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On direct products of S-posets satisfying flatness properties

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Abstract: In this paper we characterize pomonoids over which various flatness properties of S-posets are preserved under direct products.

1. Introduction

A monoid S that is also a partially ordered set, in which the binary operation and the order relation are compatible, is called a pomonoid. A right S-poset A_S is a right S-act A equipped with a partial order $\leq$ and, in addition, for all $s, t \in S$ and $a, b \in A$, if $s \leq t$ then $as \leq at$, and if $a \leq b$ then $as \leq bs$. An S-subposet of a right S-poset A is a subset of A that is closed under the S-action. The definition of ideal is the same for the act case. Moreover, $X \subseteq S$ and take $(X] = \{p \in S \mid \ \exists x \in X, p \leq x\}$. Finally, an S-morphism from S-poset A to S-poset C is a monotonic map that preserves S-action.

Let A be a right S-poset, B a left S-poset. The order relation on $A_S \otimes {}_SB$ can be described as follows: $a \otimes b \leq a' \otimes b'$ holds in $A_S \otimes {}_SB$ if and only if there exist $s_1, \ldots, s_n, t_1, \ldots, t_n \in S$, $a_1, \ldots, a_n \in A_S$, $b_2, \ldots, b_n \in {}_SB$ such that

$$\begin{array}{ll} a \leq a_1 s_1 & \\ a_1 t_1 \leq a_2 s_2 & s_1 b \leq t_1 b_2 \\ \vdots & \vdots \\ a_n t_n \leq a' & s_n b_n \leq t_n b'. \end{array}$$

When $B = Sb$ and $b = b'$, in the above scheme we can replace all b_i by b. Moreover, $a \otimes b = a' \otimes b'$ if $a \otimes b \leq a' \otimes b'$ and $a' \otimes b' \leq a \otimes b$. More information about tensor products in S-posets can be found in [12]. A right S-poset A_S is weakly po-flat if $a \otimes s \leq a' \otimes t$ in $A_S \otimes S$ (equivalently, $as \leq a't$) implies that the same inequality holds also in $A_S \otimes {}_S(Ss \cup St)$ for $a, a' \in A_S, s, t \in S$. A right S-poset A_S is principally weakly po-flat if $as \leq a's$ implies that $a \otimes s \leq a' \otimes s$ in $A_S \otimes {}_SSs$ for $a, a' \in A_S, s \in S$. Weakly flat and principally weakly flat can be defined the same as the previous by replacing $\leq$ by $=$.

An S-poset A_S satisfies condition (P_w) if, for all $a, b \in A$ and $s, t \in S$, $as \leq bt$ implies $a \leq a'u$, $a'v \leq b$ for some $a' \in A$, $u, v \in S$ with $us \leq vt$. A right S-poset A_S satisfies condition (P) if, for all $a, b \in A$ and $s, t \in S$, $as \leq bt$ implies $a = a'u, b = a'v$ for some $a' \in A$, $u, v \in S$ with $us \leq vt$, and it satisfies condition (E)

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if, for all $a \in A$ and $s, t \in S$, $as \leq at$ implies $a = a'u$ for some $a' \in A$, $u \in S$ with $us \leq ut$. A right S-poset is called strongly flat if it satisfies both conditions (P) and (E). Projectivity is defined in the standard categorical manner.

In [1], Bulman-Fleming characterized monoids over which direct products of projective acts are projective. Gould in [6] then solved this problem for strongly flat and conditions (P) and (E). Meanwhile, Bulman-Fleming and McDowell [4] defined a monoid to be right coherent if every direct product of flat S-acts is flat, and they obtained some results when S^Γ is (principally) weakly flat for a monoid S. In [2], Bulman-Fleming and Gilmour discussed when $S \times S$ has certain flatness properties. Then in [9], principally weakly left coherent monoids were characterized as monoids over which direct products of nonempty families of principally weakly flat right S-acts are principally weakly flat. The reader is referred to the monograph [8] for a complete discussion of flatness properties and definition of acts over monoids. On the other hand, the investigation of S-posets was initiated by Fakhruddin in the 1980s, and recently many papers on this topic have appeared, mostly concentrating on projectivity and various notions of flatness for S-posets, such as [5, 3, 7, 11, 10]. Following Section 1, we give some preliminaries about the S-poset S^Γ, where Γ is a nonempty set and tensor product. In Section 2, we investigate products of (po-)torsion free, principally weakly and weakly (po-)flat S-posets. In Section 3, conditions (P), (E), (P_w), and strongly flatness are considered. Finally, in Section 4, products of projective S-posets are studied.

