

Existence and multiplicity of positive solutions for discrete anisotropic equations

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Abstract: In this paper we consider the Dirichlet problem for a discrete anisotropic equation with some function α , a nonlinear term f , and a numerical parameter $\lambda : \Delta \left(\alpha(k) |\Delta u(k-1)|^{p(k-1)-2} \Delta u(k-1) \right) + \lambda f(k, u(k)) = 0, k \in [1, T]$. We derive the intervals of a numerical parameter λ for which the considered BVP has at least 1, exactly 1, or at least 2 positive solutions. Some useful discrete inequalities are also derived.

Key words: Discrete boundary value problem, variational methods, Ekeland's variational principle, mountain pass theorem, Karush–Kuhn–Tucker theorem, positive solution, anisotropic problem

1. Introduction

In this paper we consider by variational methods and a critical point theory the existence and multiplicity of positive solutions for a perturbed anisotropic difference equation with dependence on a numerical parameter $\lambda > 0$ and with the Dirichlet type boundary condition, namely

$$\begin{cases} \Delta \left(\alpha(k) |\Delta u(k-1)|^{p(k-1)-2} \Delta u(k-1) \right) + \lambda f(k, u(k)) = 0, & k \in [1, T], \\ u(0) = u(T+1) = 0, \end{cases} \quad (1)$$

where $T \geq 2$ is an integer; $[1, T]$ is a discrete interval $\{1, 2, \dots, T\}$; $\Delta u(k-1) = u(k) - u(k-1)$ is the forward difference operator; $u(k) \in \mathbb{R}$ for all $k \in [1, T]$; $\alpha : [1, T+1] \rightarrow (0, +\infty)$ is a fixed function; $p : [0, T+1] \rightarrow [2, +\infty)$; $f : [1, T] \times \mathbb{R} \rightarrow (0, +\infty)$ is a continuous function, i.e. for any fixed $k \in [1, T]$ a function $f(k, \cdot)$ is continuous.

We aim to provide intervals for a parameter λ for which (1) has at least 1, exactly 1, or at least 2 positive solutions. For these results we use some known tools such as a direct variational method, mountain pass geometry, Ekeland's variational principle, and Karush–Kuhn–Tucker theorem. We also provide several inequalities useful in variational investigations of discrete anisotropic problems. Some of them are known, but we provide different proofs or else we give other estimations for constants appearing in these inequalities; see Section 3. As far as the existence of 2 positive solutions is concerned, we have already obtained some results in [9]. However, the problem considered in [9] does not depend on a numerical parameter and next in the present paper we simplify considerably some proofs. We also mention that, contrary to the present paper, solutions obtained in [9] are small in the sense that they belong to a unit ball.

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Concerning discrete anisotropic problems of type (1) or similar, there has already been some research beginning from [12], [15], where critical point theory was applied. In [2] the authors examine the existence of periodic or Neumann solutions for the discrete $p(k)$ -Laplacian. Problem (1) may be seen as discretization of mathematical models arising in the study of elastic mechanics [23], electrorheological fluids [16], or image restoration [6]. Variational continuous anisotropic problems have been considered by many methods and authors [11], for an extensive survey of such boundary value problems.

There are some related papers in the area of discrete problems. Paper [3] treats the discrete p -Laplacian problem and intervals for a nonlinear parameter are derived for which the existence and multiplicity are obtained. Let us also mention, far from being exhaustive, the following recent papers on discrete BVPs investigated via critical point theory: [1], [5], [13], [17–22]. These papers employ in the discrete setting the variational techniques already known for continuous problems, of course with necessary modifications. The tools employed cover the Morse theory, the mountain pass methodology, and linking arguments.

About the nonlinear term, we assume the following condition:

(C) There exist a function $m : [1, T] \rightarrow [2, +\infty)$ and functions $\varphi_1, \varphi_2, \psi_1, \psi_2 : [1, T] \rightarrow (0, +\infty)$ such that

$$\psi_1(k) + \varphi_1(k)|u|^{m(k)-2}u \leq f(k, u) \leq \varphi_2(k)|u|^{m(k)-2}u + \psi_2(k)$$

for all $u \geq 0$ and all $k \in [1, T]$.

From now on we will use the following notations:

$$\begin{aligned} \alpha^- &= \min_{k \in [1, T+1]} \alpha(k), & \alpha^+ &= \max_{k \in [1, T+1]} \alpha(k); \\ p^- &= \min_{k \in [0, T+1]} p(k), & p^+ &= \max_{k \in [0, T+1]} p(k); \\ m^- &= \min_{k \in [1, T]} m(k), & m^+ &= \max_{k \in [1, T]} m(k); \\ \varphi_1^- &= \min_{k \in [1, T]} \varphi_1(k), & \varphi_2^+ &= \max_{k \in [1, T]} \varphi_2(k), \\ \psi_1^- &= \min_{k \in [1, T]} \psi_1(k), & \psi_2^+ &= \max_{k \in [1, T]} \psi_2(k). \end{aligned}$$

When compared to [9] there is no additional condition placed on functions $\varphi_1, \varphi_2, \psi_1$, and ψ_2 , and so there are many functions that satisfy our conditions. Our results will depend on the relation between p^-, p^+ and m^-, m^+ . These will influence the method used in proving that a solution exists.

