{k}^q(x,\delta_{1/(2c_0)}x,t)dtdy\right)^{p/(p-q)}\right. \\
&\quad\left.\times\big(r(x)\big)^{Qp(q-1)/(p-q)}dx\right]^{1/p} < \infty.
\end{aligned}$$

Furthermore, by the condition $\hat{k} \in \hat{V}_p$ (in particular, by (3)), we have that

$$\begin{aligned}
\|\hat{K}f\|_{L^q_v(\hat{E})} &\geq \Bigg[\int\limits_{\hat{E}}\Bigg(\int\limits_{F_x} f_n(y)\hat{k}(x,y,t)dy\Bigg)^q dxdt\Bigg]^{1/q} \\
&\geq c\Bigg[\int\limits_{\hat{E}} k^q(x,\delta_{1/(2c_0)}x,t) \\
&\quad\times\Bigg(\int\limits_{E\setminus E_{r(x)}} \overline{v}_n(y)dy\Bigg)^{q/(p-q)}\big(r(x)\big)^{Qq(p-1)/(p-q)}dxdt\Bigg]^{1/q} \\
&= c\Bigg[\int\limits_{E}\Bigg(\int\limits_0^\infty \hat{k}(x,\delta_{1/(2c_0)}x,t)dt\Bigg) \\
&\quad\times\Bigg(\int\limits_{E\setminus E_{r(x)}} \overline{v}_n(y)dy\Bigg)^{q/(p-q)}\big(r(x)\big)^{Qq(p-1)/(p-q)}dx\Bigg]^{1/q} \\
&\geq c\Bigg[\int\limits_{E}\overline{v}_n(x)\Bigg(\int\limits_{E\setminus E_{r(x)}} \overline{v}_n(y)dy\Bigg)^{q/(p-q)}\big(r(x)\big)^{Qq(p-1)/(p-q)}dx\Bigg]^{1/q} \\
&= c\Bigg[\int\limits_0^\infty s^{Q-1}\Bigg(\int\limits_{E\setminus E_s} \overline{v}_n(y)dy\Bigg)^{q/(p-q)} \\
&\quad\times\Bigg(\int\limits_A \overline{v}_n(\delta_s\overline{x})d\sigma(\overline{x})\Bigg)s^{Qq(p-1)/(p-q)}dx\Bigg]^{1/q} \\
&= c\Bigg[\int\limits_0^\infty\Bigg(\int\limits_s^\infty \tau^{Q-1}\int\limits_A \overline{v}_n(\delta_\tau\overline{y})d\sigma(\overline{y})d\tau\Bigg)^{q/(p-q)}s^{Q-1} \\
&\quad\times\Bigg(\int\limits_A \overline{v}_n(\delta_s\overline{x})d\sigma(\overline{x})\Bigg)s^{Qq(p-1)/(p-q)}ds\Bigg]^{1/q} \\
&= c\Bigg[\int\limits_0^\infty\Bigg(\int\limits_s^\infty \tau^{Q-1}\int\limits_A \overline{v}_n(\delta_\tau\overline{y})d\sigma(\overline{y})d\tau\Bigg)^{p/(p-q)}s^{Qq(p-1)/(p-q)-1}ds\Bigg]^{1/q} \\
&= c\Bigg[\int\limits_E \big(r(x)\big)^{Qq(p-1)/(p-q)-Q}\Bigg(\int\limits_{E\setminus E_{r(x)}} \overline{v}_n(y)dy\Bigg)^{p/(p-q)}dx\Bigg]^{1/q} \\
&= c\Bigg[\int\limits_E \big(r(x)\big)^{Qp(q-1)/(p-q)}\Bigg(\int\limits_{E\setminus E_{r(x)}} \overline{v}_n(y)dy\Bigg)^{p/(p-q)}dx\Bigg]^{1/q}.
\end{aligned}$$

Hence, the bondedness of $\hat{K}$ implies that

$$\left[\int_E \big(r(x)\big)^{Qp(q-1)/(p-q)}\left(\int_{E\setminus E_{r(x)}} \tilde{v}_n(y)dy\right)^{p/(p-q)} dx\right]^{(p-q)/(pq)} \le c.$$

Passing to the limit as $n\to\infty$, we conclude that $D<\infty$.

Finally, Proposition D implies (i) $\Leftrightarrow$ (ii). □

Remark 2 Taking Remark 1 into account, it is possible to formulate the main results of this paper in the equivalent form in terms only of the kernel $\hat{k}$.

Remark 3 Suppose that

$$\mathcal{K}f(x)=\int_{E_{r(x)}} k(x,y)f(y)dy, \quad x\in E,$$

where

$$k(x,y)=\left(\int_0^\infty \hat{k}(x,y,t)^q dt\right)^{1/q}. \tag{8}$$

Definition A [3] Let k be a positive function on $\{(x,y)\in E\times E: r(y)<r(x)\}$ and let $1<\lambda<\infty$. We say that $k\in V_\lambda$, if

(a) there exist positive constants c_1, c_2, and c_3 such that

$$k(x,y)\le c_1 k(x,\delta_{1/(2c_0)}x) \tag{9}$$

for all $x,y\in E$ with $r(y)<r(x)/(2c_0)$;

(b)

$$k(x,y)\ge c_2 k(x,\delta_{1/(2c_0)}x) \tag{10}$$

for all $x,y\in E$ with $r(x)/(2c_0)<r(y)<r(x)$;

(c)

$$\int_{F_x} k^{\lambda'}(x,y)dy\le c_3 r^Q(x)k^{\lambda'}(x,\delta_{1/(2c_0)}x), \tag{11}$$

for all $x\in E$.

Using Minkowski integral inequality and taking into account the main results of this paper and [3], it can be checked that if $\hat{k}\in\hat{V}_p$ and $k\in V_p$, where k is defined by (8), then the boundedness/compactness of $\mathcal{K}$ from $L^p(E)$ to $L^q(E)$ implies the boundedness/compactness of $\hat{K}$ from $L^p(E)$ to $L^q(\hat{E})$. Furthermore, if $q\le p'$, and $\hat{k}\in\hat{V}_p$, then $k\in V_p$, where k is defined by (8). Indeed, let $\hat{k}\in\hat{V}_p$. Then (9) and (10) are obvious for k, while Minkowski integral inequality yields

$$\int_{F_x} k(x,y)^{p'}dy\le\left(\int_0^\infty\left(\int_{F_x}\hat{k}(x,y,t)dy\right)^{q/p'}dt\right)^{p'/q}\le c_3 r(x)^Q k(x,\delta_{1/(2c_0)}x,t)^{p'}.$$

Consequently, for this p and q, the results of this paper follow from the results of [3].

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