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Geometry of almost Cliffordian manifolds: classes of subordinated connections

Jaroslav HRDINA*, **Petr VAŠÍK**
Institute of Mathematics, Faculty of Mechanical Engineering, Brno University of Technology,
Brno, Czech Republic

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Abstract: An almost Clifford and an almost Cliffordian manifold is a G–structure based on the definition of Clifford algebras. An almost Clifford manifold based on $\mathcal{O} := \mathcal{C}l(s,t)$ is given by a reduction of the structure group $GL(km,\mathbb{R})$ to $GL(m,\mathcal{O})$, where $k = 2^{s+t}$ and $m \in \mathbb{N}$. An almost Cliffordian manifold is given by a reduction of the structure group to $GL(m,\mathcal{O})GL(1,\mathcal{O})$. We prove that an almost Clifford manifold based on $\mathcal{O}$ is such that there exists a unique subordinated connection, while the case of an almost Cliffordian manifold based on $\mathcal{O}$ is more rich. A class of distinguished connections in this case is described explicitly.

Key words: Clifford algebra, affinor structure, G–structure, linear connection, planar curves

1. Introduction

First, let us recall some facts about G-structures and their prolongations. There are 2 definitions of G–structures. The first reads that a G–structure is a principal bundle $P \to M$ with structure group G together with a soldering form θ. The second reads that it is a reduction of the frame bundle P^1M to the Lie group G. In the latter case, the soldering form θ is induced from a canonical soldering form on the frame bundle.

Now let $\mathfrak{g} \subset \wedge^2\mathbb{V}$ be the Lie algebra of the Lie group G and let $\mathbb{V}$ be a vector space. From the structure theory we know that there is a G-invariant complement $\mathcal{D}$ of $\partial(\mathfrak{g} \otimes \mathbb{V}^*)$ in $\mathbb{V} \otimes \wedge^2\mathbb{V}^*$, where ∂ is the operator of alternation; see [6]. Let us recall that the torsion of a linear connection lies in the space $\mathbb{V} \otimes \wedge^2\mathbb{V}^*$.

The almost Clifford and almost Cliffordian structures are G-structures based on Clifford algebras. The 2 most important examples are an almost hypercomplex geometry and an almost quaternionic geometry, which are based on Clifford algebra $\mathcal{C}l(0,2)$. An important geometric property of almost hypercomplex structures reads that there is no nontrivial G-invariant subspace $\mathcal{D}$ in $\mathbb{V}\otimes\wedge^2\mathbb{V}^*$, because the first prolongation $\mathfrak{g}^{(1)}$ of the Lie algebra $\mathfrak{g}$ vanishes. For almost quaternionic structure, the situation is more complicated, because $\mathfrak{g}^{(1)} = \mathbb{V}^*$; see [1]. For these reasons, in the latter case, there exists a distinguished class of linear connections compatible with the structure. Our goal is to describe some of these connections for almost Cliffordian G–structures based on Clifford algebras $\mathcal{C}l(s,t)$ generally.

*Correspondence: hrdina@fme.vutbr.cz
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2. Clifford algebras

The pair $(\mathbb{V}, Q)$, where $\mathbb{V}$ is a vector space of dimension n and Q is a quadratic form, is called a quadratic vector space. To define Clifford algebras in coordinates, we start by choosing a basis $e_i,\ i=1,...,n$ of $\mathbb{V}$ and by $I_i,\ i=1,...,n$ we denote the image of e_i under the inclusion $\mathbb{V} \hookrightarrow \mathcal{C}l(\mathbb{V}, Q)$. Then the elements I_i satisfy the relation

$$I_j I_k + I_k I_j = 2B_{jk}1,$$

where 1 is the unit in the Clifford algebra and B is a bilinear form obtained from Q by polarization. In a quadratic finite dimensional real vector space it is always possible to choose a basis e_i for which the matrix of the bilinear form B has the form

$$\begin{pmatrix} O_r & & \\ & E_s & \\ & & -E_t \end{pmatrix},\ r+s+t=n,$$

where E_k denotes the $k \times k$ identity matrix and O_k the $k \times k$ zero matrix. Let us restrict to the case $r=0$, whence B is nondegenerate. Then B defines the inner product of signature (s,t) and we call the corresponding Clifford algebra $\mathcal{C}l(s,t)$. For example, $\mathcal{C}l(0,2)$ is generated by I_1, I_2, satisfying $I_1^2 = I_2^2 = -E$ with $I_1 I_2 = -I_2 I_1$, i.e. $\mathcal{C}l(0,2)$ is isomorphic to $\mathbb{H}$.

