

Asymptotic analysis of the 2-dimensional soliton solutions for the Nizhnik–Veselov–Novikov equations

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Abstract: In this paper we present a direct approach to determining a class of solutions, the asymptotic analysis of the dromion solutions, and their asymptotic properties of the Nizhnik–Veselov–Novikov equations by means of Pfaffians. The form of the solution obtained allows a detailed asymptotic analysis of the dromion solutions and compact expression for the phase shifts and changes of amplitude as a result of interaction of the dromions to be determined.

Key words: Soliton, dromion

1. Introduction

In recent years the generalizations of integrable (1+1)-dimensional equations to (2+1) dimensions have been widely studied. The integrable generalization of the nonlinear Schrödinger (NLS) equation is the Davey–Stewartson (DS) equations [5]. The generalization of the Korteweg–de-Vries (KdV) equation has 2 possibilities, which are the Kadomtsev–Petviashvili (KP) equations [11] and the Nizhnik–Veselov–Novikov (NVN) equations [14]. The NVN equations are

$$U_t = U_{xxx} + U_{yyy} + 3(\Phi_{xx}U)_x + 3(\Phi_{yy}U)_y, \quad (1)$$

$$U = \Phi_{xy}. \quad (2)$$

These generalizations, the DS and NVN equations, have 2-dimensional localized hump solutions that decay exponentially in all directions, which are called 2-dimensional solitons or dromions. The KP equation does not have such solutions. The word dromion comes from the Greek word dromos, which means track, and it has been given [6] to these objects because they are located at the intersection of plane waves, which can be thought to form tracks. These 2-dimensional solitons, like the well-known solutions in (1 + 1)-dimensional systems, decay to zero exponentially as spatial variables tend to infinity and, as a result of interaction with other solitons, undergo a phase shift. However, unlike the (1 + 1)-dimensional situation, changes of amplitude because of interaction may also occur.

These (2+1)-dimensional generalizations also possess the usual features of (1+1)-dimension integrable equations, namely solvability by the inverse scattering transform, existence of Bäcklund transformations, and Hamiltonian formulation [1], [6], [16]. The NVN equations are a pair of coupled nonlinear equations in 2 dependent variables; U is the amplitude and Φ is related to an auxiliary field $V = \frac{1}{2}(\Phi_{xx} + \Phi_{yy})$, which is

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the velocity of an underlying mean flow. This mean flow is a feature of the 2-dimensional model having no counterpart in the 1-dimensional KdV equation. The soliton solution is localized only in the physical variable U , while the auxiliary variable V consists of a pair of perpendicular solitons. The point of intersection of the plane-wave solitons in V and the localized soliton in U coincide. The most general solution consists of 2 sets of perpendicular plane waves, M and N in number, in the V plane and $M \times N$ localized solitons in the U plane.

The NVN equations are the first member of the hierarchy describing the deformations preserving the zero energy level of 2-dimensional Schrödinger operators [21]. They also naturally arise in the theory of surfaces [18], and a modified version appears in string theory [12]. There are elliptic version of NVN equations (NVN-I) studied by Veselov and Novikov [21] and hyperbolic version of NVN equations (NVN-II) studied by Nizhnik [13].

An alternative approach has been through direct methods using the bilinear form of the DS and NVN equations. Hietarinta and Hirota [9] and Jaulent et al. [10] obtained a broader class of dromion solutions of the DS equations in terms of Wronskian determinants and as polynomials in exponentials, respectively. Athorne and Nimmo [2] and Ohta [15] obtained dromion solutions of the NVN equations in terms of Pfaffians. Gilson and Nimmo [7] studied the dromion solutions of the DS equations [3] and their asymptotic properties. Recently, there has been a number of studies in the asymptotic analysis of the NVN equations. Peng [17] and Kumara [19] used the Painlevé truncation approach. This approach converts the given evolution equation into a multilinear equation in terms of the noncharacteristic manifold. This multilinear equation can be solved in terms of lower-dimensional arbitrary functions of space and time. Zhou [22] used the Darboux transformations for the DSII equation to solve the NVN-II equation and obtained global multisoliton solutions.

In this paper, we wish to consider an alternative direct approach that uses a formulation of the solutions as Pfaffians. These solutions describe a class of solutions much broader than those obtained in the other approaches mentioned above. This paper is organized as follows. In Section 2, we introduce some properties of Pfaffians. In Section 3 we recall some results obtained in [2] and give a class of solutions of the NVN equations, which uses a formulation of the solutions as Pfaffians. In Section 4 we wish to consider asymptotic analysis of the (1,1)-dromion solution. In Section 5 we will see that the form of the solution taken allows us to carry out a detailed asymptotic analysis of the (M, N) -dromion solution and obtain compact expressions for the phase shifts and changes in amplitude that occur due to the interactions.

2. Introduction to Pfaffians

Let $A = (a_{ij})$ be an $n \times n$ skew-symmetric matrix (i.e. $a_{ij} = -a_{ji}$ and consequently $a_{ii} = 0$ for $i, j = 1, 2, \dots, n$). It is known that if n is odd, then $\det(A)$ is zero, but if n is even $\det(A)$ is a perfect square of a polynomial in the entries a_{ij} , called the *Pfaffian* of A and denoted by $\text{Pf}(A)$. Roughly speaking, a Pfaffian is the square root of the determinant of a skew-symmetric matrix. To be precise, for even n

$$\text{Pf}(A) = \begin{vmatrix} a_{12} & a_{13} & \cdots & a_{1n} \\ & a_{23} & \cdots & a_{2n} \\ & & \ddots & \vdots \\ & & & a_{n-1,n} \end{vmatrix} = \sum_{\sigma} \epsilon(\sigma) a_{\sigma(1),\sigma(2)} \cdots a_{\sigma(n-1),\sigma(n)},$$

where σ runs over the permutations of $\{1, \dots, n\}$ such that

$$\sigma(1) < \sigma(2), \sigma(3) < \sigma(4), \dots, \sigma(n-1) < \sigma(n), \quad \sigma(1) < \sigma(3) < \dots < \sigma(n-1),$$

and $\epsilon(\sigma)(= \pm 1)$ is the parity of this permutation. For example, we have $\text{Pf}(A) = a_{12}$ for $n = 2$ and

$$\text{Pf}(A) = \begin{vmatrix} a_{12} & a_{13} & a_{14} \\ & a_{23} & a_{24} \\ & & a_{34} \end{vmatrix} = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}$$

for $n = 4$. The a_{ij} is taken to be skew-symmetric, and therefore $a_{ij} = -a_{ji}$.

A classical notation for the Pfaffian of A [4] is

$$\text{Pf}(A) = (1, 2, \dots, n),$$

where $(i, j) = a_{ij}$. One expansion rule for Pfaffians is given by

$$(1, 2, \dots, n) = \sum_{i=2}^n (-1)^i (1, i)(2, 3, \dots, \hat{i}, \dots, n),$$

where \hat{i} indicates that the index underneath should be deleted. For example, for $n = 4$, we can write the Pfaffian representation as

$$(1, 2, 3, 4) = (1, 2)(3, 4) - (1, 3)(2, 4) + (1, 4)(2, 3).$$

(See [20] for more information on Pfaffians.)