Example 1 Let $m : [1, T] \rightarrow [2, +\infty)$. Put $f : [1, T] \times \mathbb{R} \rightarrow (0, +\infty)$ by the formula

$$f(k, u) = |\sin k| |u|^{m(k)-2} u \arctan u + \ln(k + 1)$$

for $(k, u) \in [1, T] \times \mathbb{R}$.

Solutions to (1) will be investigated in a space

$$H = \{u : [0, T + 1] \rightarrow \mathbb{R} : u(0) = u(T + 1) = 0\}$$

considered with a norm

$$\|u\| = \left(\sum_{k=1}^{T+1} |\Delta u(k-1)|^2 \right)^{1/2}$$

with which H becomes a Hilbert space. For $u \in H$ let

$$u_+ = \max\{u, 0\}, \quad u_- = \max\{-u, 0\}.$$

Note that

$$u_+ \geq 0 \text{ and } u_- \geq 0; \quad u = u_+ - u_-; \quad u_+ \cdot u_- = 0.$$

2. Preliminaries

In this section we provide some tools that are used throughout the paper.

Theorem 2 [14] *Let E be a reflexive Banach space. If a functional $J \in C^1(E, \mathbb{R})$ is weakly lower semicontinuous and coercive, i.e. $\lim_{\|x\| \rightarrow \infty} J(x) = +\infty$, then there exists $\bar{x} \in E$ such that $\inf_{x \in E} J(x) = J(\bar{x})$ and \bar{x} is also a critical point of J , i.e. $J'(\bar{x}) = 0$. Moreover, if J is strictly convex, then a critical point is unique.*

Theorem 3 [8] (Ekeland’s principle) *Let X be a complete metric space and $\Phi : X \rightarrow \mathbb{R}$ a lower semicontinuous function that is bounded below. Let $\varepsilon > 0$ and $\bar{u} \in X$ be given such that*

$$\Phi(\bar{u}) \leq \inf_X \Phi + \frac{\varepsilon}{2}.$$

Then given $\lambda > 0$ there exists $u_\lambda \in X$ such that

- (i) $\Phi(u_\lambda) \leq \Phi(\bar{u})$,
- (ii) $d(u_\lambda, \bar{u}) < \lambda$,
- (iii) $\Phi(u_\lambda) < \Phi(u) + \frac{\varepsilon}{\lambda}d(u, u_\lambda)$ for all $u \neq u_\lambda$.

Definition 4 *Let E be a real Banach space. We say that a functional $J : E \rightarrow \mathbb{R}$ satisfies the Palais–Smale condition if every sequence (u_n) such that $\{J(u_n)\}$ is bounded and $J'(u_n) \rightarrow 0$ has a convergent subsequence.*

Lemma 5 [7] *Let E be a Banach space and $J \in C^1(E, \mathbb{R})$ satisfy the Palais–Smale condition. Assume that there exist $x_0, x_1 \in E$ and a bounded open neighborhood Ω of x_0 such that $x_1 \notin \bar{\Omega}$ and*

$$\max\{J(x_0), J(x_1)\} < \inf_{x \in \partial\Omega} J(x).$$

Let

$$\Gamma = \{h \in C([0, 1], E) : h(0) = x_0, h(1) = x_1\}$$

and

$$c = \inf_{h \in \Gamma} \max_{s \in [0, 1]} J(h(s)).$$

Then c is a critical value of J ; that is, there exists $x^ \in E$ such that $J'(x^*) = 0$ and $J(x^*) = c$, where $c > \max\{J(x_0), J(x_1)\}$.*

Finally, we recall the Karush–Kuhn–Tucker theorem with Slater qualification conditions (for 2 constraints).

Theorem 6 [4] *Let X be a finite-dimensional Euclidean space, $\eta, \mu_1, \mu_2 : X \rightarrow \mathbb{R}$ be differentiable functions, and $S = \{x \in X : \mu_1(x) \leq 0, \mu_2(x) \leq 0\}$. Moreover, let $\bar{x} \in S$ be such that $\eta(\bar{x}) = \inf_S \eta(x)$. Then there exist numbers $\sigma_0, \sigma_1, \sigma_2 \geq 0$ such that $(\sigma_0)^2 + (\sigma_1)^2 + (\sigma_2)^2 > 0$ and*

$$\sigma_0 \eta'(\bar{x}) + \sigma_1 \mu_1'(\bar{x}) + \sigma_2 \mu_2'(\bar{x}) = 0 \text{ and } \sigma_1 \mu_1(\bar{x}) = 0, \sigma_2 \mu_2(\bar{x}) = 0.$$

3. Auxiliary inequalities

Now we recall some auxiliary materials that we use later on.