Following the classification of the Clifford algebra, Bott periodicity reads that $\mathcal{C}l(0,n) \cong \mathcal{C}l(0,q) \otimes \mathbb{R}(16p)$, where $n = 8p+q$, $q = 0,\dots,7$ and $\mathbb{R}(N)$ denotes the $N \times N$ matrices with coefficients in $\mathbb{R}$. To determine explicit matrix representations we use the periodicity conditions

$$\mathcal{C}l(0,n) \cong \mathcal{C}l(n-2,0) \otimes \mathcal{C}l(0,2),$$
$$\mathcal{C}l(n,0) \cong \mathcal{C}l(0,n-2) \otimes \mathcal{C}l(2,0),$$
$$\mathcal{C}l(s,t) \cong \mathcal{C}l(s-1,t-1) \otimes \mathcal{C}l(1,1),$$

together with the explicit matrix representations of $\mathcal{C}l(0,2)$, $\mathcal{C}l(2,0)$, $\mathcal{C}l(1,0)$, and $\mathcal{C}l(0,1)$. For example

$$\mathcal{C}l(3,0) \cong \mathcal{C}l(0,1) \otimes \mathcal{C}l(2,0),$$

where the matrix representation of $\mathcal{C}l(0,1)$ on $\mathbb{R}^{2m}$ is given by the matrices

$$\begin{pmatrix} E_m & 0 \\ 0 & E_m \end{pmatrix} \text{ and } \begin{pmatrix} 0 & E_m \\ -E_m & 0 \end{pmatrix}$$

and the matrix representation of $\mathcal{C}l(2,0)$ on $\mathbb{R}^{4m}$ is given by the matrices

$$E_{4m}, I_1 = \begin{pmatrix} 0 & -E_m & 0 & 0 \\ -E_m & 0 & 0 & 0 \\ 0 & 0 & 0 & E_m \\ 0 & 0 & E_m & 0 \end{pmatrix}, I_2 = \begin{pmatrix} 0 & 0 & E_m & 0 \\ 0 & 0 & 0 & E_m \\ E_m & 0 & 0 & 0 \\ 0 & E_m & 0 & 0 \end{pmatrix},$$

$$I_3 = I_1 I_2 = \begin{pmatrix} 0 & 0 & 0 & -E_m \\ 0 & 0 & -E_m & 0 \\ 0 & E_m & 0 & 0 \\ E_m & 0 & 0 & 0 \end{pmatrix},$$

where E_p is an identity matrix $p \times p$. Now, the matrix representation of $\mathcal{C}l(3,0)$ on $\mathbb{R}^{8m}$ is given by

$$\begin{pmatrix} E_{4m} & 0 \\ 0 & E_{4m} \end{pmatrix}, \begin{pmatrix} I_1 & 0 \\ 0 & I_1 \end{pmatrix}, \begin{pmatrix} I_2 & 0 \\ 0 & I_2 \end{pmatrix}, \begin{pmatrix} I_3 & 0 \\ 0 & I_3 \end{pmatrix},$$

$$\begin{pmatrix} 0 & E_{4m} \\ -E_{4m} & 0 \end{pmatrix}, \begin{pmatrix} 0 & I_1 \\ -I_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & I_3 \\ -I_3 & 0 \end{pmatrix};$$

for an explicit description see [5].

We now focus on the algebra $\mathcal{O} := \mathcal{C}l(s,t)$, i.e. the algebra generated by elements $I_i, i = 1, \dots, t$ (called complex units), and elements $J_j,\ j = 1, \dots, s$ (called product units), which are anticommuting, i.e. $I_i^2 = -E$, $J_j^2 = E$ and $K_iK_j = -K_jK_i,\ i \neq j$, where $K \in \{I_i, J_j\}$. On the other hand, this algebra is generated by elements $F_i,\ i = 1, \dots, k$ as a vector space. We chose a basis $F_i,\ i = 1, \dots, k$, such that $F_1 = E$, $F_i = I_{i-1}$ for $i = 2, \dots, t+1$, $F_j = J_{j-t-1}$ for $j = t+2, \dots, s+t+1$ and by all different multiples of I_i and J_j of length $2, ..., s+t$. Let us note that both complex and product units can be found among these multiple generators.

Lemma 2.1 *Let $F_1, \dots, F_k$ denote the $k = 2^{s+t}$ elements of the matrix representation of Clifford algebra $\mathcal{C}l(s,t)$ on $\mathbb{R}^k$. Then there exists a real vector $X \in \mathbb{R}^k$ such that the dimension of a linear span $\langle F_iX | i = 1, \dots, k\rangle$ is equal to k.*

Proof Let us suppose, without loss of generality, that $F_1, \dots, F_k$ are the elements constructed by means of Bott periodicity as above. Then, by induction, we prove that the matrix $F = \sum_{i=1}^k a_iF_i,\ a_i \in \mathbb{R}$, is a square matrix that has exactly one entry a_i in each column and each row. For $\mathcal{C}l(1,0)$ and $\mathcal{C}l(0,1)$, we have

$$F = \begin{pmatrix} a_1 & a_2 \\ a_2 & a_1 \end{pmatrix}, F = \begin{pmatrix} a_1 & a_2 \\ -a_2 & a_1 \end{pmatrix},$$

respectively. For the rest of the generating cases, $\mathcal{C}l(2,0)$, $\mathcal{C}l(0,2)$, and $\mathcal{C}l(1,1)$, matrix F can be obtained in very similar way.