If S is a pomonoid, the Cartesian product S^Γ is a right and left S-poset equipped with the order and the action componentwise where Γ is a nonempty set. Moreover, $(s_\gamma)_{\gamma\in\Gamma} \in S^\Gamma$ is dented simply by (s_γ), and the right S-poset $S \times S$ will be denoted by $D(S)$.

Recall that an S-poset morphism $f : A_S \to B_S$ is called *order-embedding* if $f(a) \leq f(a')$ implies $a \leq a'$, for all $a, a' \in A$. The proof of the following lemma is routine.

Lemma 1.1 *Let S be a pomonoid, Γ any nonempty set, and I a left ideal of S. Then the following are equivalent:*

(i) $S^\Gamma \otimes I \to S^\Gamma \otimes S$ *is order-embedding;*

(ii) $S^\Gamma \otimes I \to I^\Gamma$ *is order-embedding.*

Proposition 1.2 *Let S be a pomonoid and $s \in S$. Then the following are equivalent:*

(i) $f_s : S^\Gamma \otimes Ss \to (Ss)^\Gamma$ *is order-embedding for all* $\Gamma \neq \emptyset$*;*

(ii) there exist $(s_1, t_1),\ ..., (s_n, t_n) \in D(S)$ *such that*

(1) $s_i s \leq t_i s$ *for all* $1 \leq i \leq n$*, and*

(2) *if* $us \leq vs$ *for some* $u, v \in S$*, then there exist* $u_1, ..., u_n \in S$ *such that*

$$\begin{aligned} u &\leq u_1 s_1 \\ u_1 t_1 &\leq u_2 s_2 \\ &\vdots \\ u_n t_n &\leq v. \end{aligned}$$

Proof *(i)* $\Rightarrow$ *(ii)* Let $L = \{(u, v) \in D(S) |\ us \leq vs\}$, and index L by $L = \{(u_\gamma, v_\gamma) |\ \gamma \in \Gamma\}$. Since $(u_\gamma)s \leq (v_\gamma)s$ in S^Γ, then, by *(i)*, $(u_\gamma) \otimes s \leq (v_\gamma) \otimes s$ in $S^\Gamma \otimes Ss$. So there exist $s_1, \ldots, s_n, t_1, \ldots, t_n \in$

S, $(u_\gamma^1), \ldots, (u_\gamma^n) \in S^\Gamma$ such that

$$\begin{array}{cc} (u_\gamma) \leq (u_\gamma^1)s_1 & \\ (u_\gamma^1)t_1 \leq (u_\gamma^2)s_2 & s_1 s \leq t_1 s \\ \vdots & \vdots \\ (u_\gamma^n)t_n \leq (v_\gamma) & s_n s \leq t_n s. \end{array}$$

So the result is easily checked.

