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On the structure of some modules over generalized soluble groups

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Abstract: Let R be a ring and G a group. An R-module A is said to be *Artinian-by-(finite rank)* if $\mathbf{Tor}_R(A)$ is Artinian and $A/\mathbf{Tor}_R(A)$ has finite R-rank. We study a module A over a group ring RG such that $A/C_A(H)$ is Artinian-by-(finite rank) (as an R-module) for every proper subgroup H.

Key words: Modules, group rings, modules over group rings, generalized soluble groups, radical groups, Artinian modules, generalized radical groups, modules of finite rank

1. Introduction

Let R be a ring, G a group, and A an RG-module. The modules over group rings are classic objects of study with well-established links to various areas of algebra. The case where G is a finite group has been studied in sufficient details for a long time. For the case where G is an infinite group, the situation is different. Investigation of modules over polycyclic-by-finite groups was initiated in the classical works of Hall [4, 5]. Nowadays the theory of modules over polycyclic-by-finite groups is highly developed and rich in interesting results. This was largely due to the fact that a group ring RG of a polycyclic-by-finite group G over a Noetherian ring R is also Noetherian. The group rings over some other groups (even well-studied ones such as, for instance, the Chernikov groups), do not always have such good properties as to be Noetherian. Therefore, their study requires some different approaches and restrictions. For instance, the classical finiteness conditions are largely employed and popular. The very first restrictions here were those that came from ring theory, namely the conditions like "to be Noetherian" and "to be Artinian". Noetherian and Artinian modules over group rings are also very well investigated. Many aspects of the theory of Artinian modules over group rings were treated in [7]. Recently the so-called finitary approach began to be employed intensively in the theory of infinite dimensional linear groups, where it brings many interesting and promising results.

If H is a subgroup of G, then consider the centralizer $C_A(H) = \{a \in A \mid ah = a \text{ for each element } h \in H\}$ of H in A. Clearly $C_A(H)$ is an RH-submodule of A and H really acts on $A/C_A(H)$. The R-factor-module $A/C_A(H)$ is called the *cocentralizer of* H *in* A. Then $H/C_H(A/C_A(H))$ is isomorphic to a subgroup of an automorphism group of an R-module $A/C_A(H)$. It is not hard to see that $C_H(A/C_A(H))$ is abelian, and therefore the structure of the automorphism group of the R-module $A/C_A(H)$ defines the structure of the whole group H.

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Let $\mathfrak{M}$ be a class of R-modules. We say that A is an $\mathfrak{M}$*-finitary module over* RG, if $A/C_A(x) \in \mathfrak{M}$ for each element $x \in G$. If R is a field, $C_G(A) = \langle 1 \rangle$, and $\mathfrak{M}$ is a class of all finite dimensional vector spaces over R, then we come to the finitary linear groups. The theory of finitary linear groups is quite well developed (see the survey in [10]). Wehrfritz began to consider the cases where $\mathfrak{M}$ is the class of finite R-modules [12, 14, 15, 17], where $\mathfrak{M}$ is the class of Noetherian R-modules [13], and where $\mathfrak{M}$ is the class of Artinian R-modules [15, 16, 17, 18, 19]. The Artinian-finitary modules have been considered also in [8]. The notion of a minimax module extends the notions of Noetherian and Artinian modules. An R-module A is said to be *minimax* if A has a finite series of submodules, whose factors are either Noetherian or Artinian. It is not hard to show that if R is an integral domain, then every minimax R-module A includes a Noetherian submodule B such that A/B is Artinian. The first natural case here is the case where $R = \mathbb{Z}$ is the ring of all integers. This case has very important applications in generalized soluble groups. Every $\mathbb{Z}$-minimax module M has the following important property: $\mathbf{r}_{\mathbb{Z}}(M)$ is finite and $\mathbf{Tor}(M)$ is an Artinian $\mathbb{Z}$-module.