We now restrict to Clifford algebras of type $\mathcal{C}l(s,0),\ s > 2$, and show the induction step by means of the periodicity condition

$$\mathcal{C}l(s,0) \cong \mathcal{C}l(0,s-2) \otimes \mathcal{C}l(2,0).$$

The rest of the cases according to the Clifford algebra identification above can be proved similarly and we leave it to the reader. Let $G_i,\ i = 1, \dots, l$ denote the l elements of the matrix representation of Clifford algebra $\mathcal{C}l(0,s-2)$ with the required property, i.e. the matrix $G = \sum_{i=1}^l g_iG_i$ is a square matrix with exactly one entry g_i in each column and each row, i.e.

$$G := \begin{pmatrix} g_{\sigma_1(1)} & \cdots & g_{\sigma_1(l)} \\ \vdots & & \vdots \\ g_{\sigma_l(1)} & \cdots & g_{\sigma_l(l)} \end{pmatrix},$$

where σ_i are all permutations of $\{1, \dots, l\}$. The matrix of $\mathcal{C}l(2,0)$ is

$$H := \begin{pmatrix} a_1 & -a_2 & a_3 & -a_4 \\ -a_2 & a_1 & -a_4 & a_3 \\ a_3 & a_4 & a_1 & a_2 \\ a_4 & a_3 & a_2 & a_1 \end{pmatrix}.$$

The matrix for the representation of Clifford algebra $\mathcal{C}l(s,0)$ is then composed as follows:

$$F := \begin{pmatrix} g_{\sigma_1(1)}H & \dots & g_{\sigma_1(l)}H \\ \vdots & & \vdots \\ g_{\sigma_l(1)}H & \dots & g_{\sigma_l(l)}H \end{pmatrix}.$$

Finally, if matrix G has exactly one g_i in each column and each row, matrix F is a square matrix with exactly one $a_j g_i$, where $j = 1, \dots, 4, i = 1, \dots, l$ in each column and each row.

Now, let $F = \sum_{i=1}^{k} b_i F_i$ be a $k \times k$ matrix constructed as above and let e_i denote the standard basis of $\mathbb{R}^k$. Then the vector

$$v_i := F e_i^T$$

is the i-th column of the matrix F and thus it is composed of k different entries b_i. If the dimension of $\langle F_i X | i = 1, \dots, k \rangle$ is less then k, then the vector v has to be zero and thus all b_i have to be zero. □

Definition 2.2 *Let P^1M be a bundle of linear frames over M (the fiber bundle P^1M is a principal bundle over M with the structure group $GL(n,\mathbb{R})$). Reduction of the bundle P^1M to the subgroup $G \subset GL(n,\mathbb{R})$ is called a G-*structure*.*

Definition 2.3 *If M is an km–dimensional manifold, where $k = 2^{s+t}$ and $m \in \mathbb{N}$, then an almost Clifford manifold is given by a reduction of the structure group $GL(km,\mathbb{R})$ of the principal frame bundle of M to*

$$GL(m,\mathcal{O}) = \{A \in GL(km,\mathbb{R}) | AI_i = I_i A, AJ_j = J_j A\},$$

where $\mathcal{O}$ is an arbitrary Clifford algebra and $I_i, i = 1, \dots, t,\ I_i^2 = -E$ and $J_j,\ j = 1, \dots, s,\ J_j^2 = E$ is the set of anticommuting affinors such that the free associative unitary algebra generated by $\langle I_i, J_j, E \rangle$ is isomorphically equivalent to $\mathcal{O}$.

In particular, on the elements of this reduced bundle one can define affinors in the form of $F_1, \dots, F_k$ globally.

3. A-planar curves and morphisms

The concept of planar curves is a generalization of a geodesic on a smooth manifold equipped with certain structure. In [7] the authors proved a set of facts about structures based on 2 different affinors. Following [3, 4], a manifold equipped with an affine connection and a set of affinors $A = \{F_1, \dots, F_l\}$ is called an A-structure and a curve satisfying $\nabla_{\dot{c}}\dot{c} \in \langle F_1(\dot{c}), \dots, F_l(\dot{c}) \rangle$ is called an A-planar curve.

Definition 3.1 *Let M be a smooth manifold such that $\dim(M) = m$. Let A be a smooth ℓ-dimensional $(\ell < m)$ vector subbundle in $T^*M \otimes TM$ such that the identity affinor $E = id_{TM}$ restricted to T_xM belongs to $A_xM \subset T_x^*M \otimes T_xM$ at each point $x \in M$. We say that M is equipped with an ℓ-dimensional A–*structure.*

It is easy to see that an almost Clifford structure is not an A–structure, because the affinors in the form of $F_0, \dots, F_\ell \in A$ have to be defined only locally.

Definition 3.2 *The A–structure where A is isomorphically equivalent to a Clifford algebra $\mathcal{O}$ is called an almost Cliffordian manifold.*

The classical concept of F–planar curves defines the F*-planar* curve as the curve $c : \mathbb{R} \to M$ satisfying the condition

$$\nabla_{\dot{c}}\dot{c} \in \langle \dot{c}, F(\dot{c}) \rangle,$$

where F is an arbitrary affinor. Clearly, geodesics are F-planar curves for all affinors F, because $\nabla_{\dot{c}}\dot{c} \in \langle \dot{c} \rangle \subset \langle \dot{c}, F(\dot{c}) \rangle$.