$(ii) \Rightarrow (i)$ Let $\Gamma \neq \emptyset$, and let $(u_\gamma), (v_\gamma) \in S^\Gamma$ be such that $(u_\gamma)s \leq (v_\gamma)s$ in $(Ss)^\Gamma$. By (ii), there exist (s_1, t_1), ..., $(s_n, t_n) \in D(S)$ such that $s_i s \leq t_i s$ for all $1 \leq i \leq n$, and there exist $u_\gamma^l, ..., u_\gamma^n \in S$ for all $\gamma \in \Gamma$ such that

$$\begin{array}{c} u_\gamma \leq u_\gamma^1 s_1 \\ u_\gamma^1 t_1 \leq u_\gamma^2 s_2 \\ \vdots \\ u_\gamma^n t_n \leq v_\gamma. \end{array}$$

Thus $(u_\gamma) \otimes s \leq (u_\gamma^1)s_1 \otimes s \leq (u_\gamma^1) \otimes s_1 s \leq (u_\gamma^1) \otimes t_1 s \leq (u_\gamma^1)t_1 \otimes s \leq (u_\gamma^2)s_2 \otimes s \leq ... \leq (v_\gamma) \otimes s$ in $S^\Gamma \otimes Ss$, as required. □

2. Po-torsion free, principally weakly (po-)flat and weakly po-flat

In this section we consider direct products of (po-)torsion free, principally weakly, and weakly (po-)flat S-posets. Specifically, when S^Γ is principally weakly and weakly (po-)flat is studied. First, we begin our investigation with the weakest of the flatness properties. An element c of a pomonoid S will be called *right po-cancelable* if, for all $s, t \in S$, $sc \leq tc$ implies $s \leq t$. A right S-poset A_S is called po-torsion (torsion) free if, for $a, a' \in A$ and a right po-cancelable (cancelable) element c of S, from $ac \leq a'c$ $(ac = a'c)$ it follows that $a \leq a'$ $(a = a')$. The proof of the following result is immediately evident.

Proposition 2.1 *For any pomonoid S direct products of po-torsion (torsion) free S-posets are again po-torsion (torsion) free.*

Recall that a pomonoid S is called a left PSF pomonoid if all principal left ideals of a pomonoid S are strongly flat. Let S be a pomonoid. An element $u \in S$ is called *right semi-po-cancelable* if for $s, t \in S, su \leq tu$ implies that there exists $r \in S$ such that $ru = u$, $sr \leq tr$. In [11], it is shown that a pomonoid S is left PSF pomonoid if and only if every element of S is right semi-po-cancelable.

Lemma 2.2 *([11]) Over a left PSF pomonoid S a right S-poset A_S is principally weakly po-flat if and only if for any $a, a' \in A_S, s \in S$, if $as \leq a's$, then there exists $r \in S$ such that $rs = s$ and $ar \leq a'r$.*

Proposition 2.3 *If S is a left PSF pomonoid, then the S-poset S^n is principally weakly po-flat for each $n \in \mathbb{N}$.*

Proof Suppose that $(x_1,\ldots,x_n)s \leq (y_1,\ldots,y_n)s$. Since $x_1s \leq y_1s$ and S is left PSF pomonoid, there is $r_1 \in S$ such that $r_1s = s$ and $x_1r_1 \leq y_1r_1$. By the equality $x_2r_1s \leq y_2r_1s$ we get $r_2 \in S$ such that $r_2s = s$ and $x_2r_1r_2 \leq y_2r_1r_2$. Continuing this process, we obtain $r_1,\ldots,r_n \in S$ with $r_is = s$ and $x_ir_1\ldots r_i \leq y_ir_1\ldots r_i$ for each $1 \leq i \leq n$. Put $r = r_1\ldots r_n$. Thus $(x_1,\ldots,x_n)r \leq (y_1,\ldots,y_n)r$ and $rs = s$. Applying Lemma 2.2, we obtain our assertion. □

Since principally weakly po-flat implies principally weakly flat, over a left PSF pomonoid S, S^n is also principally weakly flat.

Using Lemma 1.1 and Proposition 1.2, we get the following proposition.