Let R be an integral domain and A be an R-module. An analogue of the concept of a dimension for modules over integral domains is the concept of R-rank. One of the essential differences of R-modules and vector spaces is that some elements of A can have a nonzero annihilator in the ring. Put

$$\mathbf{Tor}_R(A) = \{a \in A \mid \mathbf{Ann}_R(a) \neq \langle 0 \rangle\}.$$

It is not hard to see that $\mathbf{Tor}_R(A)$ is an R-submodule of A. Actually, the concept of R-rank works only for the factor-module $A/\mathbf{Tor}_R(A)$. In particular, the finiteness of R-rank does not affect the submodule $\mathbf{Tor}_R(A)$. We say that an R-module A is an *Artinian-by-(finite rank)* if $\mathbf{Tor}_R(A)$ is artinian and $A/\mathbf{Tor}_R(A)$ has finite R-rank. In particular, if an Artinian-by-(finite rank) module A is R-torsion-free, then it could be embedded into a finite dimensional vector space (over the field of fractions of R). If A is R-periodic, then it is Artinian.

Let G be a group, A an RG-module, and $\mathfrak{M}$ a class of R-modules. Put

$$\mathcal{C}_{\mathfrak{M}}(G) = \{H \mid H \text{ is a subgroup of } G \text{ such that } A/C_A(H) \in \mathfrak{M}\}.$$

If A is an $\mathfrak{M}$-finitary module, then $\mathcal{C}_{\mathfrak{M}}(G)$ contains every cyclic subgroup (moreover, every finitely generated subgroup whenever $\mathfrak{M}$ satisfies some natural restrictions). It is clear that the structure of G depends significantly on which important subfamilies of the family $\Lambda(G)$ of all proper subgroups of G include $\mathcal{C}_{\mathfrak{M}}(G)$. The first natural question that arises here is the following: What is the structure of a group G in which $\Lambda(G) = \mathcal{C}_{\mathfrak{M}}(G)$ (in other words, the cocentralizer of every proper subgroup of G belongs to $\mathfrak{M}$)? In the current article, we consider the case when $R = \mathbb{Z}$ and $\mathfrak{M}$ is the class of all Artinian-by-(finite rank) modules. This examination is conducted for the groups belonging to the very wide class of the locally generalized radical groups.

Recall that a group G is called a *generalized radical* if G has an ascending series whose factors are locally nilpotent or locally finite.

The main result of our paper is the following:

Theorem *Let G be a locally generalized radical group and A a $\mathbb{Z}G$-module. If the factor-module $A/C_A(H)$ is Artinian-by-(finite rank) for every proper subgroup H of G, then either $A/C_A(G)$ is Artinian-by-(finite rank) or $G/C_G(A)$ is a cyclic or quasicyclic p-group for some prime p.*

Corollary M *Let G be a locally generalized radical group and A a $\mathbb{Z}G$-module. If a factor-module $A/C_A(H)$ is minimax for every proper subgroup H of G, then either $A/C_A(G)$ is minimax or $G/C_G(A)$ is a cyclic or quasicyclic p-group for some prime p.*

Corollary N *Let G be a locally generalized radical group and A a $\mathbb{Z}G$-module. If a factor-module $A/C_A(H)$ is finitely generated for every proper subgroup H of G, then either $A/C_A(G)$ is finitely generated or $G/C_G(A)$ is a cyclic or quasicyclic p-group for some prime p.*

Corollary A *Let G be a locally generalized radical group and A a $\mathbb{Z}G$-module. If a factor-module $A/C_A(H)$ is Artinian for every proper subgroup H of G, then either $A/C_A(G)$ is Artinian or $G/C_G(A)$ is a cyclic or quasicyclic p-group for some prime p.*

We also will show that for every quasicyclic group one can find a $\mathbb{Z}G$-module A such that $C_G(A) = \langle 1 \rangle$ but the factor-module $A/C_A(H)$ is Artinian-by-(finite rank) for every proper subgroup H of G.

