Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2014) 38: 311 – 317
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doi:10.3906/mat-1304-31

A reproducing kernel for a Hilbert space related to harmonic Bergman space on a domain outside compact set

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Received: 17.04.2013 • **Accepted:** 28.07.2013 • **Published Online:** 27.01.2014 • **Printed:** 24.02.2014

Abstract: In this paper for $1 \le p < \infty$ we introduce a space $\mathcal{A}^p(\Omega\backslash K)$ of all functions $u \in b^p(\Omega\backslash K)$ such that there exist $v \in b^p(\Omega)$ and $w \in b^p(\mathbb{R}^n\backslash K)$ such that $u = v + w$ on $\Omega\backslash K$, and we give a characterization of it. For the case $p = 2$ we get a reproducing kernel for a Hilbert space $\mathcal{A}^2(\Omega\backslash K)$, after which we obtain a characterization and its useful properties.

Key words: Bergman spaces, harmonic function, reproducing kernel, removable singularity

1. Introduction

The theory of Bergman spaces is connected to the theory of reproducing kernels (see [3, 4, 5]). The theory of reproducing kernels related to harmonic Bergman space is considered for example in [7] for the special case of the unit ball, or for the case of smooth bounded domains in [6], and in their references. The theory of reproducing kernels on Hilbert spaces is not new, and it can be found in [1] or elsewhere. In this paper we introduce a new reproducing kernel on the Hilbert space whose definition is closely related to harmonic Bergman space on domains outside compact sets, and we obtain its useful properties. The theory of domains outside a compact set can now be considered in the framework of this space and the new reproducing kernel introduced in this paper. 1

2. Definition of $\mathcal{A}^p(\Omega\backslash K)$ and its properties

Let Ω be an open subset of the Euclidean space $\mathbb{R}^n$ and $K \subset \Omega$ a compact set. For $1 \le p < \infty$, the harmonic Bergman space $b^p(\Omega)$ is the set of all harmonic functions on Ω that belong to $L^p(\Omega)$. In [2] we can find the following decomposition theorem.

Theorem 1 *1. $(n > 2)$: Let Ω be an open subset of $\mathbb{R}^n$ and K be a compact subset of Ω. If u is harmonic on $\Omega\backslash K$, then u has a unique decomposition of the form*

$$u = v + w,$$

where v is harmonic on Ω and w is a harmonic function on $\mathbb{R}^n\backslash K$ satisfying $\lim_{x\to\infty} w(x) = 0$.
2. $(n = 2)$: Let Ω be an open subset of $\mathbb{R}^n$ and K be a compact subset of Ω. If u is harmonic on $\Omega\backslash K$, then

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2010 *AMS Mathematics Subject Classification:* 32A36, 46E22.

u has a unique decomposition of the form

$$u = v + w,$$

where v is harmonic on Ω and w is a harmonic function on $\mathbb{R}^2 \backslash K$ satisfying $\lim_{x\to\infty} w(x) - b\log|x| = 0$ for some constant b.

In [8], an analogous result was proven for solutions of parabolic equations. In this paper we consider the set $\mathcal{A}^p(\Omega\backslash K)$ of all functions $u \in b^p(\Omega\backslash K)$ possessing unique decomposition $u = v+w$, $v \in b^p(\Omega)$, $w \in b^p(\mathbb{R}^n\backslash K)$, and the condition on w at infinity remains the same (i.e. stronger conditions than those given in the previous theorem).

From the definition, it is obvious that $\mathcal{A}^p(\Omega\backslash K)$ is a vector subspace of $b^p(\Omega\backslash K)$ and that $b^p(\Omega)$ is a vector subspace of $\mathcal{A}^p(\Omega\backslash K)$. We define a norm in $\mathcal{A}^p(\Omega\backslash K)$ as follows.