(A.1) [15] For every $u \in H$ with $\|u\| \leq 1$ we have

$$\sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} \geq T^{-\frac{p^+-2}{2}} \|u\|^{p^+}.$$

(A.2) [10] For every $u \in H$ and for every $m \geq 2$ we have

$$\sum_{k=1}^{T+1} |\Delta u(k-1)|^m \leq 2^m \sum_{k=1}^T |u(k)|^m.$$

(A.3) [18] For every $u \in H$ and for any $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ we have

$$\|u\|_C = \max_{k \in [1, T]} |u(k)| \leq (T+1)^{\frac{1}{q}} \left(\sum_{k=1}^{T+1} |\Delta u(k-1)|^p \right)^{1/p}.$$

The following inequalities will also be of use.

Lemma 7

(A.4) For every $u \in H$ and for every $m > 1$ we have

$$\sum_{k=1}^T |u(k)|^m \leq T(T+1)^{m-1} \sum_{k=1}^{T+1} |\Delta u(k-1)|^m.$$

(A.5) For every $u \in H$ and for every $m \geq 1$ we have

$$\sum_{k=1}^{T+1} |\Delta u(k-1)|^m \leq (T+1) \|u\|^m.$$

(A.6) For every $u \in H$ and for every $m \geq 2$ we have

$$\sum_{k=1}^{T+1} |\Delta u(k-1)|^m \geq (T+1)^{\frac{2-m}{2}} \|u\|^m.$$

(A.7) For every $u \in H$ with $\|u\| > 1$

$$\sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} \geq T^{\frac{2-p^-}{2}} \|u\|^{p^-} - (T+1).$$

(A.8) For every $u \in H$ we have

$$\sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} \leq (T+1) \|u\|^{p^+} + (T+1).$$

Proof We will show that (A.4) holds true. Let $m, q > 1$ be such that $\frac{1}{m} + \frac{1}{q} = 1$. By (A.3) we have

$$|u(k)|^m \leq (T + 1)^{\frac{m}{q}} \sum_{i=0}^T |\Delta u(i)|^m \text{ for every } k \in [1, T].$$

Thus summing for k from 1 to T we get

$$\begin{aligned} \sum_{k=1}^T |u(k)|^m &= |u(1)|^m + |u(2)|^m + \dots + |u(T)|^m \leq \\ &(T + 1)^{\frac{m}{q}} \sum_{i=0}^T |\Delta u(i)|^m + (T + 1)^{\frac{m}{q}} \sum_{i=0}^T |\Delta u(i)|^m + \dots + (T + 1)^{\frac{m}{q}} \sum_{i=0}^T |\Delta u(i)|^m = \\ &T(T + 1)^{\frac{m}{q}} \sum_{i=0}^T |\Delta u(i)|^m = T(T + 1)^{\frac{m}{q}} \sum_{k=1}^{T+1} |\Delta u(k - 1)|^m = \\ &T(T + 1)^{m-1} \sum_{k=1}^{T+1} |\Delta u(k - 1)|^m. \end{aligned}$$

To see (A.5) note that for every $k \in [0, T]$ we have

$$|\Delta u(k)|^2 \leq \sum_{i=0}^T |\Delta u(i)|^2,$$

and so

$$|\Delta u(k)|^m \leq \left(\left(\sum_{i=0}^T |\Delta u(i)|^2 \right)^{\frac{1}{2}} \right)^m.$$

Thus

$$\sum_{k=0}^T |\Delta u(k)|^m \leq (T + 1) \left(\left(\sum_{k=0}^T |\Delta u(k)|^2 \right)^{\frac{1}{2}} \right)^m$$

and

$$\sum_{k=1}^{T+1} |\Delta u(k - 1)|^m \leq (T + 1) \|u\|^m.$$

To see (A.6) note that for $m > 2$ the Hölder inequality implies

$$\begin{aligned} \sum_{k=0}^T |\Delta u(k)|^2 &\leq \left(\sum_{k=0}^T 1^{\frac{m-2}{m}} \right)^{\frac{m-2}{m}} \left(\sum_{k=0}^T (|\Delta u(k)|^2)^{\frac{m}{2}} \right)^{\frac{2}{m}} \\ &= (T + 1)^{\frac{m-2}{m}} \left(\sum_{k=0}^T |\Delta u(k)|^m \right)^{\frac{2}{m}}. \end{aligned}$$

The above inequality leads to

$$\left(\sum_{k=1}^{T+1} |\Delta u(k - 1)|^2 \right)^{\frac{1}{2}} \leq (T + 1)^{\frac{m-2}{2m}} \left(\sum_{k=1}^{T+1} |\Delta u(k - 1)|^m \right)^{\frac{1}{m}}.$$