Proposition 2.4 *The following are equivalent for a pomonoid S:*

(i) S_S^Γ is principally weakly po-flat for each nonempty set Γ;

(ii) For any $s \in S$, the mapping $f_s : S^\Gamma \otimes Ss \longrightarrow (Ss)^\Gamma$ is order-embedding for each nonempty set Γ;

(iii) For any $s \in S$ there exist $(s_1,t_1),\ldots,(s_n,t_n) \in D(S)$ such that

(1) $s_is \leq t_it$ for all $1 \leq i \leq n$, and

(2) if $us \leq vs$ $(u,v \in S)$, then there exist $u_1,...,u_n \in S$ such that

$$\begin{aligned} u &\leq u_1s_1\\ u_1t_1 &\leq u_2s_2\\ &\vdots\\ u_nt_n &\leq v. \end{aligned}$$

In [11], it is shown that a right S-poset A_S is weakly po-flat if and only if it is principally weakly po-flat and satisfies condition (W):

If $as \leq a't$ for $a,a' \in A_S$, $s,t \in S$, then there exist $a'' \in A_S$, $p \in Ss$ and $q \in St$ such that $p \leq q$, $as \leq a''p$, $a''q \leq a't$.

For each $(p,q) \in D(S)$, $\{(u,v) \in D(S) |\ \exists w \in S, u \leq wp, wq \leq v\}$ is a left S-poset and will be denoted by $\widehat{S(p,q)}$ from now on. Clearly $\widehat{S(p,q)}$ contains the cyclic S-poset $S(p,q)$. Moreover, if $Ss \cap (St] \neq \emptyset$, $\{(as,a't) |\ as \leq a't\}$ is denoted by $H(s,t)$.

Proposition 2.5 *The diagonal S-poset $D(S)$ is weakly po-flat if and only if it is principally weakly po-flat and $Ss \cap (St] \neq \emptyset$ or for each $(as,a't)$ and $(bs,b't)$ in $H(s,t)$ there exist $(p,q) \in H(s,t)$ such that $(as,a't),(bs,b't) \in \widehat{S(p,q)}$.*

Proof We show that, for any pomonoid S, $D(S)$ satisfying condition (W) is equivalent to the second condition of this proposition. First suppose that $D(S)$ satisfies condition (W), and let $s,t \in S$, and $Ss \cap (St] \neq \emptyset$. Suppose that $(as,a't)$, $(bs,b't) \in H(s,t)$. Then we have $(a,b)s \leq (a',b')t$. By condition (W), $(a,b)s \leq (a'',b'')p$, $(a'',b'')q \leq (a',b')t$ for some $p \in Ss, q \in St, p \leq q$, and $(a'',b'') \in D(S)$. Therefore, $as \leq a''p$, $a''q \leq a't$, $bs \leq b''p, b''q \leq b't$, and so $(as,a't),(bs,b't) \in \widehat{S(p,q)}$.

Now suppose that $(a,b)s \leq (a',b')t$ for (a,b), $(a',b') \in D(S)$, $s,t \in S$. Then $Ss \cap (St] \neq \emptyset$, and since $as \leq a't$ and $bs \leq b't$, by assumption $(as,a't),(bs,b't) \in \widehat{S(p,q)}$ for some $(p,q \in H(s,t)$. So

$p \in Ss,\ q \in St,\ p \leq q$, and there exist $a'', b'' \in S$ such that $as \leq a''p,\ a''q \leq a't,\ bs \leq b''p, b''q \leq b't$. Then $(a,b)s \leq (a'',b'')p,\ (a'',b'')q \leq (a',b')t$ and so $D(S)$ satisfies condition (W). □

Definition 2.6 Let S be a pomonoid. A finitely generated left S-poset ${}_SB$ is called *finitely definable (FD)* if the S-morphism $S^\Gamma \otimes B \to B^\Gamma$ is order-embedding for all nonempty sets Γ.

Theorem 2.7 *The following are equivalent for a pomonoid S:*

(i) *S^Γ is weakly po-flat right S-poset for each $\Gamma \neq \emptyset$;*

(ii) *every finitely generated left ideal of S is FD;*

(iii) *Ss is FD for each $s \in S$, and*

for every $s,t \in S$, if $Ss \cap (St] \neq \emptyset$, then $H(s,t) \subseteq \widehat{S(p,q)}$ for some $(p,q) \in H(s,t)$.