Definition 1 *Let $1 \le p < \infty$ and $u \in \mathcal{A}^p(\Omega\backslash K)$, and then define*

$$\|u\|^p_{\mathcal{A}^p(\Omega\backslash K)} = \|v\|^p_{b^p(\Omega)} + \|w\|^p_{b^p(\mathbb{R}^n\backslash K)}.$$

With this norm, $\mathcal{A}^p(\Omega\backslash K)$ becomes a normed space. We will now prove that this is a Banach space and that $b^p(\Omega)$ is its closed subspace.

Lemma 1 *Let $1 \le p < \infty$ and $u \in \mathcal{A}^p(\Omega\backslash K)$ is arbitrarily chosen. Then*

$$\|u\|_{b^p(\Omega\backslash K)} \le 2^{\frac{p-1}{p}} \|u\|_{\mathcal{A}^p(\Omega\backslash K)}.$$

Proof The result follows from the following inequalities:

$$\begin{aligned}
\|u\|^p_{b^p(\Omega\backslash K)} = \|v+w\|^p_{b^p(\Omega\backslash K)} &\le \left(\|v\|_{b^p(\Omega\backslash K)} + \|w\|_{b^p(\Omega\backslash K)}\right)^p \\
&\le 2^{p-1}\left(\|v\|^p_{b^p(\Omega\backslash K)} + \|w\|^p_{b^p(\Omega\backslash K)}\right) \\
&\le 2^{p-1}\left(\|v\|^p_{b^p(\Omega)} + \|w\|^p_{b^p(\mathbb{R}^n\backslash K)}\right) \\
&= 2^{p-1}\|u\|^p_{\mathcal{A}^p(\Omega\backslash K)}.
\end{aligned}$$

□

Theorem 2 *Suppose $x \in \Omega\backslash K$. Then*

$$|u(x)| \le \frac{2^{\frac{p-1}{p}} \|u\|_{\mathcal{A}^p(\Omega\backslash K)}}{V(B)^{1/p}\, d(x, \partial(\Omega\backslash K))^{n/p}}$$

for every $u \in \mathcal{A}^p(\Omega\backslash K)$.

Proof The proof follows immediately from Lemma 1, the fact that $\mathcal{A}^p(\Omega\backslash K) \subseteq b^p(\Omega\backslash K)$, and from Proposition 8.1 in [2]. □

Lemma 2 *Let $1 \le p < \infty$, $n \ge 2$. If $w \in b^p(\mathbb{R}^n\backslash K)$, then $\lim_{x\to\infty} w(x) = 0$.*

Proof The assertion follows directly from the fact that $\lim_{|x|\to\infty} d\left(x, \partial\left(\mathbb{R}^n \backslash K\right)\right) = \infty$. □

Theorem 3 *Let* $1 \leq p < \infty$. $\mathcal{A}^p\left(\Omega \backslash K\right)$ *with a norm* $\|.\|_{\mathcal{A}^p(\Omega\backslash K)}$ *is a Banach space.*

Proof Let (u_m) be a Cauchy sequence in $\mathcal{A}^p\left(\Omega \backslash K\right)$. From Lemma 1 it follows that (u_m) is a Cauchy sequence in $b^p\left(\Omega \backslash K\right)$. $b^p\left(\Omega \backslash K\right)$ is a Banach space, so there exists $u \in b^p\left(\Omega \backslash K\right)$ such that $u_m \to u$ in $b^p\left(\Omega \backslash K\right)$. Also,

$$\|u_m - u_k\|^p_{\mathcal{A}^p(\Omega\backslash K)} = \|v_m - v_k\|^p_{b^p(\Omega)} + \|w_m - w_k\|^p_{b^p(\mathbb{R}^n\backslash K)},$$

