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Some results on $\mathcal{T}$-noncosingular modules

Rachid TRIBAK*
Centre Régional des Métiers de L'Education et de la Formation (CRMEF)-Tanger, Tangier 90000, Morocco

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Abstract: The notion of $\mathcal{T}$-noncosingularity of a module has been introduced and studied recently. In this article, a number of new results of this property are provided. It is shown that over a commutative semilocal ring R such that $Jac(R)$ is a nil ideal, every $\mathcal{T}$-noncosingular module is semisimple. We prove that for a perfect ring R, the class of $\mathcal{T}$-noncosingular modules is closed under direct sums if and only if R is a primary decomposable ring. Finitely generated $\mathcal{T}$-noncosingular modules over commutative rings are shown to be precisely those having zero Jacobson radical. We also show that for a simple module S, $E(S) \oplus S$ is $\mathcal{T}$-noncosingular if and only if S is injective. Connections of $\mathcal{T}$-noncosingular modules to their endomorphism rings are investigated.

Key words: Small submodules, $\mathcal{T}$-noncosingular modules, endomorphism rings

1. Introduction

The concept of $\mathcal{T}$-noncosingularity of a module was introduced and studied recently by Tütüncü and Tribak in 2009 [20] as a dual notion of the $\mathcal{K}$-nonsingularity that was introduced and studied by Rizvi-Roman [14, 15]. It was shown in [21] that every dual Baer module is $\mathcal{T}$-noncosingular and that every $\mathcal{T}$-noncosingular lifting module is dual Baer. We note also that dual Rickart modules were introduced and studied by Lee et al. in 2011 [12] and it is easy to see that every dual Rickart module is $\mathcal{T}$-noncosingular. These links of the $\mathcal{T}$-noncosingularity with the dual Rickart and dual Baer properties are the motivations for the investigations in this paper. We obtain some new useful properties of this kind of module.

Throughout, R will denote an associative ring with unity, $Jac(R)$ will denote the Jacobson radical of R, and $Z(R)$ will stand for the right singular ideal of R. For an R-module M, we write $E(M)$ and $Rad(M)$ for the injective hull and the Jacobson radical of M, respectively. If N is a submodule of an R-module M, then the notation $N \ll M$ means that N is small in M.

In Section 2 we investigate general properties of $\mathcal{T}$-noncosingular modules. We provide conditions for a $\mathcal{T}$-noncosingular module to have zero Jacobson radical. Among other results, we show that every finitely generated $\mathcal{T}$-noncosingular module over a commutative ring has zero Jacobson radical. The class of commutative rings R for which every cyclic R-module is $\mathcal{T}$-noncosingular is characterized as that of von Neumann regular rings, while the class of commutative rings R for which every finitely generated $\mathcal{T}$-noncosingular R-module is semisimple is shown to be precisely that of semilocal rings. It is also shown that over a commutative semilocal ring R such that $Jac(R)$ is a nil ideal, every $\mathcal{T}$-noncosingular R-module is semisimple.

Section 3 is devoted to some results on direct sums of $\mathcal{T}$-noncosingular modules. We show that for a

*Correspondence: tribak12@yahoo.com

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simple module S, $E(S) \oplus S$ is $\mathcal{T}$-noncosingular if and only if S is injective. We prove that for a perfect ring R, the class of $\mathcal{T}$-noncosingular modules is closed under direct sums if and only if R is a primary decomposable ring.

The focus of our investigations in Section 4 is on connections of a $\mathcal{T}$-noncosingular module to its endomorphism ring.

2. Some properties of $\mathcal{T}$-noncosingular modules

Definition 2.1 *Let M and N be 2 modules.*

(1) We say that M is $\mathcal{T}$-noncosingular relative to N if $\forall \varphi \in Hom_R(M, N)$, $\operatorname{Im}\varphi \ll N$ implies $\varphi = 0$.

(2) The module M is called $\mathcal{T}$-noncosingular if M is $\mathcal{T}$-noncosingular relative to M (or, equivalently, $\forall \varphi \in End_R(M)$, $\operatorname{Im}\varphi \ll M \Rightarrow \varphi = 0$).

Many examples of $\mathcal{T}$-noncosingular modules are exhibited in [21] and [20]. Before presenting another example, we recall that a module M is called *radical* if M has no maximal submodules, i.e. $Rad(M) = M$.