Proof The equivalence of (i) and (ii) is clear.

$(i) \Rightarrow (iii)$ The first part is obvious. Let $s,t \in S$ such that $Ss \cap (St] \neq \emptyset$. Index the set $H(s,t)$ by $H(s,t) = \{(u_\gamma s, v_\gamma t) |\ \gamma \in \Gamma\}$. Since $S^\Gamma \otimes (Ss \cup St) \to (Ss \cup St)^\Gamma$ is order-embedding and $(u_\gamma)s \leq (v_\gamma)t$, then $(u_\gamma) \otimes s \leq (v_\gamma) \otimes t$ in $S^\Gamma \otimes (Ss \cup St)$. So there exist $s_1, \ldots, s_n, t_1, \ldots, t_n \in S,\ (u_\gamma^1), \ldots, (u_\gamma^n) \in S^\Gamma, b_2, \ldots, b_n \in Ss \cup St$ such that

$$\begin{array}{ll} (u_\gamma) \leq (u_\gamma^1)s_1 & \\ (u_\gamma^1)t_1 \leq (u_\gamma^2)s_2 & s_1 s \leq t_1 b_2 \\ \vdots & \vdots \\ (u_\gamma^n)t_n \leq (v_\gamma) & s_n b_n \leq t_n t. \end{array}$$

Let k be the smallest integer such that $b_k \in St$. So $b_{k-1} \in Ss$ and $s_{k-1}b_{k-1} \leq t_{k-1}b_k$. Take $p = s_{k-1}b_{k-1}$ and $q = t_{k-1}b_k$. Thus $(u_\gamma)s \leq (u_\gamma^1)s_1 s \leq (u_\gamma^1)t_1 b_2 \leq (u_\gamma^2)s_2 b_2 \leq \ldots \leq (u_\gamma^{k-1})s_{k-1}b_{k-1} \leq (u_\gamma^{k-1})t_{k-1}b_k \leq \ldots \leq (v_\gamma)t$. Then $(u_\gamma)s \leq (u_\gamma^{k-1})p,\ (u_\gamma^{k-1})q \leq (v_\gamma)t$, and so $H(s,t) \subseteq \widehat{S(p,q)}$.

$(iii) \Rightarrow (ii)$ Let I be a left ideal of S, and $(u_\gamma)s \leq (v_\gamma)t$ for some $(u_\gamma),(v_\gamma) \in S^\Gamma,\ s,t \in I$. By (iii), $H(s,t) \subseteq \widehat{S(p,q)}$ for some $(p,q) \in H(s,t)$. Thus there exists $w_\gamma \in S$ such that $u_\gamma s \leq w_\gamma p,\ w_\gamma q \leq v_\gamma t$ for each $\gamma \in \Gamma$. Take $p = cs, q = dt$ for some $c,d \in S$. Since Ss and St are FD and $(u_\gamma)s \leq (w_\gamma c)s, (w_\gamma d)t \leq (v_\gamma)t$, we have $(u_\gamma) \otimes s \leq (w_\gamma c) \otimes s, (w_\gamma d) \otimes t \leq (v_\gamma) \otimes t$ in $S^\Gamma \otimes Ss$ and $S^\Gamma \otimes St$, respectively. Therefore, $(u_\gamma) \otimes s \leq (w_\gamma c) \otimes s = (w_\gamma) \otimes cs \leq (w_\gamma) \otimes dt = (w_\gamma d) \otimes t \leq (v_\gamma) \otimes t$ in $S^\Gamma \otimes (Ss \cup St)$, as required. □

3. Conditions (P) and (P_w), and strongly flat

In this section a characterization of pomonoids over which direct products of S-posets satisfying conditions (P), (E), and (P_w) again satisfy that condition is given. First, we focus our attention on finite direct products of S-posets satisfying conditions (P), (E), and (P_w).