Now, for any tangent vector $X \in T_xM$ we shall write $A_x(X)$ for the vector subspace

$$A_x(X) = \{F_i(X) | F_i \in A_xM\} \subset T_xM$$

and call it the *A–hull of the vector X*. Similarly, the *A–hull of a vector field* is a subbundle in TM obtained pointwise. For example, the A–hull of an almost quaternionic structure is

$$A_x(X) = \{aX + bI(X) + cJ(X) + dK(X) | a, b, c, d \in \mathbb{R}\}.$$

Definition 3.3 *Let M be a smooth manifold equipped with an A–structure and a linear connection ∇. A smooth curve $c : \mathbb{R} \to M$ is said to be A–*planar *if*

$$\nabla_{\dot{c}}\dot{c} \in A(\dot{c}).$$

One can easily check that the class of connections

$$[\nabla]_A = \nabla + \sum_{i=1}^{\dim A} \Upsilon_i \otimes F_i, \tag{1}$$

where Υ_i are one-forms on M, share the same class of A–planar curves, but we have to describe them more carefully for Cliffordian manifolds.

Theorem 3.4 *Let M be a smooth manifold equipped with an almost Cliffordian structure, i.e. an A–structure, where $A = \mathcal{C}l(s,t)$, $\dim(M) \geq 2^{(s+t+1)}$, and let ∇ be a linear connection such that $\nabla A = 0$. The class of connections $[\nabla]$ preserving A, sharing the same torsion and A–planar curves, is isomorphic to T^*M and the isomorphism has the following form:*

$$\Upsilon \mapsto \nabla + \sum_{i=1}^{k} \epsilon_i (\Upsilon \circ F_i) \odot F_i, \tag{2}$$

where $\langle F_1, \ldots, F_k \rangle = A$, $k = 2^{s+t}$, as a vector space, $\epsilon_i \in \{\pm 1\}$, and Υ is a one-form on M.

Proof First, let us consider the difference tensor

$$P(X,Y) = \bar{\nabla}_X(Y) - \nabla_X(Y)$$

and one can see that its value is symmetric in each tangent space because both connections share the same torsion. Since both ∇ and $\bar{\nabla}$ preserve F_i, $i = 1, \ldots, k$, the difference tensor P is Clifford linear in the second variable. By symmetry it is thus Clifford bilinear and we can proceed by induction. Let $X = \dot{c}$ and the deformation $P(X,X)$ equals to $\sum_{i=1}^{k} \Upsilon_i(X)F_i(X)$ because c is A-planar with respect to ∇ and $\bar{\nabla}$. In this case, we shall verify.

First, for $s = 1, t = 0$,

$$P(X,X) = a(X)X + b(JX)JX,$$
$$P(X,X) = J^2P(X,X) = P(JX,JX) = a(JX)JX + b(X)X.$$

The difference of the first row and the second row implies $a(X) = b(X)$ and $a(JX) = b(JX)$ because we can suppose that X, JX are linearly independent. For $s = 0, t = 1$,

$$P(X,X) = a(X)X + b(IX)IX,$$
$$-P(X,X) = I^2P(X,X) = P(IX,IX) = a(IX)IX - b(X)X.$$

The sum of the first row and the second row implies $a(X) = b(X)$ and $a(IX) = -b(IX)$ because we can suppose that X, IX are linearly independent.

Let us suppose that the property holds for a Clifford algebra $\mathcal{C}l(s,t)$, $k = 2^{s+t}$, i.e.

$$P(X,X) = \sum_{i=1}^{k} \epsilon_i(\Upsilon(F_i(X)))F_i(X),$$

where $\epsilon_i \in \{\pm 1\}$.

For $\mathcal{C}l(s,t+1)$ we have

$$P(X,X) = \sum_{i=1}^{k} \epsilon_i(\Upsilon(F_i(X)))F_i(X) + \sum_{i=1}^{k} (\xi_i(F_iS(X)))F_iS(X),$$

and

$$S^2P(X,X) = \sum_{i=1}^{k} \epsilon_i(\Upsilon(F_i(SX)))F_i(SX) + \sum_{i=1}^{k} (\xi_i(F_i(X)))F_i(X).$$

The sum of the first row and the second row implies

$$\epsilon_i\Upsilon(F_i(X)) = -\xi_i(F_iX) \text{ and } \epsilon_i\Upsilon(F_i(SX)) = -\xi_i(F_iSX),$$

because we can suppose that F_iX are linearly independent. The case of $\mathcal{C}l(s+1,t)$ is calculated in the same way.

Now, $P(X,X) = \sum_{i=1}^{k} \epsilon_i(\Upsilon(F_i(X)))F_i(X)$ and one shall compute

$$P(X,Y) = \frac{1}{2}(\sum_{i=1}^{k} \epsilon_i\Upsilon(F_i(X+Y))F_i(X+Y) - \sum_{i=1}^{k} \epsilon_i\Upsilon(F_i(X))F_i(X) - \sum_{i=1}^{k} \epsilon_i\Upsilon(F_i(Y))F_i(Y))$$

by polarization.