2. Some preparatory results

Lemma 1 *Let R be a ring, G a group, and A an RG ğmodule. If L and H are subgroups of G, whose cocentralizers are Artinian-by-(finite rank) modules, then $A/C_A(\langle H, L \rangle)$ is also Artinian-by-(finite rank).*

Proof The equation $C_A(\langle H, L \rangle) = C_A(H) \cap C_A(L)$ together with Remak's Theorem imply an embedding $A/C_A(\langle H, L \rangle) \hookrightarrow A/C_A(H) \oplus A/C_A(L)$. Let $U/C_A(H) = \mathbf{Tor}(A/C_A(H))$ and $V/C_A(L) = \mathbf{Tor}(A/C_A(L))$. Then $\mathbf{Tor}(A/C_A(H) \oplus A/C_A(L)) = U/C_A(H) \oplus V/C_A(L)$ is an Artinian R-module and

$$(A/C_A(H) \oplus A/C_A(L))/\mathbf{Tor}(A/C_A(H) \oplus A/C_A(L)) \cong A/U \oplus A/V$$

is an R-torsion-free module of finite R-rank. Thus, $A/C_A(H) \oplus A/C_A(L)$ is an Artinian-by-(finite rank) R-module. Therefore, its submodule $A/C_A(\langle H, L \rangle)$ is also Artinian-by-(finite rank).

A group G is said to be $\mathfrak{F}$*-perfect* if G does not include proper subgroups of finite index.

Let G be a generalized radical group. Then either G has an ascendant locally nilpotent subgroup or it has an ascendant locally finite subgroup. In the first case, the locally nilpotent radical $\mathbf{Lnr}(G)$ of G is nonidentity. In the second case, it is not hard to see that G includes a nontrivial normal locally finite subgroup. Clearly, in every group G, the subgroup $\mathbf{Lfr}(G)$ generated by all normal locally finite subgroups is the largest normal locally finite subgroup (the *locally finite radical*). Thus, every generalized radical group has an ascending series of normal subgroups with locally nilpotent or locally finite factors.

Observe also that a periodic generalized radical group is locally finite, and hence a periodic locally generalized radical group is also locally finite.

Let q be a prime and A is an additive abelian q-group. For each positive integer n we define the nth layer $\Omega_n(A)$ by the following rule: $\Omega_n(A) = \{a \in A \mid q^n a = 0\}$. Clearly, $\Omega_n(A)$ is a characteristic subgroup of A.

Futhermore, by $\mathbf{Dr}_{\lambda \in \Lambda} G_\lambda$, we denote a direct product of groups G_λ, $\lambda \in \Lambda$. □

Lemma 2 *Let G be a locally generalized radical group and A be a $\mathbb{Z}G$-module. Suppose that A includes a $\mathbb{Z}G$-submodule B, which is Artinian-by-(finite rank). Then the following assertions hold:*

(i) $G/C_G(B)$ is soluble-by-finite.

(ii) If $G/C_G(B)$ is periodic, then it is nilpotent-by-finite.

(iii) If $G/C_G(B)$ is $\mathfrak{F}$-perfect and periodic, then it is abelian. Moreover, $[[B, G], G] = \langle 0 \rangle$.