Thus for $m \geq 2$ we have

$$\sum_{k=1}^{T+1} |\Delta u(k-1)|^m \geq (T+1)^{\frac{2-m}{2}} \|u\|^m.$$

The relation (A.7) we obtain by (A.6). Indeed,

$$\begin{aligned} \sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} &= \sum_{\{k \in [1, T+1]: |\Delta u(k-1)| < 1\}} |\Delta u(k-1)|^{p(k-1)} + \sum_{\{k \in [1, T+1]: |\Delta u(k-1)| \geq 1\}} |\Delta u(k-1)|^{p(k-1)} \geq \\ & \sum_{\{k \in [1, T+1]: |\Delta u(k-1)| < 1\}} |\Delta u(k-1)|^{p^+} + \sum_{\{k \in [1, T+1]: |\Delta u(k-1)| \geq 1\}} |\Delta u(k-1)|^{p^-} = \\ & \sum_{k=1}^{T+1} |\Delta u(k-1)|^{p^-} - \left(\sum_{\{k \in [1, T+1]: |\Delta u(k-1)| < 1\}} |\Delta u(k-1)|^{p^-} - \sum_{\{k \in [1, T+1]: |\Delta u(k-1)| < 1\}} |\Delta u(k-1)|^{p^+} \right) \geq \\ & \sum_{k=1}^{T+1} |\Delta u(k-1)|^{p^-} - \sum_{k=1}^{T+1} 1 \geq T^{\frac{2-p^-}{2}} \|u\|^{p^-} - (T+1). \end{aligned}$$

And the relation (A.8) we obtain by (A.5)

$$\begin{aligned} \sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} &= \sum_{\{k \in [1, T+1]: |\Delta u(k-1)| < 1\}} |\Delta u(k-1)|^{p(k-1)} + \sum_{\{k \in [1, T+1]: |\Delta u(k-1)| \geq 1\}} |\Delta u(k-1)|^{p(k-1)} \leq \\ & \sum_{\{k \in [1, T+1]: |\Delta u(k-1)| < 1\}} |\Delta u(k-1)|^{p^-} + \sum_{\{k \in [1, T+1]: |\Delta u(k-1)| \geq 1\}} |\Delta u(k-1)|^{p^+} = \\ & \sum_{k=1}^{T+1} |\Delta u(k-1)|^{p^+} + \sum_{\{k \in [1, T+1]: |\Delta u(k-1)| < 1\}} |\Delta u(k-1)|^{p^-} - \sum_{\{k \in [1, T+1]: |\Delta u(k-1)| < 1\}} |\Delta u(k-1)|^{p^+} \leq \\ & \sum_{k=1}^{T+1} |\Delta u(k-1)|^{p^+} + \sum_{k=1}^{T+1} 1 \leq (T+1) \|u\|^{p^+} + (T+1). \end{aligned}$$

The proof of Lemma 7 is complete. □

4. Variational framework

In this section we connect positive solutions to (1) with critical points of a suitably chosen action functional. Next, we derive a type of a maximum principle and then we prove that under (C) and an additional assumption this functional satisfies the (PS) condition. Let

$$F(k, u) = \int_0^u f(k, s) ds \text{ for } u \in \mathbb{R} \text{ and } k \in [1, T]. \tag{2}$$

Let us define a functional $J_\lambda : H \rightarrow \mathbb{R}$ by the formula

$$J_\lambda(u) = \sum_{k=1}^{T+1} \frac{\alpha(k)}{p(k-1)} |\Delta u(k-1)|^{p(k-1)} - \lambda \sum_{k=1}^T F(k, u_+(k)).$$

The functional J_λ is continuously Gâteaux differentiable and its Gâteaux derivative J'_λ at u reads

$$\begin{aligned} \langle J'_\lambda(u), v \rangle &= \sum_{k=1}^{T+1} \alpha(k) |\Delta u(k-1)|^{p(k-1)-2} \Delta u(k-1) \Delta v(k-1) - \\ &\lambda \sum_{k=1}^T f(k, u_+(k)) v(k) \end{aligned} \tag{3}$$

for all $v \in H$. Suppose that u is a critical point to J_λ , i.e. $\langle J'_\lambda(u), v \rangle = 0$ for all $v \in H$. Summing by parts and taking boundary values into account, see [10], we observe that

$$\sum_{k=1}^{T+1} \Delta \left(\alpha(k) |\Delta u(k-1)|^{p(k-1)-2} \Delta u(k-1) \right) v(k) + \lambda \sum_{k=1}^T f(k, u_+(k)) v(k) = 0.$$

Since $v \in H$ is arbitrary we see that u satisfies (1).

Now we will provide some results that are used in the proof of the Main Theorem. The following lemma may be viewed as a kind of a discrete maximum principle. These results follow partially as in [9].

Lemma 8 *Let $\lambda > 0$. Assume that $u \in H$ is a solution of the equation*

$$\begin{cases} \Delta \left(\alpha(k) |\Delta u(k-1)|^{p(k-1)-2} \Delta u(k-1) \right) + \lambda f(k, u_+(k)) = 0, k \in [1, T], \\ u(0) = u(T+1) = 0, \end{cases} \tag{4}$$

then $u(k) > 0$ for all $k \in [1, T]$ and moreover u is a positive solution of (1).