Example 2.2 *Let M be a simple radical module. That is, M is a nonzero radical module that has no nonzero radical submodules (e.g., we can consider the $\mathbb{Z}$-modules $\mathbb{Z}(p^\infty)$ and $\mathbb{Q}$, where p is a prime number). Let φ be a nonzero endomorphism of M. Then $\operatorname{Im}\varphi \cong M/\operatorname{Ker}\varphi$ and so $Rad(\operatorname{Im}\varphi) = \operatorname{Im}\varphi$. Therefore, $\operatorname{Im}\varphi = M$. Hence, M is $\mathcal{T}$-noncosingular.*

A ring R is called a *right V-ring* if every simple right R-module is injective. This is equivalent to the condition that for any right R-module M, we have $Rad(M) = 0$. Recall that a module M is called $\mathcal{K}$-*nonsingular* if, for every $0 \neq \varphi \in End_R(M)$, $\operatorname{Ker}\varphi$ is not essential in M (see [15]). The next example shows the existence of a $\mathcal{T}$-noncosingular module that is not $\mathcal{K}$-nonsingular and provides a $\mathcal{K}$-nonsingular module that is not $\mathcal{T}$-noncosingular.

Example 2.3 *(1) Let R be a right V-ring that is not semisimple (e.g., we can take $R = \prod_{i=1}^{\infty} F_i$ with $F_i = F$ is a field for all $i \geq 1$). By [20, Proposition 2.13], every R-module is $\mathcal{T}$-noncosingular. On the other hand, from [15, Corollary 2.21] it follows that R has a module M that is not $\mathcal{K}$-nonsingular.*

(2) Let F be a field and set $R = \begin{bmatrix} F & 0 \\ F & F \end{bmatrix}$. Then $Jac(R) = \begin{bmatrix} 0 & 0 \\ F & 0 \end{bmatrix}$, and hence R_R is not $\mathcal{T}$-noncosingular by [20, Corollary 2.7]. On the other hand, we have $Z(R_R) = 0$ by [4, Corollary 4.3]. Applying [15, Corollary 2.4], we conclude that R_R is $\mathcal{K}$-nonsingular.

In [20, Proposition 2.3] it was showed that the $\mathcal{T}$-noncosingularity is inherited by direct summands. Next, we show that the $\mathcal{T}$-noncosingularity property does not always transfer from a module to each of its submodules and factor modules.

Example 2.4 *(1) Note that the $\mathbb{Z}$-module $\mathbb{Z}/8\mathbb{Z}$ is not $\mathcal{T}$-noncosingular, while $\mathbb{Z}$ is $\mathcal{T}$-noncosingular (see [20, Proposition 2.10]).*

(2) Consider the $\mathbb{Z}$-module $M = \mathbb{Z}(2^\infty) \oplus \mathbb{Z}(2^\infty)$. Then M has a submodule N, which is isomorphic to $\mathbb{Z}(2^\infty) \oplus \mathbb{Z}/2\mathbb{Z}$. Then M is $\mathcal{T}$-noncosingular as every factor module of M is injective, while N is not $\mathcal{T}$-noncosingular by [20, Example 2.12].

Recall that a ring R is said to be a *right H-ring* if, whenever S_1 and S_2 are simple R-modules such that $Hom_R(E(S_1), E(S_2)) \neq 0$, then $S_1 \cong S_2$. It is well known that every commutative Noetherian ring is an H-ring (see, e.g., [16]). Next, we deal with the $\mathcal{T}$-noncosingularity of injective hulls of simple modules. First note that for any prime number p, the $\mathbb{Z}$-module $E(\mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}(p^\infty)$ is $\mathcal{T}$-noncosingular.

Proposition 2.5 *Assume that R is a ring that has a unique simple right R-module (up to isomorphism) or R is a right H-ring. If S is a simple R-module such that $E(S)$ is $\mathcal{T}$-noncosingular, then S is injective or $Rad(E(S)) = E(S)$.*

Proof Suppose that S is not injective and $Rad(E(S)) \neq E(S)$. Then $S \ll E(S)$ and $E(S)$ has a maximal submodule N. Let S' denote the simple R-module $E(S)/N$. Taking the canonical projection $\pi : E(S) \to S'$ and the inclusion map $\alpha : S' \to E(S')$, the homomorphism $\alpha\pi : E(S) \to E(S')$ is nonzero and $\mathrm{Im}(\alpha\pi) = S'$. By hypothesis, we get that $S' \cong S$. Hence, there exists a nonzero endomorphism φ of $E(S)$ such that $\mathrm{Im}\,\varphi = S \ll E(S)$. This contradicts the fact that M is $\mathcal{T}$-noncosingular. □

Corollary 2.6 *Let m be a maximal ideal of a commutative Artinian ring R. Then $E(R/m)$ is $\mathcal{T}$-noncosingular if and only if R/m is injective.*

Proof Note that $Rad(E(R/m)) \neq E(R/m)$ and R is Noetherian by [2, Theorem 15.20 and Corollary 15.21]. Hence, R is an H-ring, and the result follows from Proposition 2.5. □

We recall that a ring R is called a *right max* ring if $Rad(M) \neq M$ for all nonzero right R-modules M.

Corollary 2.7 *Let R be a right max local ring with maximal right ideal m. The following are equivalent:*

(i) *$E(R/m)$ is $\mathcal{T}$-noncosingular;*

(ii) *R/m is injective;*

(iii) *R is a division ring.*

Proof (i) $\Rightarrow$ (ii) By Proposition 2.5 and the fact that R is a right max ring.