Proof Without loss of generality we can suppose that $C_G(B) = \langle 1 \rangle$. We recall that the additive group of the Artinian $\mathbb{Z}$-module is Chernikov, that is $K = \mathbf{Tor}_{\mathbb{Z}}(B)$ includes a divisible subgroup D, which is a direct sum of quasicyclic subgroups such that K/T is finite. The additive group of B/K is torsion-free and has finite $\mathbb{Z}$-rank. In particular, the set $\Pi(D)$ is finite, let us say $\Pi(D) = \{p_1, \ldots, p_n\}$. Clearly D is G-invariant. Being $\mathbb{Z}$-periodic, D is a direct sum of its primary components, that is $D = D_1 \oplus \ldots \oplus D_n$ where D_j is a Sylow p_j-subgroup of D, $1 \leq j \leq n$. We note that every subgroup D_j is G-invariant, $1 \leq j \leq n$. Let $q = p_j$. The factor-group $G/C_G(D_j)$ is isomorphic to a subgroup of $\mathbf{GL}_m(R_q)$ where R_q is the ring of integer q-adic numbers and m satisfies $q^m = |\mathbf{\Omega}_1(D_j)|$. Let F_q be a field of fractions of R_q, and then $G/C_G(D_j)$ is isomorphic to a subgroup of $\mathbf{GL}_m(F_q)$. Note that $\mathbf{char}(F_q) = 0$. Being locally generalized radical, $G/C_G(D_j)$ does not include the noncyclic free subgroup; thus, an application of Tits' Theorem (see, for example, [11, Corollary 10.17]) shows that $G/C_G(D_j)$ is soluble-by-finite. If G is periodic, then $G/C_G(D_j)$ is finite (see, for example, [11, Theorem 9.33]). It is valid for each j, $1 \leq j \leq n$. We have $C_G(D) = \bigcap_{1 \leq j \leq n} C_G(D_j)$, and therefore using Remak's Theorem we obtain the embedding

$$G/C_G(D) \hookrightarrow \mathbf{Dr}_{1 \leq j \leq n} G/C_G(D_j),$$

which shows that $G/C_G(D)$ is also soluble-by-finite (respectively, for periodic G it is finite). Since K/D is finite, $G/C_G(K/D)$ is finite. Finally, $G/C_G(B/K)$ is isomorphic to a subgroup of $\mathbf{GL}_r(\mathbb{Q})$, where $\mathbf{r} = \mathbf{r}_{\mathbb{Z}}(B/K)$. Using again the fact that $G/C_G(A/K)$ does not include the noncyclic free subgroup and Tits' Theorem or Theorem 9.33 of [11] (for periodic G), we obtain that $G/C_G(B/K)$ is soluble-by-finite (respectively finite whenever G is periodic). Put

$$Z = C_G(D) \cap C_G(K/D) \cap C_G(B/K).$$

Then G/Z is embedded in $G/C_G(D) \cap G/C_G(K/D) \cap G/C_G(B/K)$, and in particular, G/Z is soluble-by-finite (respectively finite). If $x \in Z$, then x acts trivially on every factor of the series $\langle 0 \rangle \leq D \leq K \leq A$. Then Z is nilpotent [6]. It follows that G is soluble-by-finite (respectively, for periodic G, it is nilpotent-by-finite). This completes the proof of (i) and (ii).

Now we prove (iii). Suppose now that G is an $\mathfrak{F}$-perfect group. Again consider the series of G-invariant subgroups $\langle 0 \rangle \leq K \leq B$. Being abelian and Chernikov, K is a union of ascending series

$$\langle 0 \rangle = K_0 \leq K_1 \leq ... \leq K_n \leq K_{n+1} \leq ...$$

of G-invariant finite subgroups K_n, $n \in \mathbb{N}$. Then the factor-group $G/C_G(K_n)$ is finite, $n \in \mathbb{N}$. Since G is $\mathfrak{F}$-perfect, $G = C_G(K_n)$ for each $n \in \mathbb{N}$. The equation $K = \bigcup_{n \in \mathbb{N}} K_n$ implies that $G = C_G(K)$. By the above, $G/C_G(B/K)$ is soluble-by-finite, and being $\mathfrak{F}$-perfect, it is soluble. Then $G/C_G(B/K)$ includes normal subgroups U, V such that $C_G(B/K) \leq U \leq V$, $U/C_G(B/K)$ is isomorphic to a subgroup of $\mathbf{UT}_r(\mathbb{Q})$, and V/U includes a free abelian subgroup of finite index [1, Theorem 2]. Since $G/C_G(B/K)$ is $\mathfrak{F}$- perfect, it follows that $G/C_G(B/K)$ is torsion-free. Being periodic, $G/C_G(B/K)$ must be identity. In other words, $G = C_G(B/K)$. Hence, G acts trivially on every factor of the series $\langle 0 \rangle \leq K \leq A$, so that $[[B, G], G] = \langle 0 \rangle$ and we obtain that G is abelian [6]. The result is proved. □