Proof Note that

$$\Delta u(k-1) \Delta u_-(k-1) \leq 0 \text{ for every } k \in [1, T+1].$$

Assume that $u \in H$ is a solution of (4). Taking $v = u_-$ in (3) we obtain

$$\sum_{k=1}^{T+1} \alpha(k) |\Delta u(k-1)|^{p(k-1)-2} \Delta u(k-1) \Delta u_-(k-1) = \lambda \sum_{k=1}^T f(k, u_+(k)) u_-(k).$$

Since the term on the left is nonpositive and the one on the right is nonnegative, this equation holds true if both terms are equal to zero, which leads to $u_-(k) = 0$ for all $k \in [1, T]$. Then $u = u_+$. Moreover, $u(k) \neq 0$ for all $k \in [1, T]$. Indeed, assume that there exists $k \in [1, T]$ such that $u(k) = 0$. Then, by (4) we have

$$\alpha(k+1) u(k+1)^{p(k)-1} + \alpha(k) u(k-1)^{p(k-1)-1} + \lambda f(k, 0) = 0;$$

Since $\lambda > 0$ and $f(k, 0) > 0$, we have a contradiction. Thus $u(k) \neq 0$ for all $k \in [1, T]$, and it follows u is a positive solution of (1).

We will prove that J_λ satisfies the Palais–Smale condition. □

Lemma 9 *Assume that (C) holds with $m^- > p^+$. Then for any $\lambda > 0$ the functional J_λ satisfies the Palais–Smale condition.*

Proof Assume that $\{u_n\}$ is such that $\{J_\lambda(u_n)\}$ is bounded and $J'_\lambda(u_n) \rightarrow 0$. Since H is finitely dimensional, it is enough to show that $\{u_n\}$ is bounded. Let $\{u_{n_k}\}$ be such a subsequence of the sequence $\{u_n\}$ whose all elements are non-negative and $\{u_{n_l}\}$ be such subsequence of $\{u_n\}$ whose all elements are non-positive. Either of this sequences must have an infinite number of elements. Assume that $\{u_n\}$ is unbounded. Note that either $\{u_{n_k}\}$ or $\{u_{n_l}\}$ is then unbounded, up to a subsequence that we assume to be chosen. Suppose that $\{u_{n_k}\}$ is unbounded. Then by (2), by the condition (C), and by relations (A.2), (A.6), (A.8) we have

$$J_\lambda(u_{n_k}) \leq \frac{\alpha^+}{p^-} \left((T+1) \|u_{n_k}\|^{p^+} + (T+1) \right) - \lambda \left(\frac{\varphi_1^-}{m^+} 2^{-m^-} (T+1)^{\frac{2-m^-}{2}} \|u_{n_k}\|^{m^-} + \psi_1^- \sum_{k=1}^T u_{n_k}(k) \right).$$

Since $m^- > p^+$ and since $\|u_{n_k}\| \rightarrow +\infty$, it follows that $J_\lambda(u_{n_k}) \rightarrow -\infty$. Thus we obtain a contradiction with the assumption $\{J_\lambda(u_n)\}$ is bounded since in this case also $\{J_\lambda(u_{n_k})\}$ is bounded.

Now suppose $\{u_{n_l}\}$ is unbounded. Then by the relation (A.7) we observe that

$$J_\lambda(u_{n_l}) \geq \frac{\alpha^-}{p^+} \left(T^{\frac{2-p^-}{2}} \|u_{n_l}\|^{p^-} - (T+1) \right).$$

Since $\|u_{n_l}\| \rightarrow +\infty$, $J_\lambda(u_{n_l}) \rightarrow +\infty$. Thus, there is a contradiction with the assumption $\{J_\lambda(u_n)\}$ is bounded. It follows that $\{u_{n_l}\}$ is bounded. Hence the sequence $\{u_n\}$ is bounded. □

Remark 10 Note that with the assumptions of Lemma 9 the functional J_λ is neither coercive nor anticoercive.

5. Existence of a solution

In this section we consider the existence of at least one solution to the problem under consideration. We apply a direct variational method and a mountain pass geometry and Ekeland’s variational principle. Results depend on a relation between functions $k \rightarrow m(k)$ and $k \rightarrow p(k)$. Uniqueness is also undertaken.

5.1. Case $m^+ < p^-$

In this section we apply a direct variational approach.

Theorem 11 Let $m^+ < p^-$. Assume that the condition (C) holds. Then for all $\lambda > 0$ the problem (1) has at least one positive solution.

Proof Fix $\lambda > 0$. Since H is finite dimensional and since J_λ is Gâteaux differentiable and continuous it suffices to show that it is coercive. By the condition (C) and relations (A.3), (A.4), (A.5), and (A.7) for sufficiently large $\|u\|$ we obtain

$$\begin{aligned} J_\lambda(u) &\geq \frac{\alpha^-}{p^+} \left(T^{\frac{2-p^-}{2}} \|u\|^{p^-} - (T+1) \right) - \lambda \left(\frac{\varphi_2^+}{m^-} \sum_{k=1}^T |u_+(k)|^{m(k)} + \psi_1^+ \sum_{k=1}^T |u_+(k)| \right) \geq \\ &\frac{\alpha^-}{p^+} T^{\frac{2-p^-}{2}} \|u\|^{p^-} - \frac{\alpha^-}{p^+} (T+1) - \lambda \frac{\varphi_2^+}{m^-} T (T+1)^{m^+} \|u_+\|^{m^+} - \lambda T \psi_1^+ \max_{k \in [1, T]} |u_+(k)| \\ &\frac{\alpha^-}{p^+} T^{\frac{2-p^-}{2}} \|u\|^{p^-} - \frac{\alpha^-}{p^+} (T+1) - \lambda \frac{\varphi_2^+}{m^-} T (T+1)^{m^+} \|u_+\|^{m^+} - \lambda T (T+1)^{\frac{1}{2}} \psi_1^+ \|u_+\|. \end{aligned}$$