(ii) $\Rightarrow$ (iii) By hypothesis, every simple R-module is injective. Thus, R is a right V-ring and $m = Rad(R) = 0$. Therefore, R is a division ring.

(iii) $\Rightarrow$ (i) This is obvious. □

Proposition 2.8 *Let M be a module with $Rad(M) \neq 0$ and let N be a nonzero small submodule of M. If K is a module that is isomorphic to N, then the module $M \oplus K$ is not $\mathcal{T}$-noncosingular.*

Proof By hypothesis, there exists an isomorphism $\varphi : K \to N$. Let $\pi : M \oplus K \to K$ be the canonical projection and let $\mu : N \to M$ and $\rho : M \to M \oplus K$ be the inclusion maps. Then $\rho\mu\varphi\pi$ is a nonzero endomorphism of $M \oplus K$ such that $\mathrm{Im}(\rho\mu\varphi\pi) = N \oplus 0 \ll M \oplus K$. □

It is easy to see that every module with zero Jacobson radical is $\mathcal{T}$-noncosingular and that the converse is not true, in general (e.g., for any prime integer p, the $\mathbb{Z}$-module $\mathbb{Z}(p^\infty)$ is $\mathcal{T}$-noncosingular, but $Rad(\mathbb{Z}(p^\infty)) = \mathbb{Z}(p^\infty)$). In the next 3 results we present conditions under which the converse holds.

Proposition 2.9 *Let M be a module such that every nonzero submodule contains a simple submodule. If $M \oplus S$ is $\mathcal{T}$-noncosingular for every simple small submodule $S \leq M$, then $Rad(M) = 0$.*

Proof Assume that $Rad(M) \neq 0$. Then $Rad(M)$ contains a simple submodule S. Thus, $S \ll M$. From Proposition 2.8 it follows that $M \oplus S$ is not $\mathcal{T}$-noncosingular. This completes the proof. □

Definition 2.10 *A module M is said to be retractable if for every submodule $N \leq M$, $Hom(M, N) \neq 0$.*

Retractable modules have been studied extensively by different authors (see, e.g., [6, 7, 8, 9, 17]).

Proposition 2.11 *Let M be a retractable module. If M is $\mathcal{T}$-noncosingular, then $Rad(M) = 0$.*

Proof Suppose that $Rad(M) \neq 0$. Then M contains a nonzero submodule N such that $N \ll M$. Since M is retractable, there exists a nonzero endomorphism $f : M \to M$ with $\operatorname{Im} f \subseteq N$. This contradicts the $\mathcal{T}$-noncosingularity of M. □

A ring R is said to be *right semi-Artinian* if every nonzero right R-module contains a simple submodule. Recall that if R is any ring, then a right R-module M is *nonsingular* if $mE \neq 0$ for every nonzero element m of M and essential right ideal E of R.

Corollary 2.12 *If M is a nonzero module that satisfies one of the following conditions:*

(i) *M is a module over a commutative semi-Artinian ring,*

(ii) *M is a projective module over a commutative Noetherian ring,*

(iii) *M is a finitely generated module over a commutative ring,*

(iv) *M is a nonsingular module over a right self-injective ring,*

then M is $\mathcal{T}$-noncosingular if and only if $Rad(M) = 0$.

Proof By Proposition 2.11 [5, Theorems 2.7 and 2.8] and [17, Proposition 1.17 and Corollary 2.12]. □

Corollary 2.13 *The following are equivalent for a commutative ring R:*

(i) *Every cyclic R-module is $\mathcal{T}$-noncosingular;*

(ii) *R is a von Neumann regular ring.*

Proof (i) $\Rightarrow$ (ii) Let I be an ideal of R. By hypothesis, the R-module R/I is $\mathcal{T}$-noncosingular. Then $Rad(R/I) = 0$ by Corollary 2.12. So R is a V-ring (see [19, Theorem 22.1]). Thus, R is von Neumann regular since R is commutative (see [19, Theorem 22.4]).

(ii) $\Rightarrow$ (i) This follows from [20, Proposition 2.13] and [19, Theorem 22.4]. □

Following [18], a module M is called *noncosingular* if $\overline{Z}(M) = \cap\{N \mid M/N$ is small in its injective hull$\} = M$. That is, for every nonzero module N and every nonzero homomorphism $f : M \to N$, $\operatorname{Im} f$ is not a small submodule of N. This is obviously equivalent to the condition that M is $\mathcal{T}$-noncosingular relative to N for every module N. Clearly, every noncosingular module is $\mathcal{T}$-noncosingular. It is easy to check that if S is a simple module that is not injective, then S is $\mathcal{T}$-noncosingular but not noncosingular. In the next result, we give conditions under which the $\mathcal{T}$-noncosingularity of a module implies its noncosingularity. The following condition was studied in [1] as a dual notion of the retractability.