Corollary *Let G be a group and A a $\mathbb{Z}G$-module. If the factor-module $A/C_A(G)$ is Artinian-by-(finite rank), then every locally generalized radical subgroup of $G/C_G(A)$ is soluble-by-finite, and every periodic subgroup of $G/C_G(A)$ is nilpotent-by-finite.*

Indeed, Lemma 2 shows that $G/C_G(A/C_A(G))$ is soluble-by-finite. Every element $x \in C_G(A/C_A(G))$ acts trivially in the factors of the series $\langle 0 \rangle \leq C_A(G) \leq A$. It follows that $C_G(A/C_A(G))$ is abelian. Suppose now that $H/C_G(A)$ is a periodic subgroup. Since $A/C_A(G)$ is Artinian-by-(finite rank), A has a series of H-invariant subgroups $\langle 0 \rangle \leq C_A(G) \leq D \leq K \leq A$ where $D/C_A(G)$ is a divisible Chernikov subgroup, K/D is finite, and A/K is torsion-free and has finite $\mathbb{Z}$-rank. In Lemma 2 we have already proven that $G/C_G(D/C_A(G))$, $G/C_G(K/D)$, and $G/C_G(A/K)$ are finite. Let $Z = C_G(D/C_A(G)) \cap C_G(K/D) \cap C_G(A/K)$. Then G/Z is finite. If $x \in Z$, then x acts trivially on every factor of the series $\langle 0 \rangle \leq C_A(G) \leq D \leq K \leq A$. Therefore, Z is nilpotent [6].

The next result is well known, but we were not able to find an appropriate reference, so we prove it here.

Lemma 3 *Let G be an abelian group. Suppose that $G \neq KL$ for arbitrary proper subgroups K, L. Then G is a cyclic or quasicyclic p-group for some prime p.*

Proof If G is finite, then it is not hard to see that G is a cyclic p-group for some prime p. Therefore, suppose that G is infinite. If G is periodic, then obviously G is a p-group for some prime p. Let B be a basic subgroup of G; that is, B is a pure subgroup of G such that B is a direct product of cyclic p-subgroups and G/B is divisible. The existence of such subgroups follows from [3, Theorem 32.3]. Since G/B is divisible, $G/B = \mathbf{Dr}_{\lambda \in \Lambda} D_\lambda$ where D_λ is a quasicyclic subgroup for every $\lambda \in \Lambda$ (see, for example, [3, Theorem 23.1]). Our condition shows that G/B is a quasicyclic group. In particular, if $B = \langle 1 \rangle$, then G is a quasicyclic group. Assume that $B \neq \langle 1 \rangle$. If B is a bounded subgroup, then $G = B \times C$ for some subgroup C (see, for example, [3, Theorem 27.5]), and we obtain a contradiction. Suppose that B is not bounded. Then B includes a subgroup $C = \mathbf{Dr}_{n \in \mathbb{N}} \langle c_n \rangle$ such that $B = C \times U$ for some subgroup U and $|c_n| = p^n$, $n \in \mathbb{N}$. Let $E = \langle c_n^{-1} \cdot c_{n+1}^p \mid n \in \mathbb{N} \rangle$. Then the factor-group C/E is quasicyclic, so that B/EU is also quasicyclic. It follows that G/EU is a direct product of 2 quasicyclic subgroups, which yields a contradiction. This shows that $B = \langle 1 \rangle$, which proves our result. □

Corollary *Let G be a soluble group. Suppose that G is not finitely generated and $G \neq \langle K, L \rangle$ for arbitrary proper subgroups K, L. Then $G/[G,G]$ is a quasicyclic p-group for some prime p.*

If G is a group, then by $\mathbf{Tor}(G)$ we will denote the maximal normal periodic subgroup of G. We recall that if G is a locally nilpotent group, then $\mathbf{Tor}(G)$ is a (characteristic) subgroup of G and $G/\mathbf{Tor}(G)$ is torsion-free.