where $u_m = v_m + w_m$ on $\Omega \backslash K$ is a decomposition mentioned in the definition of $\mathcal{A}^p\left(\Omega \backslash K\right)$. Since (u_m) is a Cauchy sequence in $\mathcal{A}^p\left(\Omega \backslash K\right)$, it follows that (v_m) is a Cauchy sequence in $b^p\left(\Omega\right)$ and (w_m) is a Cauchy sequence in $b^p\left(\mathbb{R}^n \backslash K\right)$. Since $b^p\left(\Omega\right)$ and $b^p\left(\mathbb{R}^n \backslash K\right)$ are Banach spaces, it follows that there are $v \in b^p\left(\Omega\right)$ and $w \in b^p\left(\mathbb{R}^n \backslash K\right)$ such that $v_m \to v$ in $b^p\left(\Omega\right)$ and $w_m \to w$ in $b^p\left(\mathbb{R}^n \backslash K\right)$. Now define $u' = v + w$. By Lemma 2, $u' \in \mathcal{A}^p\left(\Omega \backslash K\right)$. Now,

$$\|u_m - u'\|^p_{\mathcal{A}^p(\Omega\backslash K)} = \|v_m - v\|^p_{b^p(\Omega)} + \|w_m - w\|^p_{b^p(\mathbb{R}^n\backslash K)},$$

so $u_m \to u'$ in $\mathcal{A}^p\left(\Omega \backslash K\right)$. Since $u_m \to u$ in $b^p\left(\Omega \backslash K\right)$ and, by Lemma 1, $u_m \to u'$ in $b^p\left(\Omega \backslash K\right)$, it yields $u = u' \in \mathcal{A}^p\left(\Omega \backslash K\right)$. This implies that $\mathcal{A}^p\left(\Omega \backslash K\right)$ is a Banach space, and so the theorem is proven. □

Lemma 3 *Let* $1 \leq p < \infty$. *The norm* $\|.\|_{b^p(\Omega)}$ *on* $b^p\left(\Omega\right)$ *is equal to the norm* $\|.\|_{\mathcal{A}^p(\Omega\backslash K)}$ *restricted to* $b^p\left(\Omega\right)$.

Proof Let $u \in b^p\left(\Omega\right)$ be arbitrarily chosen. Then $u = v + w$, where $v = u$ on Ω and $w = 0$ on $\mathbb{R}^n \backslash K$. The condition at infinity on w is obviously satisfied. Thus, $u \in \mathcal{A}^p\left(\Omega \backslash K\right)$ and $\|u\|^p_{\mathcal{A}^p(\Omega\backslash K)} = \|u\|^p_{b^p(\Omega)} + \|0\|^p_{b^p(\mathbb{R}^n\backslash K)} = \|u\|^p_{b^p(\Omega)}$, which proves this lemma. □

Theorem 4 *Let* $1 \leq p < \infty$. *Then* $b^p\left(\Omega\right)$ *is a closed subspace of* $\mathcal{A}^p\left(\Omega \backslash K\right)$.

Proof The previous lemma implies that the topology on $b^p\left(\Omega\right)$ is the same as subspace topology from $\mathcal{A}^p\left(\Omega \backslash K\right)$. Because a subspace of a complete metric space is complete if and only if it is closed in subspace topology, it follows that $b^p\left(\Omega\right)$ is closed in $\mathcal{A}^p\left(\Omega \backslash K\right)$. □

Theorem 5

$$\mathcal{A}^p\left(\Omega \backslash K\right) = V \oplus W, \tag{1}$$

where $V = b^p\left(\Omega\right)|_{\Omega\backslash K}$ *and* $W = b^p\left(\mathbb{R}^n \backslash K\right)|_{\Omega\backslash K}$.

Proof Lemma 2 and the definition of $\mathcal{A}^p\left(\Omega \backslash K\right)$ imply that it is sufficient to prove that $V \cap W = \{0\}$. Let $u \in V \cap W$ be arbitrarily chosen. There exists $v \in b^p\left(\Omega\right)$ such that $u\left(x\right) = v\left(x\right)$ for all $x \in \Omega \backslash K$. Also, there exists $w \in b^p\left(\mathbb{R}^n \backslash K\right)$ such that $u\left(x\right) = w\left(x\right)$ for all $x \in \Omega \backslash K$. So, $v\left(x\right) = w\left(x\right)$ for all $x \in \Omega \backslash K$. Let us define the function

$$\psi(x) = \begin{cases} v\left(x\right) & \text{for} \quad x \in \Omega \\ w\left(x\right) & \text{for} \quad x \in \mathbb{R}^n \backslash K. \end{cases}$$