Definition 2.14 *A module M is called coretractable if, for any proper submodule K of M, there exists a nonzero homomorphism $f : M \to M$ with $f(K) = 0$, that is, $Hom_R(M/K, M) \neq 0$.*

Proposition 2.15 *Let M be a coretractable injective module. If M is $\mathcal{T}$-noncosingular, then M is noncosingular.*

Proof Suppose that there exists a proper submodule X of M such that $M/X \ll E(M/X)$. Let $\pi : M \to M/X$ be the canonical projection. Since M is coretractable, there exists a nonzero homomorphism $\varphi : M/X \to M$. Since M is injective, φ can be extended to a homomorphism $\psi : E(M/X) \to M$. Taking the inclusion map $\alpha : M/X \to E(M/X)$, $\psi\alpha\pi$ is a nonzero endomorphism of M. Since $\alpha\pi(M) \ll E(M/X)$, $\psi\alpha\pi(M) \ll M$ by [13, Lemma 4.2(3)]. This contradicts the $\mathcal{T}$-noncosingularity of M. Hence, M is noncosingular. □

The next proposition can be regarded as the dual of [15, Proposition 2.18]. First we prove the following elementary known result.

Lemma 2.16 *Let N be a small submodule of a module M. If L is a submodule of M such that $(L+N)/N \ll M/N$, then $L \ll M$.*

Proof Let X be a submodule of M such that $L + X = M$. Then, $[(L+N)/N] + [(X+N)/N] = M/N$. By hypothesis, we have $(X + N)/N = M/N$. Therefore, $X = M$ as $N \ll M$. So, $L \ll M$. □

Proposition 2.17 *Let M be a module that has a projective cover $f : P \to M$. If P is $\mathcal{T}$-noncosingular, then so is M.*

Proof By hypothesis, $f : P \to M$ is an epimorphism with $Q = \operatorname{Ker} f \ll P$. Thus, $P/Q \cong M$. To prove the $\mathcal{T}$-noncosingularity of M, let $\varphi \in End(P/Q)$ such that $\operatorname{Im}\varphi \ll P/Q$. Consider the natural epimorphism $\pi : P \to P/Q$. Since P is projective, there exists a homomorphism $\psi : P \to P$ such that $\varphi\pi = \pi\psi$. Therefore, $\pi\psi(P) = \varphi(P/Q) \ll P/Q$. So $\psi(P) \ll P$ by Lemma 2.16. But P is $\mathcal{T}$-noncosingular. Then $\psi = 0$, and hence $\varphi\pi = 0$. This implies that $\varphi = 0$. Thus, M is $\mathcal{T}$-noncosingular. □

Proposition 2.18 *The following are equivalent for a ring R:*

(i) *R_R is $\mathcal{T}$-noncosingular;*

(ii) *Every projective R-module is $\mathcal{T}$-noncosingular;*

(iii) *Every R-module having a projective cover is $\mathcal{T}$-noncosingular.*

Proof (i) $\Rightarrow$ (ii) By [20, Corollary 2.7], $Jac(R) = 0$. Let P be a projective R-module. Hence $Rad(P) = P(Jac(R)) = 0$ by [2, Proposition 17.10]. So, P is $\mathcal{T}$-noncosingular.

(ii) $\Rightarrow$ (iii) This follows from Proposition 2.17.

(iii) $\Rightarrow$ (i) This is obvious. □

Definition 2.19 *(1) A module M has D_1 property (or is called lifting) if for every submodule $N \leq M$, there exists a direct summand K of M with $K \subseteq N$ and $N/K \ll M/K$. M has D_3 property if for any direct summands K, L of M with $M = K + L$, $K \cap L$ is a direct summand of M.*

(2) A module M satisfying D_1 and D_3 is called quasi-discrete.

Definition 2.20 *A module M is said to have the strong summand sum property, SSSP, if the sum of any family of direct summands is a direct summand of M. M is said to have the summand intersection property, SIP, if the intersection of any 2 direct summands is a direct summand of M.*

In the next result, we provide an application of $\mathcal{T}$-noncosingularity to quasi-discrete modules. It can be regarded as the dual of [15, Proposition 4.1].

Proposition 2.21 *Let M be a quasi-discrete module. If M is $\mathcal{T}$-noncosingular, then M has $SSSP$ and SIP.*

Proof Since M is $\mathcal{T}$-noncosingular lifting, M has $SSSP$ by [21, Theorems 2.1 and 2.14]. To prove SIP, let K_1 and K_2 be 2 direct summands of M. Then $K = K_1 + K_2$ is a direct summand of M. Since M has (D_3), K has (D_3) by [13, Lemma 4.7]. Therefore, $K_1 \cap K_2$ is a direct summand of K. Hence, $K_1 \cap K_2$ is a direct summand of M. □

Recall that a ring R is said to be *semilocal* if the factor ring $R/Jac(R)$ is semisimple.