Thus the functional J_λ is coercive on H . The assumptions of Theorem 2 are satisfied and by Lemma 8 the problem (1) has a positive solution. \square

Corollary 12 *Let $m^+ < p^-$. Assume that the condition (C) holds. Assume additionally that for any $k \in [1, T]$ the function $x \rightarrow f(k, x)$ is nonincreasing. Then for all $\lambda > 0$ the problem (1) has exactly one positive solution.*

Proof Fix $\lambda > 0$. By Theorem 11 there exists at least one positive solution. Set

$$J_1(u) = \sum_{k=1}^{T+1} \frac{\alpha(k)}{p(k-1)} |\Delta u(k-1)|^{p(k-1)}, \quad J_2(u) = - \sum_{k=1}^T F(k, u_+(k)).$$

Then $J_\lambda = J_1 + \lambda J_2$. Note that J_1 is strictly convex and J_2 is convex. Thus J_λ as a sum of a strictly convex and a convex functional is strictly convex and thus it has a unique critical point. \square

5.2. Case $m^- > p^+$

Define

$$\lambda_0 = \frac{\frac{\alpha^-}{p^+} T^{-\frac{p^+-2}{2}} (T+1)^{-\frac{p^+}{2}}}{\sum_{k=1}^T \left(\frac{\varphi_2(k)}{m(k)} + \psi_2(k) \right)}. \tag{5}$$

Using a mountain pass geometry we see that

Theorem 13 *Let $m^- > p^+$. Suppose that the condition (C) is satisfied. Then for any $\lambda \in (0, \lambda_0)$ the problem (1) has at least one positive solution.*

Proof Fix $\lambda \in (0, \lambda_0)$. We will show that the assumptions of Lemma 5 hold. By Lemma 9 the functional J_λ satisfied the Palais–Smale condition. Let

$$\Omega := \left\{ u \in H : \|u\| \leq (T+1)^{-\frac{1}{2}} \right\}.$$

Note that for $u \in \Omega$ by (A.3) it follows that for all $k \in [1, T]$

$$|u(k)| \leq \max_{s \in [1, T]} |u(s)| \leq (T+1)^{\frac{1}{2}} \|u\| \leq 1.$$

Next we see that for all $u \in \Omega$

$$\sum_{k=1}^T F(k, u_+(k)) \leq \sum_{k=1}^T \frac{\varphi_2(k)}{m(k)} + \sum_{k=1}^T \psi_2(k).$$

Therefore, by (A.1) for $u \in \partial\Omega$ we obtain

$$J_\lambda(u) \geq \frac{\alpha^-}{p^+} T^{-\frac{p^+-2}{2}} (T+1)^{-\frac{p^+}{2}} - \lambda \left(\sum_{k=1}^T \frac{\varphi_2(k)}{m(k)} + \sum_{k=1}^T \psi_2(k) \right).$$

Then we see that for all $\lambda \in (0, \lambda_0)$

$$J_\lambda(u) > 0 \text{ for all } u \in \partial\Omega.$$

Let $u_\xi \in H$ be defined as follows: $u_\xi(k) = \xi$ for $k = 1, \dots, T$ and $u_\xi(0) = u_\xi(T + 1) = 0$. Then for $\xi > 1$ we have

$$J(u_\xi) \leq \alpha^+ \left(\frac{\xi^{p(0)}}{p(0)} + \frac{\xi^{p(T)}}{p(T)} \right) - \lambda \sum_{k=1}^T \left(\frac{\varphi_1(k)\xi^{m(k)}}{m(k)} + \psi_1(k)\xi \right) \leq 2\frac{\alpha^+}{p^-}\xi^{p^+} - \lambda T\xi^{m^-} \left(\frac{\varphi_1^-}{m^+} + \psi_1^- \xi^{1-m^-} \right).$$

Since $m^- > p^+$, $\lim_{\xi \rightarrow \infty} J_\lambda(u_\xi) = -\infty$, and so there exists ξ_0 such that $u_{\xi_0} \in H \setminus \Omega$ and $J_\lambda(u_{\xi_0}) < \min_{u \in \partial\Omega} J_\lambda(u)$. The assumptions of Lemma 5 are satisfied; thus by Lemma 8 the problem (1) has at least one positive solution. \square

5.3. Case $p^- > m^-$

Recall that λ_0 is defined by (5). In this subsection we apply Ekeland’s variational principle.