3. Proof of main theorem

Again suppose that $C_G(A) = \langle 1 \rangle$. Suppose that G is a finitely generated group. Then we can choose a finite subset M such that $G = \langle M \rangle$, but $G \neq \langle S \rangle$ for every subset $S \neq M$. If $|M| > 1$, then $M = \{x\} \cup S$ where $x \notin S$ and $S \neq \varnothing$. It follows that $\langle S \rangle = U \neq G$, and thus $A/C_A(U)$ is Artinian-by-(finite rank). The factor $A/C_A(x)$ is also Artinian-by-(finite rank), and Lemma 1 shows that $\langle x, U \rangle = \langle x, S \rangle = G$ has an Artinian-by-(finite rank) cocentralizer.

Suppose that $M = \{y\}$; that is, $G = \langle y \rangle$ is a cyclic group. If y has infinite order, then $\langle y \rangle = \langle y^p \rangle \langle y^q \rangle$ where p, q are primes, $p \neq q$, and Lemma 1 again implies that $A/C_A(G)$ is Artinian-by-(finite rank). Finally, if y has finite order, but this order is not a prime power, then $\langle y \rangle$ is a product of 2 proper subgroups, and Lemma 1 implies that $A/C_A(G)$ is Artinian-by-(finite rank).

Assume now that G is not finitely generated and $A/C_A(G)$ is not Artinian-by-(finite rank). Suppose that G includes a proper subgroup of finite index. Then G includes a proper normal subgroup H of finite index. We can choose a finitely generated subgroup F such that $G = HF$. Since G is not finitely generated, $F \neq G$. It follows that cocentralizers of both subgroups H and F are Artinian-by-(finite rank). Lemma 1 shows that $FH = G$ has an Artinian-by-(finite rank) cocentralizer, and we obtain a contradiction. This contradiction shows that G is an $\mathfrak{F}$-perfect group.

If H is a proper subgroup of G, then Corollary to Lemma 2 shows that H is soluble-by-finite. In particular, G is locally (soluble-by-finite). By Theorem A of [2], G includes a normal locally soluble subgroup L such that G/L is a finite or locally finite simple group. Since G is an $\mathfrak{F}$-perfect group, then in the first case $G = L$, i.e. G is locally soluble. Consider the second case. Put $C = C_A(L)$. In a natural way, we can consider C as a $\mathbb{Z}(G/L)$-module. $C_{G/L}(C)$ is a normal subgroup of G/L. Since G/L is a simple group, then either $C_{G/L}(C)$ is the identity subgroup or $C_{G/L}(C) = G/L$. In the second case, $C \leq C_A(G)$ and $A/C_A(G)$ is Artinian-by-(finite rank). This contradiction shows that $C_{G/L}(C) =< 1 >$. Let H/L be an arbitrary proper subgroup of G/L. Then H is a proper subgroup of G, and therefore $A/C_A(H)$ is Artinian-by-(finite rank). It follows that $C/(C \cap C_A(H))$ is also Artinian-by-(finite rank). Clearly $C_C(H/L) \leq C \cap C_A(H)$, so that $C/C_C(H/L)$ is Artinian-by-(finite rank). Since H/L is periodic, it is nilpotent-by-finite by Corollary to Lemma 2. In other words, every proper subgroup of G/L is nilpotent-by-finite. Using now Theorem A of [9], we obtain that either G/L is soluble-by-finite or a p-group for some prime p. In any case, G/L cannot be an infinite simple group. This contradiction shows that G is locally soluble. Being an infinite locally soluble group, G has a nonidentity proper normal subgroup. The Corollary to Lemma 2 shows that this subgroup is soluble. It follows that G includes a nonidentity normal abelian subgroup. In turn, it follows that the locally nilpotent radical R_1 of G is nonidentity. Suppose that $G \neq R_1$. Being $\mathfrak{F}$-perfect, G/R_1 is infinite. Using the above arguments, we obtain that the locally nilpotent radical R_2/R_1 of G/R_1 is nonidentity. If $G \neq R_2$, then the locally nilpotent radical R_3/R_2 of G/R_2 is nonidentity, and so on. Using ordinary induction, we obtain that G is a radical group. Suppose that the upper radical series of G is infinite and consider its term R_ω, where ω is the first infinite ordinal. By its choice, R_ω is not soluble. Then the Corollary to Lemma 2 shows that $R_\omega = G$.