We conclude this section by describing the structure of some classes of $\mathcal{T}$-noncosingular modules over commutative semilocal rings. First we prove the following lemma.

Lemma 2.22 *Let I be a nil ideal of a commutative ring R. If M is $\mathcal{T}$-noncosingular, then $MI = 0$.*

Proof Let $a \in I$ and consider the endomorphism φ_a of M defined by $\varphi_a(x) = xa$ for all $x \in M$. Clearly, we have $Im\varphi_a = Ma$. Let X be a submodule of M such that $M = Ma + X$. By induction, we have $M = Ma^n + X$ for every integer $n \geq 1$. Then $X = M$ since the ideal I is nil. It follows that $Ma \ll M$. Thus, $Ma = 0$ as M is $\mathcal{T}$-noncosingular. □

Theorem 2.23 *Let R be a commutative semilocal ring such that $Jac(R)$ is a nil ideal of R. Then an R-module M is $\mathcal{T}$-noncosingular if and only if M is semisimple.*

Proof Let M be a $\mathcal{T}$-noncosingular module. Since R is semilocal, we have $Rad(M) = MJac(R)$ and $M/Rad(M)$ is semisimple by [2, Corollary 15.18]. Therefore, $Rad(M) = 0$ by Lemma 2.22. Thus, M is semisimple. The converse is immediate. □

Corollary 2.24 *Let R be a commutative perfect ring. Then an R-module M is $\mathcal{T}$-noncosingular if and only if M is semisimple.*

Proof By Theorem 2.23 and [2, Theorem 28.4]. □

Theorem 2.25 *The following statements are equivalent for a commutative ring R:*

(i) *Every $\mathcal{T}$-noncosingular R-module M with $Rad(M) \ll M$ is semisimple;*

(ii) *Every finitely generated $\mathcal{T}$-noncosingular module is semisimple;*

(iii) *R is semilocal.*

Proof (i) ⇒ (ii) This is clear.

(ii) ⇒ (iii) Since $Rad(R/Jac(R)) = 0$, the R-module $R/Jac(R)$ is $\mathcal{T}$-noncosingular. The result follows by (ii).

(iii) $\Rightarrow$ (i) Let M be a $\mathcal{T}$-noncosingular R-module with $Rad(M) \ll M$. Since R is semilocal, $M/Rad(M)$ is semisimple and $Rad(M) = MJac(R)$ by [2, Corollary 15.18]. If $a \in Jac(R)$ and φ_a is the endomorphism of M defined by $\varphi_a(x) = xa$ for all $x \in M$, then we have $\operatorname{Im}\varphi_a = Ma \subseteq MJac(R) \ll M$. But by $\mathcal{T}$-noncosingularity, $Ma = 0$. Thus, $Rad(M) = 0$. This implies that M is semisimple. □

3. Direct sums of $\mathcal{T}$-noncosingular modules

It is shown in [20, Example 2.12] that, in general, a direct sum of 2 $\mathcal{T}$-noncosingular modules is not $\mathcal{T}$-noncosingular. In this section we prove that for a simple module S, $E(S) \oplus S$ is $\mathcal{T}$-noncosingular if and only if S is injective (Proposition 3.4). The class of perfect rings for which arbitrary direct sums of $\mathcal{T}$-noncosingular modules are $\mathcal{T}$-noncosingular is shown to be exactly that of the primary decomposable rings (Theorem 3.7).

We begin with the next proposition, which is a direct consequence of [20, Corollary 2.7 and Proposition 2.11].

Proposition 3.1 (i) *If M is a $\mathcal{T}$-noncosingular module, then every direct sum of copies of M is a $\mathcal{T}$-noncosingular module.*

(ii) *If R is a ring with $Jac(R) = 0$, then every free R-module is $\mathcal{T}$-noncosingular.*

Next, we provide a characterization for an arbitrary direct sum of $\mathcal{T}$-noncosingular modules to be $\mathcal{T}$-noncosingular when each module is fully invariant in the direct sum.

Proposition 3.2 *Let $M = \oplus_{i\in I} M_i$ be the direct sum of fully invariant submodules M_i. Then M is $\mathcal{T}$-noncosingular if and only if M_i is $\mathcal{T}$-noncosingular for all $i \in I$.*

Proof The necessity follows from [20, Proposition 2.3]. Conversely, we need only to show that M_i is a $\mathcal{T}$-noncosingular module relative to M_j for all $i, j \in I$ with $i \neq j$ (see [20, Proposition 2.11]). Let $f : M_i \to M_j$ $(i \neq j)$ be a homomorphism. Let $\pi_i : M \to M_i$ be the projection map and $\alpha_j : M_j \to M$ be the inclusion map. Then $g = \alpha_j f \pi_i \in End_R(M)$ and $g(M) \subseteq M_j$. Since M_i is fully invariant in M, we have $g(M_i) \subseteq M_i$. So, $g(M_i) \subseteq M_i \cap M_j = 0$. Hence, $f = 0$. Consequently, M is $\mathcal{T}$-noncosingular. □

Proposition 3.3 *Let $M = N \oplus (\oplus_{i\in I} S_i)$ such that S_i ($i \in I$) are simple modules. The following are equivalent:*

(i) *M is $\mathcal{T}$-noncosingular;*

(ii) (a) *N is $\mathcal{T}$-noncosingular, and*

(b) *For every simple small submodule S of N, $S \ncong S_i$ for all $i \in I$.*

Proof (i) $\Rightarrow$ (ii) By [20, Proposition 2.3], N and $N \oplus S_i$ are $\mathcal{T}$-noncosingular modules for all $i \in I$. Proposition 2.8 now shows that condition (b) holds.