In [7] the ordered version of locally cyclic acts is called a *weakly locally cyclic* S-poset as an S-poset A such that every finitely generated S-subposet of A is contained in a cyclic S-poset. Moreover, a principal left ideal of S that is also weakly locally cyclic is called *weakly locally principal left ideal*. The set $L(a,b) := \{(u,v) \in D(S)|\ ua \leq vb\}$ is a left S-subposet of $D(S)$, and the set $(l(a,b) := \{u \in S|\ ua \leq ub\})$ is a left ideal of S.

Proposition 3.1 *For any pomonoid S the following are equivalent:*

(i) any finite product of right S-posets satisfying condition (P) (condition (E)) satisfies condition (P) (condition (E));

(ii) the diagonal S-poset $D(S)$ satisfies condition (P) (condition (E));

(iii) for every $a,b \in S$ the set $L(a,b)$ $(l(a,b))$ is either empty or a weakly locally cyclic left S-poset (weakly locally principal left ideal of S).

Proof $(i) \Rightarrow (ii)$ is clear. $(ii) \Rightarrow (iii)$ Suppose that $D(S)$ satisfies condition (P), and suppose $(u,v),(u',v') \in L(a,b)$, where $a,b \in S$. Since $ua \leq vb$ and $u'a \leq v'b$ we obtain $(u,u')a \leq (v,v')b$, by condition (P), there exist $(w,w') \in D(S)$ and $p,q \in S$ such that $(w,w')p = (u,u')$, $(w,w')q = (v,v')$, and $pa \leq qb$. From this it follows that $(u,v),(u',v') \in S(p,q) \subseteq L(a,b)$ and so $L(a,b)$ is weakly locally cyclic.

$(iii) \Rightarrow (i)$ Suppose that $A_1,...,A_n$ are right S-posets each satisfying condition (P). Let $a_i,a'_i \in A_i$ for each i, and let $u,v \in S$ and suppose $(a_1,...,a_n)u \leq (a'_1,...,a'_n)v$ in $A = \prod_{i=1}^n A_i$. For each i, from $a_iu \leq a'_iv$ and condition (P) for A_i we obtain $a''_i \in A_i$ and $p_i,q_i \in S$ such that $a''_ip_i = a_i$, $a''_iq_i = a'_i$, and $p_iu \leq q_iv$. Then $(p_i,q_i) \in L(u,v)$ for each i and so, by assumption, there exists $(p,q) \in D(S)$ such that $(p_i,q_i) \in S(p,q) \subseteq L(u,v)$. Suppose that $(p_i,q_i) = w_i(p,q)$ for $w_i \in S,\ 1 \leq i \leq n$. Then $(a_1,...,a_n) = (a''_1w_1,...,a''_nw_n)p$, $(a'_1,...,a'_n) = (a''_1w_1,...,a''_nw_n)q$, and $pu \leq qv$, proving that $A = \prod_{i=1}^n A_i$ satisfies condition (P). □

Proposition 3.2 *For any pomonoid S the following are equivalent:*

(i) any finite product of right S-posets satisfying condition (P_w) satisfies condition (P_w);

(ii) the diagonal S-poset $D(S)$ satisfies condition (P_w);

(iii) for every $a,b \in S$ the set $L(a,b)$ is either empty or for each 2 elements $(u,v),(u',v') \in L(a,b)$ there exists $(p,q) \in L(a,b)$ such that $(u,v),\ (u',v') \in \widehat{S(p,q)}$.

Proof $(i) \Rightarrow (ii)$ is clear. $(ii) \Rightarrow (iii)$ Suppose that $D(S)$ satisfies condition (P_w), and suppose that $(u,v),(u',v') \in L(a,b)$, where $a,b \in S$. Since $ua \leq vb$ and $u'a \leq v'b$ we obtain $(u,u')a \leq (v,v')b$, and condition (P_w) gives $(w,w') \in D(S)$ and $p,q \in S$ such that $(u,u') \leq (w,w')p$, $(v,v') \geq (w,w')q$, and $pa \leq qb$. So $(p,q) \in L(a,b)$ and we are done.