Assuming that vectors $F_i(X), F_i(Y)$, $i = 1, \dots, k$ are linearly independent, we compare the coefficients of X in the expansions of $P(sX, tY) = stP(X, Y)$ as above to get

$$s\Upsilon(sX + tY) - s\Upsilon(sX) = st(\Upsilon(X + Y) - \Upsilon(X)).$$

Dividing by s and putting $t = 1$ and taking the limit $s \to 0$, we conclude that $\Upsilon(X + Y) = \Upsilon(X) + \Upsilon(Y)$.

We have proven that the form Υ is linear in X and

$$(X, Y) \to \sum_{i=1}^{k} \epsilon_i(\Upsilon(F_i(X)))F_i(Y) + \sum_{i=1}^{k} \epsilon_i(\Upsilon(F_i(Y)))F_i(X)$$

is a symmetric complex bilinear map that corresponds to $P(X, Y)$ if both arguments coincide; it always agrees with P by polarization and $\bar{\nabla}$ lies in the projective equivalence class $[\nabla]$. □

4. $\mathcal{D}$–connections

Let $\mathbb{V} = \mathbb{R}^n$, $G \subset GL(\mathbb{V}) = GL(n, \mathbb{R})$ be a Lie group with Lie algebra $\mathfrak{g}$ and M be a smooth manifold of dimension n.

Definition 4.1 *The* first prolongation $\mathfrak{g}^{(1)}$ *of* $\mathfrak{g}$ *is a space of symmetric bilinear mappings* $t : \mathbb{V} \times \mathbb{V} \to \mathbb{V}$ *such that, for each fixed* $v_1 \in \mathbb{V}$, *the mapping* $v \in \mathbb{V} \mapsto t(v, v_1) \in \mathbb{V}$ *is in* $\mathfrak{g}$.

Example 4.2 *A* complex structure $(M, I), I^2 = -E$, *is a* G*-structure where* $G = GL(n, \mathbb{C})$ *with Lie algebra* $\mathfrak{g} = \{A \in \mathfrak{gl}(2n, \mathbb{R}) | AI = IA\}$. *The first prolongation* $\mathfrak{g}^{(1)}$ *is a space of symmetric bilinear mappings*

$$\mathfrak{g}^{(1)} = \{t | t : \mathbb{V} \times \mathbb{V} \to \mathbb{V}, t(IX, Y) = It(X, Y), t(Y, X) = t(X, Y)\}.$$

On the other hand, a product structure $(M, P), P^2 = E$ *is a* G*-structure where* $G = GL(n, \mathbb{R}) \oplus GL(n, \mathbb{R})$ *with Lie algebra* $\mathfrak{g} = \mathfrak{g}(n, \mathbb{R}) \oplus \mathfrak{g}(n, \mathbb{R})$. *The first prolongation* $\mathfrak{g}^{(1)}$ *is a space of symmetric bilinear mappings*

$$\mathfrak{g}^{(1)} = \{t | t : \mathbb{V}_1 \oplus \mathbb{V}_2 \times \mathbb{V}_1 \oplus \mathbb{V}_2 \to \mathbb{V}_1 \oplus \mathbb{V}_2, t(\mathbb{V}_i, \mathbb{V}_i) \in \mathbb{V}_i, t(\mathbb{V}_2, \mathbb{V}_1) = 0\}.$$

Lemma 4.3 *Let* M *be a* (km)*–dimensional Clifford manifold based on Clifford algebra* $\mathcal{O} = \mathcal{C}l(s, t)$, $k = 2^{s+t}$, $s + t > 1$, $m \in \mathbb{N}$, *i.e. a manifold equipped with* G*–structure, where*

$$G = GL(m, \mathcal{O}) = \{B \in GL(km, \mathbb{R}) | BI_i = I_iB,\ BJ_j = J_jB\},$$

and I_i *and* J_j *are algebra generators of* $\mathcal{O}$. *Then the first prolongation* $\mathfrak{g}^{(1)}$ *of Lie algebra* $\mathfrak{g}$ *of Lie group* G *vanishes.*

Proof Lie algebra $\mathfrak{g}$ of a Lie group G is of the form

$$\mathfrak{g} = \mathfrak{gl}(m, \mathcal{O}) = \{B \in \mathfrak{gl}(km, \mathbb{R}) | BI_i = I_iB,\ BJ_j = J_jB\},$$

where I_i and J_j are generators of $\mathcal{O}$, i.e. $K_{\bar{i}}K_{\bar{j}} = -K_{\bar{j}}K_{\bar{i}}$ for $K_{\bar{i}}, K_{\bar{j}} \in \{I_i, J_j\}$, $\bar{i} \neq \bar{j}$. For $t \in \mathfrak{g}^{(1)}$ and $K_{\bar{i}} \neq K_{\bar{j}}$ we have the equations