3. Dromion solution of the NVN equations

We recall some results obtained in [2]. A class of solutions of (1) and (2) is given by

$$U = 2 (\log(\text{Pf}(\theta_1, \dots, \theta_n)))_{xy}, \tag{3}$$

where θ_i are solutions of the linear equations

$$\phi_{xy} + \Phi_{xy}^{(0)} \phi = 0, \tag{4}$$

$$\phi_t = \phi_{xxx} + \phi_{yyy} + 3\Phi_{xx}^{(0)} \phi_x + 3\Phi_{yy}^{(0)} \phi_y, \tag{5}$$

and

$$\text{Pf}(\theta_1, \dots, \theta_n) = \begin{cases} (1, 2, \dots, n) & \text{n even} \\ (1, 2, \dots, n, I) & \text{n odd,} \end{cases}$$

where $(i, j) = \text{Pf}(\theta_i, \theta_j)$, $(i, I) = \theta_i$, and

$$\text{Pf}(\theta_i, \theta_j) = \int W_x[\theta_i, \theta_j] dx - W_y[\theta_i, \theta_j] dy. \tag{6}$$

In (6), $W_X[\theta_i, \theta_j] = \theta_i \theta_{jX} - \theta_{iX} \theta_j$ denotes the Wronskian of θ_i and θ_j with respect to variable $X = x$ or $X = y$ and so $\text{Pf}(\theta_i, \theta_j)$ is skew-symmetric. In particular, setting $\Phi^{(0)} = 0$ in (4) and (5) yields separable θ_i :

$$\theta_i = \theta_i^{(x)}(x, t) + \theta_i^{(y)}(y, t),$$

where $\theta_{it}^{(X)} = \theta_{iXXX}^{(X)}$ for $X = x$ or $X = y$.

To obtain plane wave soliton solutions, we choose

$$\theta_i^{(x)} = \alpha_i \exp(k_i x + k_i^3 t) \quad \text{and} \quad \theta_i^{(y)} = \beta_i \exp(l_i y + l_i^3 t),$$

and then (3) gives, in the case of $n = 1$,

$$U = -\frac{kl}{2} \operatorname{sech}^2 \left(\frac{1}{2} \left[kx - ly + (k^3 - l^3)t + \log \left(\frac{\alpha}{\beta} \right) \right] \right).$$

If k or l tends to zero then U tends to the trivial solution. The individual solitons are then a kind of “ghost” solitons [8], parallel to the x and y axes. These are given by $\theta^{(x)} = \alpha \exp(kx + k^3 t) + \beta$ and $\theta^{(y)} = \alpha + \beta \exp(ly + l^3 t)$, respectively. A single dromion solution may be thought of as a 2-soliton solution made out of 2 intersecting ghost solitons. Outside the interaction region, the solution is approximated by individual ghost solitons and the physical field U vanishes. If we take $n = 2$ with

$$\theta_1 = \theta_1^{(x)} = \alpha \exp(kx + k^3 t) + 1 \quad \text{and} \quad \theta_2 = \theta_2^{(y)} = 1 + \beta \exp(ly + l^3 t),$$

and using equation (6), the Pfaffian becomes

$$\operatorname{Pf}(\theta_1, \theta_2) = \int W_x[\theta_1, \theta_2] dx - W_y[\theta_1, \theta_2] dy \tag{7}$$

$$= -(\theta_1 \theta_2 + c - 1), \tag{8}$$

where c is some constant. From equation (3)

$$\begin{aligned} U &= 2 (\log(P(\theta_1, \theta_2)))_{xy} \\ &= \frac{2kl(c - 1)\alpha\beta}{(ce^{-\eta/2}e^{-\rho/2} + \alpha e^{\eta/2}e^{-\rho/2} + \beta e^{-\eta/2}e^{\rho/2} + \alpha\beta e^{\eta/2}e^{\rho/2})^2}, \end{aligned}$$

where $\eta = kx + k^3 t$ and $\rho = ly + l^3 t$. Here, we see that as $(x, y) \rightarrow \infty$, at least one of the terms in the denominator tends to infinity, and hence U is a localized solution, which we call the (1,1)-dromion solution of the NVN equations. This is illustrated in Figure 1.

An (M, N) -dromion solution, in which $M + N$ is even, is obtained by choosing $\theta_i = \alpha_i \exp(k_i x + k_i^3 t) + 1$ for $i = 1, \dots, M$ and $\theta_{j+M} = 1 + \beta_j \exp(l_j y + l_j^3 t)$ for $j = 1, \dots, N$. To express the solution in a compact form, we take S as the square of the Pfaffian $\operatorname{Pf}(\theta_1, \dots, \theta_{M+N})$, and hence $S = |\operatorname{Pf}(\theta_i, \theta_j)|$. Then S has the following block structure:

$$S = \begin{vmatrix} \int^x W_x[\theta_i, \theta_p] dx & -(\theta_i \theta_j + c_{ij}) \\ (\theta_p \theta_q + c_{pq}) & -\int^y W_y[\theta_j, \theta_q] dy \end{vmatrix}, \tag{9}$$

where $i, p = 1, \dots, M$ and $j, q = M + 1, \dots, M + N$. Hence, the solution is given by $U = (\log(S))_{xy}$ and the auxiliary field

$$V = \frac{1}{2}(\partial_x^2 + \partial_y^2)(\log(S)), \tag{10}$$

($\Phi = \log(S)$), which is introduced to help visualize the nature of the soliton.

The motivation for studying V is that, while U is localized, V is not in general and the point of intersection of the plane-wave solitons in the V -plane coincide with the localized solitons in the U -plane.

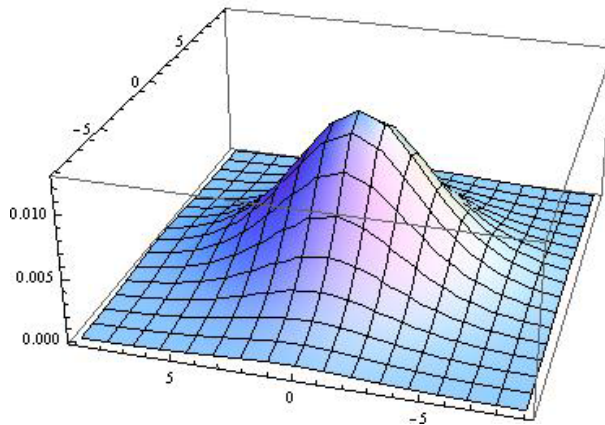


Figure 1. The (1,1)-dromion plot for the parameters given by $k = l = \frac{1}{2}$, $c = \frac{3}{2}$, and $\alpha = \beta = 1$.

4. Asymptotic analysis of the (1, 1)-dromion solution

To understand the meaning of the parameters that appear in the (M, N) -dromion solution (9), we consider the simplest case in which $M = N = 1$. The form of S is then

$$S(\theta_1, \theta_2) = (\text{Pf}(\theta_1, \theta_2))^2 \tag{11}$$

and from (7) and (8) we get

$$\text{Pf}(\theta_1, \theta_2) = -\theta_1\theta_2 + \text{constant}. \tag{12}$$

In order to simplify the representation, we use P instead of $\text{Pf}(\theta_1, \theta_2)$ and take $\theta_1 = 1 + e^\eta$ and $\theta_2 = 1 + e^\rho$; then P becomes

$$P = -c - e^\eta - e^\rho - e^{\eta + \rho}, \tag{13}$$

where c is constant and $\eta = k(x + k^2t)$, $\rho = l(y + l^2t)$. The effect of the parameters c , k , and l on the properties of the solution to the NVN equations is explained in the following theorem.

Theorem 4.1 For $c > 0$ and $k, l \neq 0$,

$$U = 2 \partial_x \partial_y (\log(P)) \quad \text{and} \quad V = (\partial_x^2 + \partial_y^2) (\log(P)) \tag{14}$$

with P given by (13) have the following properties:

1) V is the interaction of a pair of plane-wave solitons, one parallel to the y -axis, $V^{(x)}$, parametrized in terms of k , and the other parallel to the x -axis, $V^{(y)}$, parametrized in terms of l . These waves have speeds $-k^2$ and $-l^2$, and amplitudes $\frac{k^2}{4}$ and $\frac{l^2}{4}$, respectively. The relative phase shifts of the plane-wave solitons at the interaction may be expressed in terms of $F_\perp = \log(c)$, the ‘perpendicular phase shift’; for $V^{(x)}$ the relative phase shift is $\text{sgn}(l)F_\perp$ and for $V^{(y)}$ it is $\text{sgn}(k)F_\perp$.