(ii) $\Rightarrow$ (i) By (b), each S_i is $\mathcal{T}$-noncosingular relative to N. Applying [20, Proposition 2.11], we obtain that M is $\mathcal{T}$-noncosingular. □

Let R be a Dedekind domain that is not a field and P be a nonzero prime ideal of R. Let $R(P^\infty)$ denote the P-primary component of the torsion R-module K/R, where K is the quotient field of R. In [20, Example 2.12] it is proven that the R-module $R(P^\infty) \oplus R/P$ is not $\mathcal{T}$-noncosingular. That is, $E(R/P) \oplus R/P$

is not $\mathcal{T}$-noncosingular. In the next result we provide a necessary and sufficient condition for $E(S) \oplus S$ to be $\mathcal{T}$-noncosingular, where S is a simple module.

Proposition 3.4 *Let S be a simple module. Then the module $M = E(S) \oplus S$ is $\mathcal{T}$-noncosingular if and only if S is injective.*

Proof The sufficiency is obvious. Conversely, suppose that S is not injective. Then $S \ll E(S)$. Thus, M is not $\mathcal{T}$-noncosingular by Proposition 2.8. □

By combining [20, Proposition 2.13] and Proposition 3.4, we get the following result.

Corollary 3.5 *The following are equivalent for a ring R:*

(i) *Every R-module is $\mathcal{T}$-noncosingular;*

(ii) *For every simple R-module S, the module $E(S) \oplus S$ is $\mathcal{T}$-noncosingular;*

(iii) *The ring R is a right V-ring.*

Next, we present other examples that show that the property of $\mathcal{T}$-noncosingularity does not go to direct sums of $\mathcal{T}$-noncosingular modules.

Example 3.6 *(1) Let R be a right hereditary ring that is not a right V-ring. Therefore, R has a simple R-module S that is not injective (e.g., we can take a Dedekind domain R that is not a field and S any simple R-module). Then $E(S)$ and S are both $\mathcal{T}$-noncosingular R-modules by [20, Example 2.1]. However, the R-module $M = E(S) \oplus S$ is not $\mathcal{T}$-noncosingular by Proposition 3.4.*

(2) Let R be an almost DVR with maximal ideal m and quotient field Q (i.e. R is a commutative local Noetherian domain of Krull dimension 1 and the integral closure R' of R in Q is a finitely generated R-module and is a discrete valuation ring). Note that $E(R/m)$ is a simple radical R-module by [11, Proposition 4]. Therefore, $E(R/m)$ is a $\mathcal{T}$-noncosingular R-module (see Example 2.2). Further, the R-module R/m is $\mathcal{T}$-noncosingular. On the other hand, the R-module $E(R/m) \oplus R/m$ is not $\mathcal{T}$-noncosingular, since otherwise R will be a V-ring and $m = 0$ by Corollary 3.5.

Recall that a ring R is called *left (resp. right) perfect* if it is semilocal and every nonzero left R-module contains a maximal (resp. simple) submodule. A ring R is said to be *perfect* if it is right and left perfect. A perfect ring is said to be *primary* if the ring $R/Jac(R)$ is simple Artinian. A perfect ring is called *primary decomposable* if it is isomorphic to a finite product of primary rings. A module M is called *supplemented* if, for every submodule N of M, there exists a submodule $K \leq M$ such that $M = N + K$ and $N \cap K \ll K$ (see, e.g., [3], [13], and [22]). It is easy to check that if M is a module with zero Jacobson radical, then M is supplemented if and only if M is semisimple.

In the next result, we characterize the class of perfect rings R for which arbitrary direct sums of $\mathcal{T}$-noncosingular R-modules are $\mathcal{T}$-noncosingular.

Theorem 3.7 *The following assertions are equivalent for a perfect ring R:*

(i) *Every $\mathcal{T}$-noncosingular R-module is semisimple;*

(ii) *Every direct sum of $\mathcal{T}$-noncosingular R-modules is $\mathcal{T}$-noncosingular;*

(iii) *R is primary decomposable.*

Proof (i) $\Rightarrow$ (ii) This is clear.