Theorem 14 *Assume that the condition (C) holds and that $p^- > m^-$. Then for any $\lambda \in (0, \lambda_0)$ the problem (1) has at least one positive solution.*

Proof Let $\lambda \in (0, \lambda_0)$ be fixed. Let Ω be defined as in the proof of Theorem 13. Recall from the proof of Theorem 13 that for every $\lambda \in (0, \lambda_0)$ and every $u \in \partial\Omega$ we have $J_\lambda(u) > 0$. By the Weierstrass theorem we see that

$$\inf_{u \in \partial\Omega} J_\lambda(u) > 0.$$

Now take $t \in (0, (T + 1)^{-(1/2)})$, which satisfies also the following inequality:

$$t < \sqrt[p^- - m^-]{\frac{\lambda \left(\frac{\varphi_1^-}{m^+} + \psi_1^- \right)}{2\frac{\alpha^+}{p^-}}}.$$

Choose $k_0 \in [1, T]$ such that $m(k_0) = m^-$. Let $u_0 \in H$ be such a function that $u_0(k_0) = t$ and $u_0(k) = 0$ for any $k \in [1, T] \setminus \{k_0\}$. We see that

$$J_\lambda(u_0) \leq \frac{\alpha(k_0)}{p(k_0-1)}t^{p(k_0-1)} + \frac{\alpha(k_0+1)}{p(k_0)}t^{p(k_0)} - \lambda \left(\frac{\varphi_1(k_0)}{m(k_0)}t^{m(k_0)} + \psi_1(k_0)t \right) \leq 2\frac{\alpha^+}{p^-}t^{p^-} - \lambda \left(\frac{\varphi_1^-}{m^+} + \psi_1^- \right)t^{m^-} < 0.$$

Thus $J_\lambda(u_0) < 0$. Recall that $u_0 \in Int\Omega$. Therefore,

$$\inf_{u \in Int\Omega} J_\lambda(u) < 0.$$

It then follows

$$\inf_{u \in Int\Omega} J_\lambda(u) < \inf_{u \in \partial\Omega} J_\lambda(u).$$

The remaining part of the proof follows as in [15] but we provide it for reader's convenience in our setting. Let

$$0 < \varepsilon < \inf_{u \in \partial\Omega} J_\lambda(u) - \inf_{u \in \text{Int}\Omega} J_\lambda(u).$$

Applying Ekeland's variational principle to the functional $J_\lambda : \Omega \rightarrow \mathbb{R}$ we find $u_\varepsilon \in \Omega$ such that

$$J_\lambda(u_\varepsilon) < \inf_{u \in \Omega} J_\lambda(u) + \varepsilon \quad \text{and} \quad J_\lambda(u_\varepsilon) < J_\lambda(u) + \varepsilon \|u - u_\varepsilon\| \quad \text{for } u \neq u_\varepsilon.$$

Note that

$$J_\lambda(u_\varepsilon) < \inf_{u \in \Omega} J_\lambda(u) + \varepsilon \leq \inf_{u \in \text{Int}\Omega} J_\lambda(u) + \varepsilon < \inf_{u \in \partial\Omega} J_\lambda(u).$$

Thus $u_\varepsilon \in \text{Int}\Omega$. Let $\Phi_\lambda : \Omega \rightarrow \mathbb{R}$ be defined by

$$\Phi_\lambda(u) = J_\lambda(u) + \varepsilon \|u - u_\varepsilon\| \quad \text{for } u \neq u_\varepsilon.$$

It follows that u_ε is an argument of a minimum for Φ_λ and therefore

$$\frac{\Phi_\lambda(u_\varepsilon + hv) - \Phi_\lambda(u_\varepsilon)}{h} \geq 0 \tag{6}$$

for any $v \in \Omega$ and a small enough positive h . Note that the formula (6) reduces to

$$\frac{J_\lambda(u_\varepsilon + hv) - J_\lambda(u_\varepsilon)}{h} + \varepsilon \|v\| \geq 0.$$

Letting $h \rightarrow 0$ we obtain

$$\langle J'_\lambda(u_\varepsilon), v \rangle + \varepsilon \|v\| > 0,$$

and finally

$$\|J'_\lambda(u_\varepsilon)\| \leq \varepsilon.$$

Putting $\varepsilon = \frac{1}{n}$ for sufficiently large natural n , we see that there exists a sequence $\{u_n\} \subset \text{Int}\Omega$ such that

$$J_\lambda(u_n) \rightarrow \inf_{u \in \Omega} J_\lambda(u) \quad \text{and} \quad J'_\lambda(u_n) \rightarrow 0$$

The sequence $\{u_n\}$ is bounded in H , and so there exists $v_0 \in H$ such that, up to a subsequence, $\{u_n\}$ converges to v_0 in H . Thus

$$J_\lambda(v_0) = \inf_{u \in \Omega} J_\lambda(u) \quad \text{and} \quad J'_\lambda(v_0) = 0.$$

The above relations imply that v_0 is a solution of the problem (1). □

6. Multiple solutions

In this section we apply the Karush–Kuhn–Tucker conditions together with the mountain pass technique in order to obtain the existence of at least 2 solutions. We follow [9], but we obtain solutions outside the unit ball contrary to [9].