2) U decays to zero exponentially as $(x, y) \rightarrow \infty$ in any direction and the amplitude is

$$U_0 = \frac{kl}{2} \frac{\sqrt{c} - 1}{\sqrt{c} + 1} = \frac{kl}{2} \frac{e^{\frac{1}{2}F_\perp} - 1}{e^{\frac{1}{2}F_\perp} + 1}.$$

At time t the maximum or minimum on the dromion is located at

$$\begin{aligned} (x, y) &= \left(\frac{\log(c) - 2k^3t}{2k}, \frac{\log(c) - 2l^3t}{2l} \right) \\ &= \left(\frac{1}{2} (\psi_x^+ - \psi_x^-) - k^2t, \frac{1}{2} (\psi_y^+ - \psi_y^-) - l^2t \right), \end{aligned}$$

where ψ_x^\pm and ψ_y^\pm are the phase constants in the plane waves in the V -plane at $y = \pm\infty$ and $x = \pm\infty$, respectively. The trajectory of the dromion is the line

$$k^2y - l^2x = \frac{1}{2kl} \log(c) (k^3 - l^3).$$

Proof 1) To find the speeds and the amplitudes, for any value of t , we fix y and hence ρ in P :

$$P = -c - e^\eta - e^\rho - e^{\eta+\rho} = -(c + e^\rho) \left(1 + \frac{1 + e^\rho}{c + e^\rho} e^\eta \right).$$

Since ρ and c are constants, this expression for P gives the same U and V . In what follows, the symbol \simeq is used to denote functions that are ‘asymptotically equivalent’. We say that 2 expressions $P = B$ and $P = C$ are equivalent under the changes of variable (14) from P to U and V if $C = \alpha e^{\beta x + \gamma y} B$ for some constants α, β, γ so that both B and C give the same U and V . Thus, we write $A \simeq B$ ($A \simeq P$) as $X \rightarrow a$ to mean that $A \sim C$ (A is asymptotic to C) as $X \rightarrow a$ and B and C are equivalent in the sense given above. Hence, we write $P \simeq 1 + \alpha e^\eta$, where $\alpha = (1 + e^\rho)/(c + e^\rho)$ and the relation between B and C is $C = -(c + e^\rho)B$. Thus, V can be written as

$$V = (\partial_x^2 + \partial_y^2)(\log(P)) = \frac{1}{4} k^2 \operatorname{sech}^2 \left(\frac{1}{2} k \left[(x + k^2t) + \frac{1}{k} \log(\alpha) \right] \right),$$

which is a 1-dimensional plane wave soliton propagating in the x direction. Hence, the speed is $-k^2$ and the amplitude is $\frac{k^2}{4}$. If we fix x and hence η in P , the other 1-dimensional plane wave soliton perpendicular to this one, propagating in the y direction, would be of the form

$$V = \partial_y^2(\log(1 + \beta e^\rho)) = \frac{1}{4} l^2 \operatorname{sech}^2 \left(\frac{1}{2} l \left[(y + l^2t) + \frac{1}{k} \log(\beta) \right] \right),$$

where $\beta = (1 + e^\eta)/(c + e^\eta)$, the speed is $-l^2$, and the amplitude is $\frac{l^2}{4}$.

To determine the phase shift we consider the change in P as x and y change from $-\infty$ to $+\infty$. Then we obtain the phase shift in the x direction by fixing x in the limits as $y \rightarrow \pm\infty$. For $l > 0$, as $y \rightarrow \infty$,

$$P = -e^\rho (c e^{-\rho} + e^{\eta-\rho} + 1 + e^\eta) \simeq 1 + e^\eta \quad (C = -e^\rho B);$$

hence, $V = \frac{k^2}{4} \operatorname{sech}^2 \left(\frac{1}{2} k (x + k^2t) \right)$, and as $y \rightarrow -\infty$,

$$P \simeq -c - e^\eta \quad \text{and} \quad V = \frac{k^2}{4} \operatorname{sech}^2 \left(\frac{1}{2} k \left[(x + k^2t) - \frac{1}{k} \log(c) \right] \right).$$

Thus, the phase constants at $y = \pm\infty$ are $\psi_x^+ = 0$, $\psi_x^- = -\frac{1}{k} \log(c)$, and the phase shift in the plane wave parallel to the y -axis is

$$\psi_x^+ - \psi_x^- = \frac{1}{k} \log(c). \tag{15}$$

By a similar calculation, the phase shift when $l < 0$ is

$$\psi_x^+ - \psi_x^- = -\frac{1}{k} \log(c). \tag{16}$$

Hence, the *relative* phase shift (= (wave number) \times (absolute phase shift)) is $\text{sgn}(l) \log(c)$ as required.

Similarly, to get the phase shift in the y direction, we fix y in the limits as $x \rightarrow \pm\infty$. For $k > 0$, as $x \rightarrow \infty$,

$$P = -e^\eta (c e^{-\eta} + 1 + e^{\rho-\eta} + e^\rho) \simeq 1 + e^\rho \quad (C = -e^\eta B);$$

hence, $V = \frac{l^2}{4} \text{sech}^2 \left(\frac{1}{2} l(y + l^2 t) \right)$, and as $x \rightarrow -\infty$,

$$P \simeq -c - e^\rho, \quad V = \frac{l^2}{4} \text{sech}^2 \left(\frac{1}{2} l \left[(y + l^2 t) - \frac{1}{l} \log(c) \right] \right).$$

Thus, the phase constants at $x = \pm\infty$ are $\psi_y^+ = 0$ and $\psi_y^- = -\frac{1}{l} \log(c)$. Hence, the phase shift in the plane wave parallel to the x -axis is

$$\psi_y^+ - \psi_y^- = \frac{1}{l} \log(c) \tag{17}$$

and the phase shift when $k < 0$ is

$$\psi_y^+ - \psi_y^- = -\frac{1}{l} \log(c). \tag{18}$$

2) For P given by (13),

$$U = \frac{2kl(c-1)e^{\eta+\rho}}{(c+e^\eta+e^\rho+e^{\eta+\rho})^2} \tag{19}$$

$$= \frac{2kl(c-1)}{(c e^{-(\eta+\rho)/2} + e^{(\eta-\rho)/2} + e^{-(\eta-\rho)/2} + e^{(\eta+\rho)/2})^2}, \tag{20}$$

from which we see that U is exponentially localized since at least one of the exponential terms in the denominator tends to infinity as $(x, y) \rightarrow \infty$. To see this we consider the exponentials in the denominator in (20) and a ray in any direction $y = \alpha x$, where $\alpha \in \mathbb{R}$. With $y = \alpha x$ and the appropriate expressions for η and ρ , we have

$$c e^{-(\eta+\rho)/2} + e^{(\eta-\rho)/2} + e^{-(\eta-\rho)/2} + e^{(\eta+\rho)/2}.$$

Here, if $x \rightarrow \mp\infty$, the expression tends to infinity, whatever the signs of k , l , and α are. To find any critical points we need to solve the equations $U_x = 0$ and $U_y = 0$. Differentiating U in (19) with respect to x and y gives

$$U_x = \frac{2k^2 l e^{kx+k^3t+ly+l^3t} (c-1) \left(c - e^{k(x+k^2t)} + e^{l(y+l^2t)} - e^{kx+k^3t+ly+l^3t} \right)}{(c + e^{k(x+k^2t)} + e^{l(y+l^2t)} + e^{kx+k^3t+ly+l^3t})^3},$$

$$U_y = \frac{2kl^2 e^{kx+k^3t+ly+l^3t} (c-1) \left(c + e^{k(x+k^2t)} - e^{l(y+l^2t)} - e^{kx+k^3t+ly+l^3t} \right)}{\left(c + e^{k(x+k^2t)} + e^{l(y+l^2t)} + e^{kx+k^3t+ly+l^3t} \right)^3}.$$

One obvious solution of this pair of equations is $c = 1$, but this corresponds to the trivial solution $U = 0$ and is therefore excluded. Hence, solving the equations $U_x = 0$ and $U_y = 0$ for x and y is the same as solving the equations

$$c - e^{k(x+k^2t)} + e^{l(y+l^2t)} - e^{kx+k^3t+ly+l^3t} = 0,$$

$$c + e^{k(x+k^2t)} - e^{l(y+l^2t)} - e^{kx+k^3t+ly+l^3t} = 0,$$

which imply that $e^{kx+k^3t+ly+l^3t} = c$ and $e^{k(x+k^2t)} = e^{l(y+l^2t)}$.