(ii) $\Rightarrow$ (iii) Let M be a module such that $S = End_R(M)$ is a division ring. Clearly, M is an indecomposable $\mathcal{T}$-noncosingular module. Since R is perfect, every R-module contains a simple submodule. Noting that $M \oplus S$ is $\mathcal{T}$-noncosingular for every simple R-module S, we conclude from Proposition 2.9 that $Rad(M) = 0$. Since R is perfect, M is supplemented by [13, Theorem 4.41]. Thus, M is semisimple, but M is indecomposable. Then M is simple. So R is primary decomposable by [10, Theorem 1.2].

(iii) $\Rightarrow$ (i) By hypothesis, $R = R_1 \oplus \cdots \oplus R_n$ is a direct sum of perfect primary rings R_i $(1 \leq i \leq n)$. We can write $1_R = e_1 + e_2 + \cdots + e_n$, where 1_R is the identity element of R and for each i, $e_i \in R_i$. Then for each i, e_i is the identity element of the ring R_i. Let M be an R-module. Then $M = Me_1 \oplus Me_2 \oplus \cdots \oplus Me_n$. Also, Me_i can be regarded as an R_i-module as well as an R-module, and its submodules are the same in both cases, because $xe_i(r_1 + r_2 + \cdots + r_n) = xe_i r_i$, where $x \in M$ and $r_j \in R_j$ for each j, $1 \leq j \leq n$. Now assume that R has a $\mathcal{T}$-noncosingular module M that is not semisimple. Without loss of generality we can assume that $M_1 = Me_1$ is not semisimple. Note that $End_{R_1}(M_1) = End_R(M_1)$. So $(M_1)_{R_1}$ is $\mathcal{T}$-noncosingular by [20, Proposition 2.3]. Since R_1 is a perfect ring, $(M_1)_{R_1}$ is supplemented by [13, Theorem 4.41]. Therefore, $Rad_{R_1}(M_1) \neq 0$. Hence, $Rad_{R_1}(M_1)$ contains a simple submodule S_1. Moreover, $(M_1)_{R_1}$ contains a maximal submodule K_1 since R_1 is perfect. Consider the natural epimorphism $\pi : M_1 \to M_1/K_1$ and the inclusion map $\alpha : S_1 \to M_1$. Since R_1 is primary, R_1 has a unique isomorphism class of simple modules. So, there exists an isomorphism $\theta : M_1/K_1 \to S_1$. It follows that $\varphi = \alpha\theta\pi$ is a nonzero endomorphism of M_1 such that $\varphi(M_1) = S_1 \ll M_1$. This shows that $(M_1)_{R_1}$ is not $\mathcal{T}$-noncosingular, a contradiction. □

Corollary 3.8 *If R is a finite product of local perfect rings (e.g., R is commutative perfect), then every $\mathcal{T}$-noncosingular module is semisimple.*

Proof This is a direct consequence of Theorem 3.7. □

4. The endomorphism ring of a $\mathcal{T}$-noncosingular module

We conclude this paper by investigating the connection of the $\mathcal{T}$-noncosingularity of a module to its endomorphism ring. Recall that a ring R is called *reduced* if it has no nonzero nilpotent elements.

Proposition 4.1 *Let M be a quasi-discrete module with $S = End_R(M)$. If M is $\mathcal{T}$-noncosingular, then $S = S_1 \times S_2$ such that S_1 is von Neumann regular and S_2 is reduced.*

Proof Let $\nabla(M) = \{\varphi \in S \mid \operatorname{Im}\varphi \ll M\}$. By [13, Proposition 5.7], $S/\nabla(M) = S_1 \times S_2$ such that S_1 is von Neumann regular and S_2 is reduced. However, since M is $\mathcal{T}$-noncosingular, $\nabla(M) = 0$. □

Proposition 4.2 *Let P be a quasi-projective module with $S = End_R(P)$. The following are equivalent:*

(i) *P is $\mathcal{T}$-noncosingular;*

(ii) *$Jac(S) = 0$;*

(iii) *${}_SS$ is $\mathcal{T}$-noncosingular.*

Proof This follows from [20, Corollary 2.7] and the fact that $\varphi \in Jac(S)$ if and only if $\operatorname{Im}\varphi \ll P$ (see, e.g., [22, 22.2]). □

Proposition 4.2 is not true, in general, as the next example shows.

Example 4.3 *Consider the* $\mathbb{Z}$*-module* $M = \mathbb{Z}(p^{\infty})$*, where* p *is a prime number. It is well known that* $S = End_{\mathbb{Z}}(M)$ *is a local ring that is not a division ring. Then* $Jac(S) \neq 0$*, while* M *is* $\mathcal{T}$*-noncosingular.*

Definition 4.4 *A module* M *has* D_2 *property (or is called direct projective) if, for any direct summand* K *of* M *and submodule* N *of* M *with* $M/N \cong K$*,* N *is a direct summand of* M*.*

Proposition 4.5 *Let* M *be a direct projective module with* $S = End_R(M)$*. If* $Jac(S) = 0$*, then* M *is* $\mathcal{T}$*-noncosingular.*

Proof By [22, 41.19(1)]. □

Proposition 4.6 *Let* R *be a commutative ring. If* R *is* $\mathcal{T}$*-noncosingular, then* R *is* $\mathcal{K}$*-nonsingular.*

Proof By [15, Proposition 2.7], it suffices to show that R is nonsingular. Since R is $\mathcal{T}$-noncosingular, we have $Jac(R) = 0$ by [20, Corollary 2.7]. So, R is a semiprime ring. Therefore, $Z(R) = 0$ by [4, Proposition 1.27(b)]. □

The converse of Proposition 4.6 is not true, in general, as shown below.