The defined function $\psi(x)$ is harmonic on $\mathbb{R}^n$ and

$$\int_{\mathbb{R}^n} |\psi(x)|^p dx \leq \int_{\Omega} |v(x)|^p dx + \int_{\mathbb{R}^n\setminus K} |w(x)|^p dx < \infty,$$

so $\psi \in b^p(\mathbb{R}^n) = \{0\}$ (see [2]). It follows that $v = 0$ on Ω and $w = 0$ on $\mathbb{R}^n\setminus K$. So, $V \cap W = \{0\}$. The proof follows. □

Lemma 4 *Let Ω' be an open set in $\mathbb{R}^n$ such that $\Omega \subseteq \Omega'$ and let K' be a compact set such that $K' \subseteq K$. Then*

$$\mathcal{A}^p(\Omega'\setminus K') = b^p(\Omega'\setminus K') \cap \mathcal{A}^p(\Omega\setminus K).$$

Proof The proof will be done in 2 parts. " $\subseteq$ " Let $u \in \mathcal{A}^p(\Omega'\setminus K')$, and then $u = v + w$ on $\Omega'\setminus K'$, where $v \in b^p(\Omega') \subseteq b^p(\Omega)$ and $w \in b^p(\mathbb{R}^n\setminus K') \subseteq b^p(\mathbb{R}^n\setminus K)$. Therefore, we have that $u \in \mathcal{A}^p(\Omega\setminus K)$. Additionally, $\mathcal{A}^p(\Omega'\setminus K') \subseteq b^p(\Omega'\setminus K')$, so this part is done.
" $\supseteq$ " Let $u \in b^p(\Omega'\setminus K') \cap \mathcal{A}^p(\Omega\setminus K)$. The decomposition theorem for harmonic functions and the fact that $b^p(\Omega'\setminus K') \subseteq h(\Omega'\setminus K')$ yields that $u = v + w$ on $\Omega'\setminus K'$, where $v \in h(\Omega')$, $w \in h(\mathbb{R}^n\setminus K')$, $v \in b^p(\Omega)$, and $w \in b^p(\mathbb{R}^n\setminus K)$. It is sufficient to prove that $v \in L^p(\Omega')$ and $w \in L^p(\mathbb{R}^n\setminus K')$. From $u \in L^p(\Omega'\setminus K') \subseteq L^p(\Omega'\setminus\Omega)$ and $w \in L^p(\mathbb{R}^n\setminus K') \subseteq L^p(\Omega'\setminus\Omega)$ it follows that $v = u - w \in L^p(\Omega'\setminus\Omega)$. However, we have that $v \in b^p(\Omega) \subseteq L^p(\Omega)$, so we have $v \in L^p(\Omega')$. Also, we have $u \in L^p(\Omega'\setminus K') \subseteq L^p(K\setminus K')$ and $v \in L^p(\Omega) \subseteq L^p(K\setminus K')$, so it follows that $w = u - v \in L^p(K\setminus K')$. Since additionaly $w \in b^p(\mathbb{R}^n\setminus K) \subseteq L^p(\mathbb{R}^n\setminus K)$, it follows that $w \in L^p(\mathbb{R}^n\setminus K')$, so the second part and hence the lemma are proven. □

Corollary 1 *If $\mathcal{A}^p(\Omega\setminus K) = b^p(\Omega\setminus K)$, then $\mathcal{A}^p(\Omega'\setminus K') = b^p(\Omega'\setminus K')$, for every $K' \subseteq K$ and $\Omega' \supseteq \Omega$.*

Corollary 2 *If $\mathcal{A}^p(\Omega\setminus K) = b^p(\Omega\setminus K)$, then $\mathcal{A}^p(\Omega\setminus\{a\}) = b^p(\Omega\setminus\{a\})$, for every $a \in K$.*