$$t(K_{\bar{i}}X, K_{\bar{j}}X) = K_{\bar{i}}K_{\bar{j}}t(X, X),$$

$$t(K_{\bar{i}}X, K_{\bar{j}}X) = t(K_{\bar{j}}X, K_{\bar{i}}X) = K_{\bar{j}}K_{\bar{i}}t(X,X) = -K_{\bar{i}}K_{\bar{j}}t(X,X),$$

which lead to $t(X,X) = 0$. Finally, from polarization,

$$t(X,Y) = \frac{1}{2}(t(X+Y, X+Y) - t(X,X) - t(Y,Y)) = 0.$$

□

Let us shortly note that Example 4.2 covers Clifford manifolds for $\mathcal{O} = \mathcal{C}l(0,1)$ and $\mathcal{O} = \mathcal{C}l(1,0)$. Next, suppose that there is a G-invariant complement $\mathcal{D}$ to $\partial(\mathfrak{g} \otimes \mathbb{V}^*)$ in $\mathbb{V} \otimes \wedge^2\mathbb{V}^*$:

$$\mathbb{V} \otimes \wedge^2\mathbb{V}^* = \partial(\mathfrak{g} \otimes \mathbb{V}^*) \oplus \mathcal{D},$$

where

$$\partial : \mathrm{Hom}(\mathbb{V}, \mathfrak{g}) = \mathfrak{g} \otimes \mathbb{V}^* \to \mathbb{V} \otimes \wedge^2\mathbb{V}^*$$

is the Spencer operator of alternation.

Definition 4.4 *Let $\pi : P \to M$ be a G-structure. A connection ω on P is called a $\mathcal{D}-$ connection if its torsion function*

$$t^\omega : P \to \mathbb{V} \otimes \wedge^2\mathbb{V}^* = \partial(\mathfrak{g} \otimes \mathbb{V}^*) \oplus \mathcal{D}$$

has values in $\mathcal{D}$.

Theorem 4.5 *[1]*

1. *Any G-structure $\pi : P \to M$ admits a $\mathcal{D}$-connection ∇.*

2. *Let $\omega, \bar{\omega}$, be 2 $\mathcal{D}$-connections. Then the corresponding operators of covariant derivative $\nabla, \bar{\nabla}$ are related by*

$$\bar{\nabla} = \nabla + S,$$

where S is a tensor field such that for any $x \in M$, S_x belongs to the first prolongation $\mathfrak{g}^{(1)}$ of the Lie algebra $\mathfrak{g}$.

Definition 4.6 *We say that a connected linear Lie group G with Lie algebra $\mathfrak{g}$ is of type k if its k-th prolongation vanishes, i.e. $\mathfrak{g}^{(k)} = 0$ and $\mathfrak{g}^{(k-1)} \neq 0$. In this sense, any G-structure with Lie group G of type k is called a* G-structure of type k.

Theorem 4.7 *[1] Let $\pi : P \to M$ be a G-structure of type 1 and suppose that there is given a G-equivariant decomposition*

$$\mathbb{V} \otimes \wedge^2\mathbb{V} = \partial(\mathfrak{g} \otimes \mathbb{V}^*) \oplus \mathcal{D}.$$

Then there exists a unique connection, whose torsion tensor (calculated with respect to a coframe $p \in P$) has values in $\mathcal{D} \subset \mathbb{V} \otimes \wedge^2\mathbb{V}^*$.

Corollary 4.8 *Let M be a smooth manifold equipped with a G–structure, where $G = GL(n, \mathcal{O})$, $\mathcal{O} = \mathcal{C}l(s,t)$, $s + t > 1$, i.e. an almost Clifford manifold. Then the G–structure is of type 1 and there exists a unique $\mathcal{D}$–connection.*

5. An almost Cliffordian manifold

One can see that an almost Cliffordian manifold M is given as a G–structure provided that there is a reduction of the structure group of the principal frame bundle of M to

$$G := GL(m,\mathcal{O})GL(1,\mathcal{O}) = GL(m,\mathcal{O}) \times_{Z(GL(1,\mathcal{O}))} GL(1,\mathcal{O}),$$

where $Z(G)$ is a center of G. The action of G on T_xM looks like

$$QXq, \text{ where } Q \in GL(m,\mathcal{O}), q \in GL(1,\mathcal{O}),$$

where the right action of $GL(1,\mathcal{O})$ is blockwise. In this case the tensor fields in the form $F_1,\dots,F_k$ can be defined only locally. It is easy to see that the Lie algebra $\mathfrak{gl}(m,\mathcal{O})$ of a Lie group $GL(m,\mathcal{O})$ is of the form

$$\mathfrak{gl}(m,\mathcal{O}) = \{A \in \mathfrak{gl}(km,\mathbb{R}) | AI_i = I_iA, AJ_j = J_jA\}$$

and the Lie algebra $\mathfrak{g}$ of a Lie group $GL(m,\mathcal{O})GL(1,\mathcal{O})$ is of the form

$$\mathfrak{g} = \mathfrak{gl}(m,\mathcal{O}) \oplus \mathfrak{gl}(1,\mathcal{O}).$$

Let us note that the case of $\mathcal{C}l(0,3)$ was studied in a detailed way in [2].