This pair has a unique real solution $x = \frac{\log(\sqrt{c})-k^3t}{k}$, $y = \frac{\log(\sqrt{c})-l^3t}{l}$. Since $U \rightarrow 0$ as $(x, y) \rightarrow \infty$ and there is a unique critical point, this clearly must be a local maximum or minimum located at the point

$$(x, y) = \left(\frac{\log(c) - 2k^3t}{2k}, \frac{\log(c) - 2l^3t}{2l} \right).$$

By eliminating t we get the trajectory of the dromion:

$$k^2y - l^2x = \frac{1}{2kl} \log(c) (k^3 - l^3).$$

If we substitute the critical values of x and y into U , we obtain the amplitude of the dromion:

$$U_0 = \frac{kl}{2} \frac{\sqrt{c} - 1}{\sqrt{c} + 1},$$

which can be written in terms of the phase shift F_{\perp} as follows:

$$U_0 = \frac{kl}{2} \frac{e^{\frac{1}{2}F_{\perp}} - 1}{e^{\frac{1}{2}F_{\perp}} + 1}.$$

□

4.1. Summary of results for (1,1)-dromion

The main result is that there is a dromion in the U -plane and a pair of perpendicular plane waves in the V -plane. We observe further from Theorem 4.1 that:

- The dromion may have arbitrary amplitude, positive, negative, or zero. The amplitude is:
 1. positive if $kl > 0$ and $c > 1$ or $kl < 0$ and $0 \leq c < 1$,
 2. negative if $kl < 0$ and $c > 1$ or $kl > 0$ and $0 \leq c < 1$,
 3. zero if $kl = 0$ or $c = 1$
- The plane waves always have positive amplitude and exert a phase-shift on one another. In particular, the directions of these phase shifts (forward or backward) depend on the signs of k , l , and $\log(c)$. The phase shift is zero if and only if the dromion amplitude is zero.

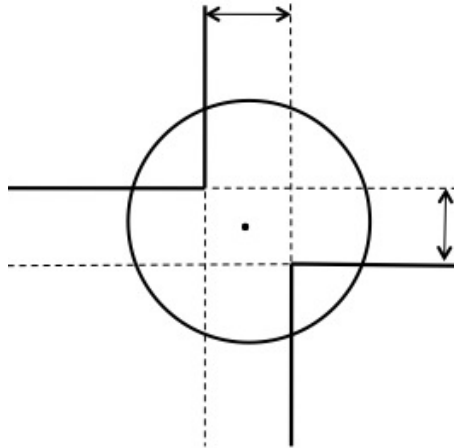


Figure 2. Phase shifts in the plane waves and the location of the dromion.

- At any fixed time, the dromion (in the U -plane) is symmetrically located between the plane waves (in the V -plane). This is illustrated schematically for the case $k > 0, l > 0, \log(c) > 0$ in Figure 2.

We next consider the general case in which V consists of $M + N$ plane waves, M plane waves parallel to the y -axis, and N plane waves parallel to the x -axis, each set of plane waves interacting like 1-dimensional multisolitons and U consisting of $M \times N$ dromions situated symmetrically at the interaction of the plane waves in the V -plane.

5. Asymptotic analysis of the (M, N) -dromion solution

In this section we consider the nature of the (M, N) -dromion solution as $t \rightarrow \pm\infty$. In order to get succinct expressions for these asymptotic forms of the solution, we order the parameters k_i and l_j in this way:

$$k_1 < k_2 < \dots < k_M \quad \text{and} \quad l_1 < l_2 < \dots < l_N. \tag{21}$$

Also, to have nonsingular solutions, we make the following choices for the arbitrary constants appearing in the solution:

$$\alpha_i = \beta_j = 1 \quad \text{for } i, j \text{ odd}, \quad \alpha_i = \beta_j = -1 \quad \text{for } i, j \text{ even}.$$

Next we write the (M, N) -dromion solution given in (9), in the case where $M + N$ is even, in the following form:

$$S = \begin{vmatrix} \Theta_1 & \Theta_2 \\ -\Theta_2^T & \Theta_3 \end{vmatrix} \tag{22}$$

where Θ_1 and Θ_3 are skew-symmetric matrices with entries

$$\Theta_{i,p} = c_{ip} - \alpha_i e^{\eta_i} + \alpha_p e^{\eta_p} + \alpha_i \alpha_p \frac{k_p - k_i}{k_p + k_i} e^{\eta_i + \eta_p} \quad 1 \leq i < p \leq M,$$

$$\Theta_{M+j, M+q} = c_{M+j, M+q} + \beta_j e^{\rho_j} - \beta_q e^{\rho_q} + \beta_j \beta_q \frac{l_j - l_q}{l_j + l_q} e^{\rho_j + \rho_q} \quad 1 \leq j < q \leq N,$$

respectively, and Θ_2 has the entries

$$\Theta_{i, M+j} = c_{i, M+j} - \alpha_i e^{\eta_i} - \beta_j e^{\rho_j} - \alpha_i \beta_j e^{\eta_i + \rho_j} \quad i = 1, \dots, M, j = 1, \dots, N,$$

where c_{ip} , c_{M+jM+q} , and c_{iM+j} are arbitrary constants. We will only be interested in c_{ij} such that the solution has no singularities. The conditions on c_{ij} that give this property will be found by considering the asymptotic form of all of the $M \times N$ dromions.

To study the (M, N) -dromion solution in the case that $M + N$ is odd, we may consider an $(M + 1, N)$ or an $(M, N + 1)$ dromion in which we set a k_i or an l_j equal to zero, respectively. In making this choice it is important that the ordering of (21) is preserved.

For the (M, N) -dromion solution in the case that $M + N$ is even, it is convenient to express the determinant S in (22) in terms of other matrices so that it can have simpler structure, namely

$$S = | C + D A - A^T D + D B D |. \tag{23}$$

The matrices in (23) have the following structure: A is a constant matrix with the (ij) th entry

$$A_{ij} = \begin{cases} (-1)^i (1 - \delta_{ij}) & \text{for } i = 1, \dots, M \\ & j = 1, \dots, M + N \\ (-1)^{i-(M+1)} (1 - \delta_{ij}) & \text{for } i = M + 1, \dots, M + N \\ & j = 1, \dots, M + N \end{cases},$$

where δ_{ij} is the Kronecker δ symbol and B is a skew-symmetric matrix with the block structure

$$B = \begin{pmatrix} K & R \\ -R^T & L \end{pmatrix}, \tag{24}$$

in which K and L are constant skew-symmetric matrices with entries

$$K_{ij} = (-1)^{i+j} \frac{k_j - k_i}{k_j + k_i} \quad 1 \leq i < j \leq M \tag{25}$$

$$L_{ij} = (-1)^{i+j+1} \frac{l_j - l_i}{l_j + l_i} \quad 1 \leq i < j \leq N; \tag{26}$$

R is a rank-1 matrix with entries

$$R_{ij} = (-1)^{i+j+1} \quad \text{for } i = 1, \dots, M, \quad j = 1, \dots, N;$$