Example 4.7 *Let* R *be a discrete valuation ring with maximal ideal* m*. It is clear that* $Z(R) = 0$*, while* $Jac(R) = m$*. So* R *is* $\mathcal{K}$*-nonsingular, but* R *is not* $\mathcal{T}$*-noncosingular by [20, Corollary 2.7] and [15, Proposition 2.7].*

Following [22, p. 261], a module M is called *semi-injective* if for any monomorphism $f : N \to M$, where N is a factor module of M, and for any homomorphism $g : N \to M$, there exists $h : M \to M$ such that $hf = g$. Note that every quasi-injective module is semi-injective.

In the next result, we provide a condition under which Proposition 4.6 holds true for modules.

Proposition 4.8 *Let* M *be a coretractable module and let* $S = End_R(M)$*. If* M *is* $\mathcal{T}$*-noncosingular, then* ${}_SS$ *is* $\mathcal{K}$*-nonsingular. The converse holds when* M *is semi-injective.*

Proof This follows from [1, Corollary 4.8] and [15, Proposition 2.7]. □

References

[1] Amini B, Ershad M, Sharif H. Coretractable modules. J Aust Math Soc 2009; 86: 289–304.

[2] Anderson FW, Fuller KR. Rings and Categories of Modules. New York: Springer-Verlag, 1974.

[3] Clark J, Lomp C, Vanaja N, Wisbauer R. Lifting Modules, Supplements and Projectivity in Module Theory. Basel: Frontiers in Mathematics, Birkhäuser Verlag, 2006.

[4] Goodearl KR. Ring Theory: Nonsingular Rings and Modules. New York: Marcel Dekker, 1976.

[5] Haghany A, Karamzadeh OAS, Vedadi MR. Rings with all finitely generated modules retractable. Bull Iranian Math Soc 2009; 35: 37–45.

[6] Haghany A, Vedadi MR. Study of semi-projective retractable modules. Algebra Colloq 2007; 14: 489–496.

[7] Khuri SM. Endomorphism rings and lattice isomorphism. J Algebra 1979; 56: 401–408.

[8] Khuri SM. Correspondence theorems for modules and their endomorphism rings. J Algebra 1989; 122: 380–396.

[9] Khuri SM. Nonsingular retractable modules and their endomorphism rings. Bull Austral Math Soc 1991; 43: 63–71.

[10] Haily A, Alaoui M. Perfect rings for which the converse of Schur's lemma holds. Publ Mat 2001; 45: 219–222.

[11] Idelhadj A, Tribak R. On injective ⊕-supplemented modules. In: Proceedings of the First Moroccan-Andalusian Meeting on Algebras and their Applications, Tétouan, Morocco, September 2001. Tétouan, Morocco: Université Abdelmalek Essaadi, 2003, pp. 166–180.

[12] Lee G, Rizvi ST, Roman CS. Dual Rickart modules. Comm Algebra 2011; 39: 4036–4058.

[13] Mohamed SH, Müller BJ. Continuous and Discrete Modules. London Math. Soc. LNS 147. Cambridge: Cambridge University Press, 1990.

[14] Rizvi ST, Roman CS. Baer and quasi-Baer modules. Comm Algebra 2004; 32: 103–123.

[15] Rizvi ST, Roman CS. On $\mathcal{K}$-nonsingular modules and applications. Comm Algebra 2007; 35: 2960–2982.

[16] Sharpe DW, Vamos P. Injective Modules. Cambridge: Cambridge University Press, 1972.

[17] Smith PF. Modules with many homomorphisms. J Pure Appl Algebra 2005; 197: 305–321.

[18] Talebi Y, Vanaja N. Torsion theory cogenerated by M-small modules. Comm Algebra 2002; 30: 1449–1460.

[19] Tuganbaev A. Rings Close to Regular. London: Kluwer Academic Publishers, 2002.

[20] Tütüncü DK, Tribak R. On $\mathcal{T}$-noncosingular modues. Bull Aust Math Soc 2009; 80: 462–471.

[21] Tütüncü DK, Tribak R. On dual Baer modules. Glasgow Math J 2010; 52: 261–269.

[22] Wisbauer R. Foundations of Module and Ring Theory. Reading: Gordon and Breach Science Publishers, 1991.