Remark 5.1 *Let $\mathcal{O}$ be the Clifford algebra $\mathcal{C}l(0,2)$. For any one-form ξ on $\mathbb{V}$ and any $X,Y \in \mathbb{V}$, the elements of the form*

$$\begin{aligned} S^\xi(X,Y) = &-\xi(X)Y - \xi(Y)X + \xi(I_1X)I_1Y + \xi(I_1Y)I_1X + \xi(I_2X)I_2Y \\ &+ \xi(I_2Y)I_2X + \xi(I_1I_2X)I_1I_2Y + \xi(I_1I_2Y)I_1I_2X \end{aligned}$$

belong to the first prolongation $\mathfrak{g}^{(1)}$ of the Lie algebra $\mathfrak{g}$ of the Lie group $GL(m,\mathcal{O})GL(1,\mathcal{O})$.

Proof We fix $X \in \mathbb{V}$ and define $S^\xi_X := S^\xi(X,Y) : \mathbb{V} \to \mathbb{V}$. We have to prove that $S^\xi_X(I_iY) = I_iS^\xi_X(Y) + \sum_{l=1}^4 a_lF_l(Y)$, for $i = 1,2$ and $S^\xi_X(Y) = S^\xi_Y(X)$. We compute directly for any X and for I_1

$$\begin{aligned} S^\xi_X(I_1Y) &= -\xi(X)I_1Y - \xi(I_1Y)X - \xi(I_1X)Y - \xi(Y)I_1X - \xi(I_2X)I_1I_2Y \\ &\quad - \xi(I_1I_2Y)I_2X + \xi(I_1I_2X)I_2Y + \xi(I_2Y)I_1I_2X \\ &= -\xi(I_1Y)X - \xi(Y)I_1X - \xi(I_1I_2Y)I_2X \\ &\quad + \xi(I_2Y)I_1I_2X + \sum_{l=1}^4 a_lF_l(Y). \end{aligned}$$

On the other hand,

$$\begin{aligned} I_1S^\xi_X(Y) &= -\xi(X)I_1Y - \xi(Y)I_1X - \xi(I_1X)Y - \xi(I_1Y)X + \xi(I_2X)I_1I_2Y \\ &\quad + \xi(I_2Y)I_1I_2X - \xi(I_1I_2X)I_2Y - \xi(I_1I_2Y)I_2X \\ &= -\xi(Y)I_1X - \xi(I_1Y)X + \xi(I_2Y)I_1I_2X \\ &\quad - \xi(I_1I_2Y)I_2X + \sum_{l=1}^4 a_lF_l(Y) \end{aligned}$$

and

$$
\begin{aligned}
S^{\xi}_X(I_1Y) - I_1S^{\xi}_X(Y) = & \\
& = -\,\xi(I_1Y)X - \xi(Y)I_1X - \xi(I_1I_2Y)I_2X + \xi(I_2Y)I_1I_2X \\
& -(-\xi(Y)I_1X - \xi(I_1Y)X + \xi(I_2Y)I_1I_2X - \xi(I_1I_2Y)I_2X) \\
& + \sum_{l=1}^{4} \bar{a}_l F_l(Y) = \sum_{l=1}^{4} \bar{a}_l F_l(Y).
\end{aligned}
$$

By the same process for I_2 we obtain

$$
\begin{aligned}
S^{\xi}_X(I_2Y) - I_2S^{\xi}_X(Y) = & \\
& = -\,\xi(I_2Y)X + \xi(I_1I_2Y)I_1X - \xi(Y)I_2X - \xi(I_1Y)I_1I_2X \\
& -(-\xi(Y)I_2X - \xi(I_1Y)I_1I_2X - \xi(I_2Y)X + \xi(I_1I_2Y)I_1X) \\
& + \sum_{l=1}^{4} \bar{a}_l F_l(Y) = \sum_{l=1}^{4} \bar{a}_l F_l(Y).
\end{aligned}
$$

Finally, we have to prove the symmetry, but this is obvious. □

Lemma 5.2 *Let $\mathcal{C}l(s,t)$ be the Clifford algebra, $n = s + t$, and let us denote by F_i the affinors obtained from the generators of $\mathcal{C}l(s,t)$. Then there exist $\varepsilon_i \in \{\pm 1\}$, $i = 1,...,n$ such that for $A \in V^*$, the tensor $S^A \in V \times V \to V$ defined by*

$$
S^A(X,Y) = \sum_{i=1}^{n} \varepsilon_i A(F_iX)F_iY, \;\; X, Y \in V \tag{3}
$$

satisfies the identity

$$
S^A(I_jX,Y) - I_jS^A(X,Y) = 0 \tag{4}
$$

for all algebra generators I_j of $\mathcal{C}l(s,t)$.