C is the general constant skew-symmetric matrix, which has the entries

$$c_{ij} \quad \text{for } 1 \leq i < j \leq M + N;$$

and

$$D = \text{diag}(e^{\eta_1}, e^{\eta_2}, \dots, e^{\eta_M}; e^{\rho_1}, e^{\rho_2}, \dots, e^{\rho_N}) \tag{27}$$

is a diagonal matrix, where

$$\eta_i = k_i(x + k_i^2 t) \quad \text{for } i = 1, \dots, M, \tag{28}$$

$$\rho_j = l_j(y + l_j^2 t) \quad \text{for } j = 1, \dots, N. \tag{29}$$

To determine the asymptotic form of the solution we fix the m th (x, t) -dependent plane wave and the n th (y, t) -dependent plane wave and we call the corresponding dromion the (m, n) th. We write S given by (23) in terms of $\hat{x} = x + k_m^2 t$ and $\hat{y} = y + l_n^2 t$ so that the (m, n) th dromion is independent of t (i.e. is stationary) when \hat{x} and \hat{y} are fixed. We also write the expressions (28) and (29) in terms of \hat{x} and \hat{y} such that we have

$$\eta_i = k_i(\hat{x} + (k_i^2 - k_m^2)t), \quad \rho_j = l_j(\hat{y} + (l_j^2 - l_n^2)t).$$

We will see that the asymptotic form of the solution as $t \rightarrow \pm\infty$ is a dromion. The (M, N) -dromion solution consists of $M \times N$ dromions separate asymptotically as $t \rightarrow \pm\infty$. Unlike solitons, however, amplitudes are not necessarily preserved. The study of these limits is rather technical but in the end we will obtain compact expressions for the change in amplitude of the dromions due to interaction. Considering the limits of S with \hat{x} and \hat{y} fixed, we have as $t \rightarrow -\infty$

$$\eta_i \rightarrow \begin{cases} +\infty & \text{for } i < m \\ -\infty & \text{for } i > m \end{cases} \quad \text{and} \quad \rho_j \rightarrow \begin{cases} +\infty & \text{for } j < n \\ -\infty & \text{for } j > n, \end{cases}$$

and as $t \rightarrow \infty$

$$\eta_i \rightarrow \begin{cases} -\infty & \text{for } i < m \\ +\infty & \text{for } i > m \end{cases} \quad \text{and} \quad \rho_j \rightarrow \begin{cases} -\infty & \text{for } j < n \\ +\infty & \text{for } j > n, \end{cases}$$

while η_m and ρ_n are t -independent and the limits of the exponentials are as $t \rightarrow -\infty$

$$\begin{aligned} e^{-\eta_i} &\rightarrow 0 \quad (i < m) & e^{-\rho_j} &\rightarrow 0 \quad (j < n) \\ e^{\eta_i} &\rightarrow 0 \quad (i > m) & e^{\rho_j} &\rightarrow 0 \quad (j > n) \end{aligned}$$

and as $t \rightarrow \infty$

$$\begin{aligned} e^{\eta_i} &\rightarrow 0 \quad (i < m) & e^{\rho_j} &\rightarrow 0 \quad (j < n) \\ e^{-\eta_i} &\rightarrow 0 \quad (i > m) & e^{-\rho_j} &\rightarrow 0 \quad (j > n). \end{aligned}$$

To exploit these limits we must use appropriate equivalent forms for S given by (23). We factorize the diagonal matrix D given by (27) so that the factors or their inverses have finite limits as $t \rightarrow \mp\infty$, namely

$$D = D_- D_o D_+,$$

where

$$\begin{aligned} D_- &= \text{diag}(1, \dots, 1, 1, e^{\eta_{m+1}}, \dots, e^{\eta_M}; 1, \dots, 1, 1, e^{\rho_{n+1}}, \dots, e^{\rho_N}), \\ D_o &= \text{diag}(1, \dots, 1, e^{\eta_m}, 1, \dots, 1; 1, \dots, 1, e^{\rho_n}, 1, \dots, 1), \\ D_+ &= \text{diag}(e^{\eta_1}, \dots, e^{\eta_{m-1}}, 1, 1, \dots, 1; e^{\rho_1}, \dots, e^{\rho_{n-1}}, 1, 1, \dots, 1). \end{aligned}$$

Hence we get, as $t \rightarrow -\infty$,

$$\begin{aligned} D_- &\rightarrow \text{diag}(1, \dots, 1, 1, 0, \dots, 0; 1, \dots, 1, 1, 0, \dots, 0) \\ D_+^{-1} &\rightarrow \text{diag}(0, \dots, 0, 1, 1, \dots, 1; 0, \dots, 0, 1, 1, \dots, 1), \end{aligned}$$

and as $t \rightarrow \infty$,

$$\begin{aligned} D_+ &\rightarrow \text{diag}(0, \dots, 0, 1, 1, \dots, 1; 0, \dots, 0, 1, 1, \dots, 1) \\ D_-^{-1} &\rightarrow \text{diag}(1, \dots, 1, 1, 0, \dots, 0; 1, \dots, 1, 1, 0, \dots, 0). \end{aligned}$$

To find the asymptotic forms of S as $t \rightarrow \mp\infty$, we take out factors of D_+ or D_- so that it is expressed solely in terms of matrices having finite limits. As $t \rightarrow -\infty$

$$\begin{aligned} S &= |C + DA - A^T D + DBD| \\ &= |C + D_+ D_- D_o A - A^T D_- D_o D_+ + D_+ D_o D_- B D_- D_o D_+| \\ &= |D_+ ||D_+^{-1} C D_+^{-1} + D_- D_o A D_+^{-1} - D_+^{-1} A^T D_- D_o + D_o D_- B D_- D_o ||D_+| \end{aligned}$$

and so the limit is

$$S_- \sim |D_+^{-1} C D_+^{-1} + D_- D_o A D_+^{-1} - D_+^{-1} A^T D_o D_- + D_- D_o B D_o D_-|. \tag{30}$$

As $t \rightarrow \infty$

$$S = |D_- ||D_-^{-1} C D_-^{-1} + D_+ D_o A D_-^{-1} - D_-^{-1} A^T D_+ D_o + D_o D_+ B D_+ D_o ||D_-|$$

and the limit is

$$S_+ \sim |D_-^{-1} C D_-^{-1} + D_+ D_o A D_-^{-1} - D_-^{-1} A^T D_o D_+ + D_+ D_o B D_o D_+|. \tag{31}$$

We see from (30) and (31) that S_- and S_+ are the determinants of skew-symmetric matrices and are hence the squares of the Pfaffians P_- and P_+ , respectively. By expanding P_- and P_+ by their m th and $(M+n)$ th lines, one finds that

$$\begin{aligned} P_- &= P_1 + P_2 e^{\eta m} + P_3 e^{\rho n} + P_4 e^{\eta m + \rho n}, \\ P_+ &= P_5 + P_6 e^{\eta m} + P_7 e^{\rho n} + P_8 e^{\eta m + \rho n}. \end{aligned}$$

A necessary and sufficient condition that P_- and P_+ have no zeros, and hence $U = 2(\log P_{\pm})_{xy}$ has no singularities, is that P_1, \dots, P_4 and P_5, \dots, P_8 have the same sign. Furthermore, an overall change of sign in P_{\pm} does not change U , and so, since U is supposed to be nonsingular, without loss of generality we may write

$$\begin{aligned} P_- &= |P_1| + |P_2| e^{\eta m} + |P_3| e^{\rho n} + |P_4| e^{\eta m + \rho n}, \\ P_+ &= |P_5| + |P_6| e^{\eta m} + |P_7| e^{\rho n} + |P_8| e^{\eta m + \rho n}, \end{aligned} \tag{32}$$

where the Pfaffians P_i ($i = 1..8$) satisfy the relation $P_i^2 = S_i$ and are defined in terms of minors of A , B , and C , and S_i are the skew-symmetric determinants. It can be expressed (see Appendix) that each of the S_i ($i = 1..8$) can be factorized into skew-symmetric determinants defined in terms of minors of B and C ; to be precise,

$$\begin{aligned} S_1 &= C_{\gg; \gg} B_{<; <}, & S_2 &= C_{>; \gg} B_{\leq; <}, \\ S_3 &= C_{\gg; >} B_{<; \leq}, & S_4 &= C_{>; >} B_{\leq; \leq}, \\ S_5 &= C_{\leq; \leq} B_{>; >}, & S_6 &= C_{<; \leq} B_{\gg; >}, \\ S_7 &= C_{\leq; <} B_{>; \gg}, & S_8 &= C_{<; <} B_{\gg; \gg}. \end{aligned} \tag{33}$$