Since R_n is a proper subgroup of G, $A/C_A(R_n)$ is Artinian-by-(finite rank), $n \in \mathbb{N}$. R_n is normal in G, and therefore $C_A(R_n)$ is a $\mathbb{Z}G$-submodule. Lemma 3 shows that $G/C_G(A/C_A(R_n))$ is abelian. Suppose that there exists a positive integer m such that $G \neq C_G(A/C_A(R_m))$, and then $[G,G]$ is a proper subgroup of G. An application of the Corollary to Lemma 2 shows that $[G,G]$ is soluble, and thus even G is soluble. This contradiction proves the equality $G = C_G(A/C_A(R_n))$. In other words, $[A,G] \leq C_A(R_n)$. Since it is valid for each $n \in \mathbb{N}$, $[A,G] \leq \bigcap_{n\in\mathbb{N}} C_A(R_n)$. The equation $G = \bigcup_{n\in\mathbb{N}} R_n$ implies that $C_A(G) = \bigcap_{n\in\mathbb{N}} C_A(R_n)$. Hence, $[A,G] \leq C_A(G)$. Thus, G acts trivially on both factors $C_A(G)$ and $A/C_A(G)$, which follows that G is abelian [6]. Contradiction. This contradiction proves that G is soluble.

Let $D = [G,G]$. Then by the Corollary to Lemma 3 G/D is a quasicyclic p-group for some prime p. It follows that G has an ascending series of normal subgroups

$$D = K_0 \leq K_1 \leq ... \leq K_n \leq K_{n+1} \leq ...$$

such that K_n/D is a cyclic group of order p^n, $n \in \mathbb{N}$, and $G = \bigcup_{n\in\mathbb{N}} K_n$. Every subgroup K_n is proper and normal in G, and therefore $C_A(K_n)$ is a $\mathbb{Z}G$-submodule and $A/C_A(K_n)$ is Artinian-by-(finite rank). Lemma 2 shows that $[[A,G],G] \leq C_A(K_n)$. It is valid for each $n \in \mathbb{N}$, and therefore $[[A,G],G] \leq \bigcap_{n\in\mathbb{N}} C_A(K_n)$. The equation $G = \bigcup_{n\in\mathbb{N}} K_n$ implies that $C_A(G) = \bigcap_{n\in\mathbb{N}} C_A(R_n)$. Hence, $[[A,G],G] \leq C_A(G)$. It follows that G acts trivially on factors $C_A(G)$, $[A,G]/C_A(G)$ and $A/[A,G]$. It follows that G is nilpotent of class at most 2 [6].