Proof Let us consider the gradation of the Clifford algebra $\mathcal{C}l(s,t) = \mathcal{C}l^0 \oplus \mathcal{C}l^1 \oplus ... \oplus \mathcal{C}l^n$ with respect to the generators of $\mathcal{C}l(s,t)$. Then we can define gradually: for $E \in \mathcal{C}l^0$ we choose $\varepsilon = 1$. If the identity (4) should be satisfied for the terms in (3), then it must hold that

$$
\varepsilon_0 A(I_jX)Y = \varepsilon_i A(I_jX)I_jI_jY \text{ for all } I_j,
$$

i.e.

$$
\varepsilon_i = \begin{cases} 1 & \text{for } I_j^2 = 1, \\ -1 & \text{for } I_j^2 = -1. \end{cases}
$$

For $F_i \in \mathcal{C}l^v$ the following equality holds:

$$
\varepsilon_i A(F_iI_jX)F_iY = \varepsilon_k A(F_kX)I_jF_kY,
$$

and thus $F_i = I_j F_k$. Note that F_k can be an element of both $\mathcal{C}l^{v+1}$ and $\mathcal{C}l^{v-1}$. WLOG we choose I_j such that $F_k \in \mathcal{C}l^{v+1}$. Now 2 possibilities can appear: either

$$F_i I_j = F_k, \tag{5}$$

which leads to $I_j F_k I_j = F_k$ and thus $\varepsilon_k = \varepsilon_i$, or

$$F_i I_j = -F_k, \tag{6}$$

which leads to $I_j F_k I_j = -F_k$ and thus $\varepsilon_k = -\varepsilon_i$.

This concludes the definition of ε_i such that the identity (4) holds. To prove the consistency, we have to show that the value of ε_k does not depend on I_j, i.e. for the generators I such that $I^2 = 1$ and J such that $J^2 = -1$, the resulting coefficient ε_k obtained after 2 consequent steps of the algorithm with the alternate use of both I and J, does not depend on the order. Thus let us consider the following cases:

(a) $F_i = IF_k$, which results in the possibilities $IF_kI = F_k$, see (5), which leads to $\varepsilon_k = \varepsilon_i$, or $IF_kI = -F_k$, see (6), which leads to $\varepsilon_k = -\varepsilon_i$.

(b) $F_j = JF_k$,which similarly leads to either $JF_kJ = F_k$ implying $\varepsilon_k = \varepsilon_j$, or $JF_kJ = -F_k$ implying $\varepsilon_k = -\varepsilon_j$.

Applying the processes (a) and (b) alternately, we obtain:

$$JIF_kJ = \begin{cases} -IF_k & \Rightarrow \varepsilon_l = -\varepsilon_i \\ IF_k & \Rightarrow \varepsilon_l = \varepsilon_i \end{cases}$$

for $F_l = JIF_k$ and

$$IJF_kI = \begin{cases} -JF_k & \Rightarrow \varepsilon_l = -\varepsilon_j \\ JF_k & \Rightarrow \varepsilon_l = \varepsilon_j \end{cases}$$

for $F_l = IJF_k$. Obviously, the corresponding cases give the same result of ε_k. □

Theorem 5.3 *Let $\mathcal{O}$ be the Clifford algebra $\mathcal{C}l(s,t)$. For any one–form ξ on $\mathbb{V}$ and any $X, Y \in \mathbb{V}$, the elements of the form*

$$S^\xi_X(Y) = \sum_{i=1}^{k} \epsilon_i(\xi(F_iX)F_iY + \xi(F_iY)F_iX), \ k = 2^{s+t},$$

where the coefficients ϵ_i depend on the type of $\mathcal{O}$, belong to the first prolongation $\mathfrak{g}^{(1)}$ of the Lie algebra $\mathfrak{g}$ of the Lie group $GL(m,\mathcal{O})GL(1,\mathcal{O})$.

Proof One can easily see that S^ξ is symmetric and we have to prove the second condition, i.e. $S^\xi_X I_i Y - I_i S^\xi_X Y \in \mathcal{O}(Y)$, i.e.

$$S^\xi_X I_i Y - I_i S^\xi_X Y = \sum_{j=1}^{k} \bar{\epsilon}_j \xi(F_jX)F_jY + \sum_{j=1}^{k} \bar{\epsilon}_j \xi(F_jY)F_jX$$
$$- \sum_{j=1}^{k} \bar{\epsilon}_j I_i \xi(F_jX)F_jY - \sum_{j=1}^{k} \bar{\epsilon}_j I_i \xi(F_jY)F_jTX.$$

From Lemma 5.2 we have

$$S_X^{\xi} I_i Y - I_i S_X^{\xi} Y = \sum_{j=1}^{k} \bar{\epsilon}_j \xi(F_j X) F_j Y - \sum_{j=1}^{k} \bar{\epsilon}_j I_i \xi(F_j Y) F_j T X = \sum_{j=0}^{k} \psi_i F_j Y.$$

□

Corollary 5.4 *Let M be an almost Cliffordian manifold based on Clifford algebra $\mathcal{O} = \mathcal{C}l(s,t)$, where* $\dim(M) \geq 2^{(s+t+1)}$*, i.e. a smooth manifold equipped with G–structure, where $G = GL(n, \mathcal{O})GL(1, \mathcal{O})$ or equivalently an A–structure where $A = \mathcal{O}$. Then the class of $\mathcal{D}$–connections preserving A and sharing the same A–planar curves is isomorphic to* $(\mathbb{R}^{km})^*$.

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