The subscript notation in (33) is used to denote certain principal minors of an $(M + N) \times (M + N)$ -dimensional matrix for fixed m and n . For instance, $X_{<,\leq}$ means the minor formed from rows and columns $1, \dots, m - 1, M + 1, \dots, M + n$ of the corresponding matrix X . In general, let W be a skew-symmetric $(M + N) \times (M + N)$ -dimensional matrix with the entries

$$W = \begin{pmatrix} X & Y \\ -Y^T & Z \end{pmatrix}, \tag{34}$$

where X is an $M \times M$, Y is an $M \times N$, and Z is an $N \times N$ dimensional block matrix. For a given m, n ($1 \leq m \leq M$ and $1 \leq n \leq N$), and inequalities \prec_1, \prec_2 (where \prec_1 and \prec_2 can be $<, \leq, >, \text{ and } \geq$), we define \tilde{W} to be a particular submatrix of W :

$$\tilde{W} = \begin{pmatrix} (X)_{\substack{i \prec_1 m \\ j \prec_1 m}} & (Y)_{\substack{i \prec_1 m \\ j \prec_2 M+n}} \\ (-Y^T)_{\substack{i \prec_2 M+n \\ j \prec_1 m}} & (Z)_{\substack{i \prec_2 M+n \\ j \prec_2 M+n}} \end{pmatrix},$$

and then $W_{\prec_1;\prec_2} = \det(\tilde{W})$. We also define the Pfaffian

$$w_{\prec_1;\prec_2} = \begin{cases} Pf(\tilde{W}) & \text{for even dimensions} \\ Pf \begin{pmatrix} \tilde{W} & \epsilon \\ -\epsilon^T & 0 \end{pmatrix} & \text{for odd dimensions,} \end{cases}$$

where ϵ is the column matrix with all entries equal to 1. The minors of B in (33) can also be factorized (see Appendix) into other minor skew-symmetric determinants, defined in terms of parameters k_i and l_j , which are given in (25) and (26), respectively. Hence, we have the fully factorized form of the skew-symmetric determinants S_i :

$$\begin{aligned} S_1 &= C_{\geq;\geq} K_{<} L_{<}, & S_2 &= C_{>,\geq} K_{\leq} L_{<}, \\ S_3 &= C_{\geq;,>} K_{<} L_{\leq}, & S_4 &= C_{>,>} K_{\leq} L_{\leq}, \\ S_5 &= C_{\leq;,\leq} K_{>} L_{>}, & S_6 &= C_{<,\leq} K_{\geq} L_{>}, \\ S_7 &= C_{\leq;,<} K_{>} L_{\geq}, & S_8 &= C_{<,<} K_{\geq} L_{\geq}, \end{aligned} \tag{35}$$

where the single subscript notations are defined in a similar fashion.

In general, for a given m and n ($1 \leq m \leq M$ and $1 \leq n \leq N$) and inequality \prec (where \prec can be $<, \leq, >, \text{ and } \geq$), we define \tilde{K} to be the submatrix of the $(M \times M)$ block-matrix K in (24):

$$\tilde{K} = (K)_{\substack{i \prec m \\ j \prec m}}.$$

Then $K_{\prec} = \det(\tilde{K})$ and we also define the Pfaffian

$$k_{\prec} = \begin{cases} Pf(\tilde{K}) & \text{for even dimensions} \\ Pf \begin{pmatrix} \tilde{K} & \epsilon \\ -\epsilon^T & 0 \end{pmatrix} & \text{for odd dimensions,} \end{cases}$$

where $\epsilon = (-1, 1, \dots, \pm 1)$. Similarly, we define \tilde{L} to be the submatrix of the $(N \times N)$ block-matrix L in (24):

$$\tilde{L} = (L)_{\substack{i \prec n \\ j \prec n}},$$

$$L_{\prec} = \det(\tilde{L}) \text{ and } l_{\prec} = \begin{cases} Pf(\tilde{L}) & \text{for even dimensions} \\ Pf \begin{pmatrix} \tilde{L} & \epsilon \\ -\epsilon^T & 0 \end{pmatrix} & \text{for odd dimensions.} \end{cases}$$

Next we write the Pfaffians P_i ($i=1 \dots 8$) (32) of the skew-symmetric determinants S_i ($i=1 \dots 8$) in terms of minor Pfaffians of the corresponding minor determinants formed from C , B in (33) and C , K , L in (35). We will denote the Pfaffians with the lowercase letters of the corresponding uppercase letters that have been used for denoting skew-symmetric determinants. For instance, $c_{\leq; <}$ is the Pfaffian corresponding to the skew-symmetric determinant $C_{\leq; <}$. From (33) we have the Pfaffians $|P_i| = (S_i)^{\frac{1}{2}}$:

$$\begin{aligned} |P_1| &= |c_{\geq; \geq} b_{< <}|, & |P_2| &= |c_{> > \geq} b_{\leq; <}|, \\ |P_3| &= |c_{\geq; >} b_{< < \leq}|, & |P_4| &= |c_{> > >} b_{\leq; \leq}|, \\ |P_5| &= |c_{\leq; \leq} b_{> > >}|, & |P_6| &= |c_{< < \leq} b_{\geq; >}|, \\ |P_7| &= |c_{\leq; <} b_{> > \geq}|, & |P_8| &= |c_{< < <} b_{\geq; \geq}|, \end{aligned}$$

and the Pfaffians in the fully factorized form, from (35):

$$\begin{aligned} |P_1| &= |c_{\geq; \geq} k_{< l_{<}}|, & |P_2| &= |c_{> > \geq} k_{\leq l_{<}}|, \\ |P_3| &= |c_{\geq; >} k_{< l_{\leq}}|, & |P_4| &= |c_{> > >} k_{\leq l_{\leq}}|, \\ |P_5| &= |c_{\leq; \leq} k_{> l_{>}}|, & |P_6| &= |c_{< < \leq} k_{\geq l_{>}}|, \\ |P_7| &= |c_{\leq; <} k_{> l_{\geq}}|, & |P_8| &= |c_{< < <} k_{\geq l_{\geq}}|. \end{aligned}$$

Here the Pfaffian c determines the phase shifts F_{\perp} between the interacting perpendicular plane waves and determines the amplitude of the dromions. The Pfaffian b determines the phase shifts between the 2 sets of parallel plane waves in the V -plane. These interpretations may be made because, if we choose the entries of Pfaffian c such that the perpendicular phase shift and hence the amplitudes of the dromions vanish, then the only phase shifts we get, determined by the Pfaffian b , are the parallel phase shifts experienced by the parallel plane waves in the V -plane. This is achieved by setting all arbitrary constants in c to be 1, so that all minor Pfaffians $c_{<_1; <_2}$ are equal to 1.