If G is abelian, then Lemma 3 shows that G is a cyclic or quasicyclic pğgroup for some prime p. Suppose that G is nonabelian. Let $T = \mathbf{Tor}(G)$. If we suppose that $T \neq G$, then G/T is a non-identity torsion-free nilpotent group. In particular, G/T has a nonidentity torsion-free abelian factor group, which contradicts the Corollary to Lemma 3. This contradiction shows that G is a periodic group. Moreover, G is a p-group. Since G is nilpotent of class 2, then $[G,G] \leq \zeta(G)$. In particular, $G/\zeta(G)$ is a quasicyclic group. In this case, $[G,G]$ is a Chernikov subgroup (see, for example, [7, Theorem 23.1]). It follows that the whole group G is Chernikov. Being $\mathfrak{F}$-perfect, G is abelian, which completes the proof.

We will construct the following example showing that for every quasicyclic p-group G there exists a $\mathbb{Z}G$ğmodule A such that every proper subgroup of G has an Artinian-by-(finite rank) cocentralizer, but the cocentralizer of the whole group G is not Artinian-by-(finite rank).

Let p be a prime and G a quasicyclic p-group; that is, $G = \langle g_n \mid n \in \mathbb{N}\rangle$, where $g_1^p = 1$, $g_2^p = g_1$, ..., $g_{n+1}^p = g_n$. Let $A_j = \langle a_j\rangle$ be an additively written infinite cyclic group, $j \in \mathbb{N}$, and $B = \langle b_n \mid n \in \mathbb{N}\rangle$ an additively written quasicyclic p-group; that is, $pb_1 = 0$, $pb_2 = b_1$, ..., $pb_{n+1} = b_n$, $n \in \mathbb{N}$. Put $A = \bigoplus_{j\in\mathbb{N}} A_j \oplus B$. Let γ_1 be an automorphism of A, satisfying the following conditions:

$$\gamma_1(a_1) = a_1 + b_1, \gamma_1(a_j) = a_j, \text{ whenever } j > 1, \gamma_1(b_n) = b_n \text{ for all } n \in \mathbb{N}.$$

Then $\gamma_1^p = \varepsilon$ is an identity automorphism of A. Consider the automorphism γ_2 of A, defined by the following rule:

$$\gamma_2(a_1) = a_1 + b_2, \gamma_2(a_2) = a_2 + b_1, \gamma_2(a_j) = a_j, \text{ whenever } j > 2, \gamma_2(b_n) = b_n \text{ for all } n \in \mathbb{N}.$$

Then $\gamma_2^p = \gamma_1$. In a similar way, if k is a positive integer, then define the automorphism γ_k of A by the following rule:

$$\gamma_k(a_1) = a_1 + b_k, \gamma_k(a_2) = a_2 + b_{k-1}, ..., \gamma_k(a_k) = a_k + b_1, \gamma_k(a_j) = a_j, \text{ whenever } j > k,$$

$$\gamma_k(b_n) = b_n \text{ for all } n \in \mathbb{N}.$$

It is not hard to prove that $\gamma_1^p = \varepsilon$, $\gamma_2^p = \gamma_1$, ..., $\gamma_{k+1}^p = \gamma_k$, $k \in \mathbb{N}$. In other words, $\Gamma = \langle \gamma_k \mid k \in \mathbb{N}\rangle$ is a quasicyclic p-group. Let f be an isomorphism of G on Γ such that $f(g_k) = \gamma_k$, $k \in \mathbb{N}$. Define the action of G on A by the rule $ag_k = \gamma_k(a)$, $k \in \mathbb{N}$. In a natural way, A becomes a $\mathbb{Z}G$-module. Furthermore, $C_A(g_k) = \bigoplus_{j>k} A_j \oplus B$, so that $A/C_A(g_k) = A_1 \oplus ... \oplus A_k$, and in particular, $A/C_A(g_k)$ is a free $\mathbb{Z}$-module and its $\mathbb{Z}$-rank is k, $k \in \mathbb{N}$. At the same time, $C_A(G) = B$, and therefore $A/C_A(G)$ has infinite $\mathbb{Z}$-rank, and hence is not Artinian-by-(finite rank).

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