$(iii) \Rightarrow (i)$ Suppose that $A_1,...,A_n$ are right S-posets each satisfying condition (P_w). Suppose $a_i,a'_i \in A_i$ for each i, and let $u,v \in S$ be such that $(a_1,...,a_n)u \leq (a'_1,...,a'_n)v$ in $A = \prod_{i=1}^n A_i$. For each i, from $a_iu \leq a'_iv$ and condition (P_w) for A_i we obtain $a''_i \in A_i$ and $p_i,q_i \in S$ such that $a_i \leq a''_ip_i$, $a'_i \geq a''_iq_i$, and $p_iu \leq q_iv$. Then $(p_i,q_i) \in L(u,v)$ and so, by assumption, there exists $(p,q) \in L(u,v)$ such that $(p_i,q_i) \in \widehat{S(p,q)}$ for each $1 \leq i \leq n$. So $p_i \leq w_ip$, $q_i \geq w_iq$ for some $w_i \in S,\ 1 \leq i \leq n$. Then $(a_1,...,a_n) \leq (a''_1w_1,...,a''_nw_n)p$, $(a'_1,...,a'_n) \geq (a''_1w_1,...,a''_nw_n)q$, and $pu \leq qv$, proving that $A = \prod_{i=1}^n A_i$ satisfies condition (P_w). □

Now, we are going to discuss direct products of any arbitrary nonempty family of S-posets satisfying conditions (P), (E), and (P_w).

Theorem 3.3 *The following are equivalent for a pomonoid S:*

(i) the direct product of every nonempty family of right S-posets satisfying condition (P) (condition (E)) satisfies condition (P) (condition (E));

(ii) $(S^\Gamma)_S$ satisfies condition (P) (condition (E)) for every nonempty set Γ;

(iii) for every $a,b\in S$ the set $L(a;b)$ $(l(a,b))$ is either empty or a cyclic left S-poset (principal left ideal of S).

Proof $(i)\Rightarrow(ii)$ is clear. $(ii)\Rightarrow(iii)$ Suppose that $a,b\in S$ and $L(a,b)\neq\emptyset$. Index the set $L(a,b)$ by $L(a,b)=\{(u_\gamma,v_\gamma)|\ \gamma\in\Gamma\}$. Let $\overrightarrow{u},\overrightarrow{v}$ be the elements of S^Γ whose γth components are u_γ,v_γ respectively. Then $\overrightarrow{u}a\leq\overrightarrow{v}b$ in S^Γ and as S^Γ satisfies condition (P) by assumption, we have that $ua\leq vb$, $\overrightarrow{u}=\overrightarrow{z}p$ and $\overrightarrow{v}=\overrightarrow{z}q$ for some $p,q\in S$ and $\overrightarrow{z}\in S^\Gamma$. Thus $(p,q)\in L(a,b)$ so that $(p,q)=(u_j,v_j)$ for some $j\in\Gamma$. If $\gamma\in\Gamma$, then $(u_i\gamma,v_\gamma)=z_\gamma(p,q)=z_\gamma(u_j,v_j)$ where z_γ is the γth component of $\overrightarrow{z}$. This gives that $L(a,b)$ is cyclic. A similar argument applies for condition (E).

$(iii)\Rightarrow(i)$ Let $A=\prod_{i\in I}A_i$ be a product of right S-posets satisfying condition (P). Suppose that $\overrightarrow{x}a\leq\overrightarrow{y}b$ where $a,b\in S$ and $\overrightarrow{x}=(x_i),\overrightarrow{y}=(y_i)\in A$. For each $i\in I, x_ia\leq y_ib$ and so as A_i satisfies condition (P) there are elements $u_i,v_i\in S$ and $z_i\in A_i$ with $u_ia\leq v_ib, x_i=z_iu_i, y_i=z_iv_i$. So $(u_i,v_i)\in L(a,b)\neq\emptyset$ and by assumption it is cyclic, say $L(a,b)=S(p,q)$. Thus for each $i\in I$, $(u_i,v_i)=r_i(p,q)$ for some $r_i\in S$. We now have $pa\leq qb$ and $x_i=z_ir_ip, y_i=z_ir_iq$ for each $i\in I$. If $\overrightarrow{w}=(z_ir_i)_{i\in I}\in A$, then $\overrightarrow{x}=\overrightarrow{w}p$ and $\overrightarrow{y}=\overrightarrow{w}q$. With a similar argument for equalities of the form $\overrightarrow{x}a\leq\overrightarrow{x}b$ condition (E) implies. □