The asymptotic expressions in (32) can then be written as $t \rightarrow -\infty$ as:

$$P_{-} = |b_{< < <}| + |b_{\leq; <}| e^{\eta m} + |b_{< < \leq}| e^{\rho n} + |b_{\leq; \leq}| e^{\eta m + \rho n}, \tag{36}$$

and as $t \rightarrow \infty$ as:

$$P_{+} = |b_{> > >}| + |b_{\geq; >}| e^{\eta m} + |b_{> > \geq}| e^{\rho n} + |b_{\geq; \geq}| e^{\eta m + \rho n}. \tag{37}$$

Additionally, (32) can also be written in the following way as $t \rightarrow -\infty$:

$$\begin{aligned} P_{-} &= |k_{< l_{<}}| + |k_{\leq l_{<}}| e^{\eta m} + |k_{< l_{\leq}}| e^{\rho n} + |k_{\leq l_{\leq}}| e^{\eta m + \rho n} \\ &\sim 1 + \left| \frac{k_{\leq}}{k_{<}} \right| e^{\eta m} + \left| \frac{l_{\leq}}{l_{<}} \right| e^{\rho n} + \left| \frac{k_{\leq} l_{\leq}}{k_{<} l_{<}} \right| e^{\eta m + \rho n} \\ &= 1 + e^{\eta m + F_{\parallel x}^{-}} + e^{\rho n + F_{\parallel y}^{-}} + e^{\eta m + \rho n + F_{\parallel x}^{-} + F_{\parallel y}^{-}}, \end{aligned}$$

and as $t \rightarrow \infty$:

$$\begin{aligned} P_+ &= |k_{>l_{>}}| + |k_{\geq l_{>}}|e^{\eta_m} + |k_{>l_{\geq}}|e^{\rho_n} + |k_{\geq l_{\geq}}|e^{\eta_m + \rho_n} \\ &\sim 1 + \left| \frac{k_{\geq}}{k_{>}} \right| e^{\eta_m} + \left| \frac{l_{\geq}}{l_{>}} \right| e^{\rho_n} + \left| \frac{k_{\geq} l_{\geq}}{k_{>} l_{>}} \right| e^{\eta_m + \rho_n} \\ &= 1 + e^{\eta_m + F_{\parallel x}^+} + e^{\rho_n + F_{\parallel y}^+} + e^{\eta_m + \rho_n + F_{\parallel x}^+ + F_{\parallel y}^+}, \end{aligned}$$

where $F_{\parallel x}^{\pm}$ and $F_{\parallel y}^{\pm}$ are the relative phase shifts experienced by the (x, t) -dependent and (y, t) -dependent parallel plane waves in the V -plane, respectively. This is a case in which all of the dromions have zero amplitude and the solution for this case is $U \equiv 0$. As a consequence of these 2 Pfaffians b and c being independent, these 2 kinds of phase shifts, determined by b and c , are independent of each other.

We have now shown that the (M, N) -dromion solution U decomposes into $M \times N$ dromions determined by (32) as $t \rightarrow \mp \infty$. To identify the properties of the resulting dromions, we compare the asymptotic expressions in (32) with (13), generalizing Theorem 4.1, and give the theorem for the general case:

Theorem 5.1

$$U = \partial_x \partial_y (\log(S)) \quad \text{and} \quad V = \frac{1}{2} (\partial_x^2 + \partial_y^2) (\log(S))$$

with S given by (23) have the following properties:

- 1) V is the interaction of M plane-wave solitons parallel to the y -axis and N plane-wave solitons parallel to the x -axis, which decomposes asymptotically into $M \times N$ solutions as described in Theorem 4.1, part 1.
- 2) U decomposes asymptotically into $M \times N$ dromions as described in Theorem 4.1, part 2. The amplitude of the (m, n) th dromion is

$$U_0^- = \frac{1}{2} k_m l_n \frac{e^{\frac{1}{2} F_{\perp}^-} - 1}{e^{\frac{1}{2} F_{\perp}^-} + 1} \quad \text{and} \quad U_0^+ = \frac{1}{2} k_m l_n \frac{e^{\frac{1}{2} F_{\perp}^+} - 1}{e^{\frac{1}{2} F_{\perp}^+} + 1} \tag{38}$$

as $t \rightarrow -\infty$ and $t \rightarrow \infty$, respectively, and the perpendicular phase shifts are

$$F_{\perp}^- = \log \left| \frac{c_{>}; > c_{\geq}; \geq}{c_{\geq}; > c_{>}; \geq} \right| \quad \text{and} \quad F_{\perp}^+ = \log \left| \frac{c_{<}; < c_{\leq}; \leq}{c_{\leq}; < c_{<}; \leq} \right| \tag{39}$$

as $t \rightarrow -\infty$ and $t \rightarrow \infty$, respectively.

The location at time t of the (m, n) th dromion moves from

$$(x, y) = \left(\frac{1}{2k_m} \log \left| \frac{c_{\geq}; \geq c_{\geq}; > k_{\leq}^2}{c_{>}; \geq c_{>}; > k_{\leq}^2} \right| - k_m^2 t, \frac{1}{2l_n} \log \left| \frac{c_{\geq}; \geq c_{>}; \geq l_{\leq}^2}{c_{\geq}; > c_{>}; > l_{\leq}^2} \right| - l_n^2 t \right)$$

as $t \rightarrow -\infty$ to

$$(x, y) = \left(\frac{1}{2k_m} \log \left| \frac{c_{\leq}; \leq c_{\leq}; < k_{\geq}^2}{c_{<}; \leq c_{<}; < k_{\geq}^2} \right| - k_m^2 t, \frac{1}{2l_n} \log \left| \frac{c_{\leq}; \leq c_{<}; \leq l_{\geq}^2}{c_{\leq}; < c_{<}; < l_{\geq}^2} \right| - l_n^2 t \right),$$

and as $t \rightarrow \infty$, giving the 2-dimensional phase shift due to all interactions,

$$\left(\frac{1}{2k_m} \log \left| \frac{c_{\leq}; \leq c_{\leq}; < c_{>}; \geq c_{>}; > k_{\leq}^2 k_{\leq}^2}{c_{<}; \leq c_{<}; < c_{\geq}; \geq c_{\geq}; > k_{\leq}^2 k_{\leq}^2} \right|, \frac{1}{2l_n} \log \left| \frac{c_{\leq}; \leq c_{<}; \leq c_{\geq}; > c_{>}; > l_{\leq}^2 l_{\leq}^2}{c_{\leq}; < c_{<}; < c_{\geq}; \geq c_{>}; > l_{\leq}^2 l_{\leq}^2} \right| \right).$$

5.1. Summary of results for (M, N) -dromion

We observe from Theorem 5.1 that the summary of results for the $(1, 1)$ -dromion can be generalized to the (M, N) -dromion. The main result is that there are $M \times N$ dromions in the U -plane and $M + N$ perpendicular plane waves in the V -plane. We observe further that:

- The (m, n) th dromion may have arbitrary amplitude, positive, negative, or zero, and it varies as $t \rightarrow -\infty$ and $t \rightarrow \infty$. The amplitude is:

1. positive

(a) as $t \rightarrow -\infty$: if $k_m l_n > 0$ and $|c_{>,>c_{\geq,\geq}}| > |c_{\geq,>c_{>,\geq}}|$

or $k_m l_n < 0$ and $0 < \left| \frac{c_{>,>c_{\geq,\geq}}}{c_{\geq,>c_{>,\geq}}} \right| < 1,$

(b) as $t \rightarrow \infty$: if $k_m l_n > 0$ and $|c_{<,<c_{\leq,\leq}}| > |c_{\leq,<c_{<,\leq}}|$

or $k_m l_n < 0$ and $0 < \left| \frac{c_{<,<c_{\leq,\leq}}}{c_{\leq,<c_{<,\leq}}} \right| < 1;$

2. negative

(a) as $t \rightarrow -\infty$: if $k_m l_n < 0$ and $|c_{>,>c_{\geq,\geq}}| > |c_{\geq,>c_{>,\geq}}|$

or $k_m l_n > 0$ and $0 < \left| \frac{c_{>,>c_{\geq,\geq}}}{c_{\geq,>c_{>,\geq}}} \right| < 1,$

(b) as $t \rightarrow \infty$: if $k_m l_n < 0$ and $|c_{<,<c_{\leq,\leq}}| > |c_{\leq,<c_{<,\leq}}|$

or $k_m l_n > 0$ and $0 < \left| \frac{c_{<,<c_{\leq,\leq}}}{c_{\leq,<c_{<,\leq}}} \right| < 1;$

3. zero

(a) as $t \rightarrow -\infty$: if $k_m l_n = 0$ or $|c_{>,>c_{\geq,\geq}}| = |c_{\geq,>c_{>,\geq}}|,$

(b) as $t \rightarrow \infty$: if $k_m l_n = 0$ or $|c_{<,<c_{\leq,\leq}}| = |c_{\leq,<c_{<,\leq}}|.$

- The plane waves in the V -plane always have positive amplitude and exert a phase-shift on one another. The directions of these phase shifts (forward or backward) depend on the signs of $k_m, l_n,$ and

$$\log \left| \frac{c_{>,>c_{\geq,\geq}}}{c_{\geq,>c_{>,\geq}}} \right| \quad \text{or} \quad \log \left| \frac{c_{<,<c_{\leq,\leq}}}{c_{\leq,<c_{<,\leq}}} \right|.$$

The phase shifts in (39) are zero if and only if the corresponding dromion amplitude is zero.