Theorem 15 Let $m^- > p^+$ and let us chose $\gamma > 1$ such that $T^{\frac{2-p^-}{2}}\gamma^{p^-} > (T+1)$. Suppose that the condition (C) holds. For any

$$\lambda \in \left(0, \frac{\alpha^{-T^{\frac{2-p^-}{2}}\gamma^{p^-}} - (T+1)}{\sum_{k=1}^T \left(\frac{\varphi_2(k)}{m(k)} \left((T+1)^{\frac{1}{2}} \gamma \right)^{m(k)} + \psi_2(k) (T+1)^{\frac{1}{2}} \gamma \right)} \right) \tag{7}$$

the problem (1) has at least 2 positive solutions with at least 1 solution satisfying $\|u\| > 1$.

Proof Fix λ satisfying (7). Let

$$\Omega_1 := \{u \in H : \|u\| \leq \gamma\}; \quad \Omega_2 := \{u \in H : \|u\| \geq \zeta\},$$

where $\zeta \in (1, \gamma)$. Assume that $u_0 \in H$ is a local minimizer of J_λ in $\Omega = \Omega_1 \cap \Omega_2$. Note that for $u \in \Omega$ by (A.3) it follows that for all $k \in [1, T]$

$$|u(k)| \leq \max_{s \in [1, T]} |u(s)| \leq (T+1)^{\frac{1}{2}} \|u\| \leq (T+1)^{\frac{1}{2}} \gamma.$$

If $u_0 \in \text{Int}(\Omega)$ then $J_\lambda(u_0) < \min_{u \in \partial\Omega_1} J_\lambda(u)$ and u_0 is the element required by the mountain pass lemma. Suppose otherwise that $u_0 \in \partial\Omega_1$. Then by Theorem 6 there exist $\kappa, \sigma, \vartheta \geq 0$, $\kappa^2 + \sigma^2 + \vartheta^2 > 0$, such that for all $v \in H$

$$\sigma(\|u_0\|^2 - \gamma^2) = 0 \quad \text{and} \quad \vartheta(\zeta^2 - \|u_0\|^2) = 0 \tag{8}$$

and

$$\kappa \langle J'(u_0), v \rangle + \sigma \langle u_0, v \rangle - \vartheta \langle u_0, v \rangle = 0.$$

We note by (8) that $\vartheta = 0$. Suppose that $\sigma = 0$. Thus $\langle J'(u_0), v \rangle = 0$ and u_0 is a solution to (1). Assume that $\kappa > 0$. We may take $\kappa = 1$. Hence

$$\sum_{k=1}^{T+1} \alpha(k) |\Delta u_0(k-1)|^{p(k-1)-2} \Delta u_0(k-1) \Delta v(k-1) - \lambda \sum_{k=1}^T f(k, (u_0)_+(k)) v(k) + \sigma \sum_{k=1}^T \langle u_0(k), v(k) \rangle = 0.$$

Taking $v = u_0$, we see that

$$\sum_{k=1}^{T+1} \alpha(k) |\Delta u_0(k-1)|^{p(k-1)} + \sigma \|u_0\|^2 = \lambda \sum_{k=1}^T f(k, (u_0)_+(k)) (u_0(k)).$$

Since $u_0 \in \partial\Omega_1$, we see that $\|u_0\| = \gamma$. Thus by (A.7) we have

$$\sum_{k=1}^{T+1} \alpha(k) |\Delta u_0(k-1)|^{p(k-1)} + \sigma \|u_0\|^2 \geq \alpha^{-T^{\frac{2-p^-}{2}}\gamma^{p^-}} - (T+1).$$

On the other hand

$$\begin{aligned} & \lambda \sum_{k=1}^T f(k, (u_0)_+(k)) (u_0(k)) \leq \\ & \lambda \left(\sum_{k=1}^T \frac{\varphi_2(k)}{m(k)} |(u_0)_+(k)|^{m(k)} + \sum_{k=1}^T \psi_2(k) |(u_0)_+(k)| \right) \leq \\ & \lambda \left(\sum_{k=1}^T \frac{\varphi_2(k)}{m(k)} \left((T+1)^{\frac{1}{2}} \gamma \right)^{m(k)} + \sum_{k=1}^T \psi_2(k) (T+1)^{\frac{1}{2}} \gamma \right). \end{aligned}$$

Thus

$$\alpha - T^{\frac{2-p^-}{2}} \gamma^{p^-} - (T+1) \leq \lambda \sum_{k=1}^T \left(\frac{\varphi_2(k)}{m(k)} \left((T+1)^{\frac{1}{2}} \gamma \right)^{m(k)} + \psi_2(k) (T+1)^{\frac{1}{2}} \gamma \right),$$

a contradiction with (7). Hence $u_0 \in \text{Int}(\Omega)$ and u_0 is a local minimizer of J_λ . Thus $J_\lambda(u_0) < \min_{u \in \partial\Omega_1} J_\lambda(u)$. By the proof of Theorem 13 we know that there exists $u_1 \in H \setminus \Omega$ such that $J_\lambda(u_1) < \min_{u \in \partial\Omega_1} J_\lambda(u)$. By Lemma 5 and Lemma 9 we obtain a critical value of the functional J_λ for some $u^* \in H$. Note that u_0 and u^* are 2 different critical points of J_λ and therefore by Lemma 8 they are 2 positive solutions of the problem (1). \square

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