Theorem 3.4 *The following are equivalent for a pomonoid S:*

(i) the direct product of every nonempty family of right S-posets satisfying condition (P_w) satisfies condition (P_w);

(ii) $(S^\Gamma)_S$ satisfies condition (P_w) for every nonempty set Γ;

(iii) for every $a,b\in S$ the set $L(a,b)$ is either empty or there exists $(p,q)\in L(a,b)$ such that $L(a,b)=\widehat{S(p,q)}$.

Proof $(i)\Rightarrow(ii)$ is clear. $(ii)\Rightarrow(iii)$ Let $a,b\in S$ and $L(a,b)\neq\emptyset$. Write $L(a,b)=\{(u_\gamma,v_\gamma)|\ \gamma\in\Gamma\}$. Let $\overrightarrow{u},\overrightarrow{v}$ be the elements of S^Γ whose γth components are u_γ,v_γ respectively. Then $\overrightarrow{u}a\leq\overrightarrow{v}b$ in S^Γ and as S^Γ satisfies condition (P_w) by assumption, we have that $pa\leq qb$, $\overrightarrow{u}\leq\overrightarrow{z}p$ and $\overrightarrow{z}q\leq\overrightarrow{v}$ for some $p,q\in S$ and $\overrightarrow{z}\in S^\Gamma$. Thus $(p,q)\in L(a,b)$ and we have $u_\gamma\leq z_\gamma p$, $z_\gamma q\leq v_\gamma$ where z_γ is the γth component of $\overrightarrow{z}$. Therefore, $L(a,b)=\widehat{S(p,q)}$.

$(iii)\Rightarrow(i)$ Let $A=\prod_{i\in I}A_i$ be a product of right S-posets satisfying condition (P_w). Suppose that $Xa\leq Yb$ where $a,b\in S$ and $\overrightarrow{x}=(x_i),\overrightarrow{y}=(y_i)\in A$. For each $i\in I, x_ia\leq y_ib$ and so as A_i satisfies condition (P_w) there are elements $u_i,v_i\in S$ and $z_i\in A_i$ with $u_ia\leq v_ib$, $x_i\leq z_iu_i$, $z_iv_i\leq y_i$. So $(u_i,v_i)\in L(a,b)\neq\emptyset$ and by assumption there exists $(p,q)\in L(a,b)$ such that $L(a,b)=\widehat{S(p,q)}$. So for each $(u_i,v_i)\in L(a,b)$ there exists $r_i\in S$ with $u_i\leq r_ip$ and $r_iq\leq v_i$. Thus $pa\leq qb$ and $x_i\leq z_ir_ip$, $z_ir_iq\leq y_i$ for each $i\in I$. If

$\overrightarrow{w} = (z_i r_i)_{i \in I} \in A$, then $\overrightarrow{x} \leq \overrightarrow{w} p$ and $\overrightarrow{w} q \leq \overrightarrow{y}$. □

In light of Theorem 3.3, the following corollary holds.

Corollary 3.5 *The following are equivalent for a pomonoid S:*

(i) every product S^{Γ} is strongly flat right S-poset for a nonempty set Γ;

(ii) every product $\prod_{i \in I} A_i$ of strongly flat right S-posets A_i, $i \in I$, is strongly flat;

(iii) for all $(a,b) \in D(S)$, $L(a,b) \neq \emptyset$ or is cyclic left S-poset and $l(a,b) \neq \emptyset$ or is principal left ideal of S.

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