- At any fixed time, the (m, n) th dromion (in the U -plane) is symmetrically located between the m th plane-wave soliton parallel to the y -axis and the n th plane-wave soliton parallel to the x -axis (in the V -plane).

Appendix

In order to factorize the Pfaffian P_1 and the skew-symmetric determinant $S_1 = (P_1)^2,$ we take its skew-symmetric matrix form and denote it with $G;$ hence $\det(G) = S_1 = P_1^2.$ For convenience we write G in the

following block structure:

$$G = \begin{pmatrix} B_1 & A_1 & B_2 & A_2 \\ -A_1^T & C_1 & A_3 & C_2 \\ -B_2^T & -A_3^T & B_3 & A_4 \\ -A_2^T & -C_2^T & -A_4^T & C_3 \end{pmatrix},$$

where

$$\begin{aligned} B_1 &= (B)_{\substack{1 \leq i < m, \\ 1 \leq j < m}}, & A_1 &= (A)_{\substack{1 \leq i < m, \\ m \leq j \leq M}}, \\ B_2 &= (B)_{\substack{1 \leq i < m, \\ M+1 \leq j < M+n}}, & A_2 &= (A)_{\substack{1 \leq i < m, \\ M+n \leq j \leq M+N}}, \\ C_1 &= (C)_{\substack{m \leq i \leq M, \\ m \leq j \leq M}}, & A_3 &= (-A^T)_{\substack{m \leq i \leq M, \\ M+1 \leq j < M+n}}, \\ C_2 &= (C)_{\substack{m \leq i \leq M, \\ M+n \leq j \leq M+N}}, & B_3 &= (B)_{\substack{M+1 \leq i < M+n, \\ M+1 \leq j < M+n}}, \\ A_4 &= (A)_{\substack{M+1 \leq i < M+n, \\ M+n \leq j \leq M+N}}, & C_3 &= (C)_{\substack{M+n \leq i \leq M+N, \\ M+n \leq j \leq M+N}}. \end{aligned}$$

In order to have a simpler structure for factorization, we exploit the matrix G in the following way. We interchange columns and rows in the determinant, so that the determinants S_i can be written in the form of a block structure. The numbers of the interchanges of the columns and rows are given by r_i . We interchange the rows in the second block-row with the rows in the third block-row, so the number of interchanges is $\tilde{r} = (M - m + 1) \times (n - 1)$, and we also interchange the corresponding columns in the second block-column with the columns in the third block-column, so again the number of interchanges is $\tilde{r} = (M - m + 1) \times (n - 1)$. Hence, the number of total interchanges in matrix G is $r_1 = 2 \times (M - m + 1) \times (n - 1)$. The interchanges of the columns and rows do not affect the value of the determinant, since whenever 2 columns are interchanged the corresponding 2 rows are also interchanged, nullifying the change in sign. Therefore, the numbers r_i are even for the determinants S_i , but may not be even for the Pfaffians P_i . Therefore, we take the absolute values of the Pfaffians, namely $|P_i|$. This process is achieved simply by premultiplying the permutation matrix P by G and P^T .

$$P = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix},$$

where I and 0 are identity and zero block-matrices, respectively. Hence, we get $\tilde{G} = P^T G P$. Next we take \tilde{G} to continue with the proof, since $\det(\tilde{G}) = \det(G)$. We have

$$\begin{aligned} \tilde{G} &= \begin{pmatrix} B_1 & B_2 & A_1 & A_2 \\ -B_2^T & B_3 & -A_3^T & A_4 \\ -A_1^T & A_3 & C_1 & C_2 \\ -A_2^T & -A_4^T & -C_2^T & C_3 \end{pmatrix} \\ &= \begin{pmatrix} \tilde{B} & \tilde{A} \\ -\tilde{A}^T & \tilde{C} \end{pmatrix}, \end{aligned} \tag{40}$$

where \tilde{B} and \tilde{C} are skew-symmetric matrices, and \tilde{A} is a rank-1 matrix given by

$$\tilde{B} = \begin{pmatrix} B_1 & B_2 \\ -B_2^T & B_3 \end{pmatrix} \quad \tilde{A} = \begin{pmatrix} A_1 & A_2 \\ -A_3^T & A_4 \end{pmatrix} \quad \tilde{C} = \begin{pmatrix} C_1 & C_2 \\ -C_2^T & C_3 \end{pmatrix}.$$

Next we factorize the matrix (40) in this form:

$$\tilde{G} = \begin{pmatrix} \tilde{B} & 0 \\ -\tilde{A}^T & I \end{pmatrix} \begin{pmatrix} I & \tilde{B}^{-1}\tilde{A} \\ 0 & \tilde{C} + \tilde{A}^T\tilde{B}^{-1}\tilde{A} \end{pmatrix} \tag{41}$$

where, again, I and 0 are identity and zero block-matrices, respectively. Taking the determinants of the factorized matrices in (41), we get

$$\det(\tilde{G}) = \det(\tilde{B}) \det(\tilde{C} + \tilde{A}^T\tilde{B}^{-1}\tilde{A}).$$

The expression $\tilde{A}^T\tilde{B}^{-1}\tilde{A}$ is a zero matrix, as long as \tilde{A} is a rank-1 matrix and \tilde{B} is a skew-symmetric matrix. If \tilde{B} is skew-symmetric then \tilde{B}^{-1} is also a skew-symmetric matrix. Thus, we get

$$\det(\tilde{G}) = \det(\tilde{B}) \det(\tilde{C}), S_1 = B_{<,<C_{\geq,\geq}}, P_1^2 = b_{<,<c_{\geq,\geq}}^2, |P_1| = |b_{<,<c_{\geq,\geq}}|.$$

Furthermore, we can also factorize the matrix \tilde{B} in terms of skew-symmetric matrices B_1 and B_3 defined by the parameters k_i and l_j , respectively.

$$\begin{aligned} \tilde{B} &= \begin{pmatrix} B_1 & B_2 \\ -B_2^T & B_3 \end{pmatrix} = \begin{pmatrix} B_1 & 0 \\ -B_2^T & I \end{pmatrix} \begin{pmatrix} I & B_1^{-1}B_2 \\ 0 & B_3 + B_2^TB_1^{-1}B_2 \end{pmatrix} \\ \det(\tilde{B}) &= \det(B_1) \det(B_3 + B_2^TB_1^{-1}B_2) = \det(B_1) \det(B_3) \\ B_{<,< &= K_{< L_{<}, \quad b_{<,<}^2 = k_{<}^2 l_{<}^2, \quad |b_{<,<}| = |k_{<} l_{<}|, \end{aligned}$$

where I and 0 are respectively identity and zero matrices, B_2 is a rank-1 matrix, B_1 and B_3 are skew-symmetric matrices, and the entries for the matrices B_1 and B_3 are given in (25) and (26), respectively. Hence,

$$\begin{aligned} S_1 &= B_{<,< C_{\geq,\geq}} = K_{< L_{< C_{\geq,\geq}} \\ |P_1| &= |b_{<,< c_{\geq,\geq}}| = |k_{<} l_{< c_{\geq,\geq}}|. \end{aligned}$$

The proof for the other Pfaffians $|P_i|$ ($i=2\dots 8$) can be done in a similar way.

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