Theorem 6 *Let $a \in \Omega$ be arbitrarily chosen. Then $\mathcal{A}^p(\Omega\setminus\{a\}) = b^p(\Omega\setminus\{a\})$ if and only if a is a removable singularity for any $u \in b^p(\Omega\setminus\{a\})$.*

Proof We have

$$\mathcal{A}^p(\Omega\setminus\{a\}) = V \oplus W,$$

where $V = b^p(\Omega)|_{\Omega\setminus\{a\}}$ and $W = b^p(\mathbb{R}^n\setminus\{a\})|_{\Omega\setminus\{a\}} = \{0\}$. So, $\mathcal{A}^p(\Omega\setminus\{a\}) = b^p(\Omega\setminus\{a\})$ if and only if $b^p(\Omega)|_{\Omega\setminus\{a\}} = b^p(\Omega\setminus\{a\})$. As $b^p(\Omega)|_{\Omega\setminus\{a\}} \subseteq b^p(\Omega\setminus\{a\})$, " $\supseteq$ " follows if and only if for each $u \in b^p(\Omega\setminus\{a\})$, there exists $v \in h(\Omega)$ such that $v = u$ on $\Omega\setminus\{a\}$, i.e. if and only if for every $u \in b^p(\Omega\setminus\{a\})$, a is a removable singularity of u. So, the proof is finished. □

Theorem 7 *If $n \geq 3$ and $p \geq \frac{n}{n-2}$, then $\mathcal{A}^p(\Omega\setminus\{a\}) = b^p(\Omega\setminus\{a\})$, for every $a \in \Omega$.*

Proof It is sufficient to prove that a is a removable singularity for every $u \in b^p(\Omega\setminus\{a\})$. It can be shown (see [2]) that u has a removable singularity at a if and only if

$$\lim_{x\to a} |x-a|^{n-2}|u(x)| = 0.$$

Also, by using Proposition 8.1 in [2] we can prove that if $u \in b^p(\Omega\backslash\{a\})$, then

$$u(x) = o\left(|x-a|^{n/p}\right),$$

as $x \to a$. Now, the proof follows. □

Remark 1 *For $n = 2$, we can find a function $u \in \bigcap_{1 \leq p < \infty} b^p(B\backslash\{0\})$ such that 0 is not a removable singularity of u. So, for $n = 2$, $\mathcal{A}^p(B\backslash\{0\}) \neq b^p(B\backslash\{0\})$, for all $1 \leq p < \infty$, where B is a unit ball in $\mathbb{R}^n$ (see [2]).*

Remark 2 *It would be interesting to characterize the set M of all (n, p, Ω, K) such that $\mathcal{A}^p(\Omega\backslash K) = b^p(\Omega\backslash K)$. From the last theorem we see that $(n, p, \Omega, \{a\}) \in M$ for $n \geq 3, p \geq \frac{n}{n-2}$, and $a \in \Omega$, arbitrarily chosen. Also, from Remark 1 and Corollary 2 we have that for any $p \geq 1$ and any compact set $K \subseteq B$, where B is a unit ball in $\mathbb{R}^n$, $(2, p, B, K) \notin M$.*

3. A reproducing kernel for $\mathcal{A}^2(\Omega\backslash K)$

For $p = 2$, $\mathcal{A}^2(\Omega\backslash K)$ is a Hilbert space with inner product defined by

$$\langle u_1, u_2\rangle_{\mathcal{A}^2(\Omega\backslash K)} = \langle v_1, v_2\rangle_{b^2(\Omega)} + \langle w_1, w_2\rangle_{b^2(\mathbb{R}^n\backslash K)},$$

where $u_1 = v_1 + w_1$ and $u_2 = v_2 + w_2$ are decompositions of u_1 and u_2 in $\mathcal{A}^2(\Omega\backslash K)$, respectively.
From Lemma 2 we have that for every $x \in \Omega\backslash K$, a map $u \longmapsto u(x)$ is a bounded linear functional on $\mathcal{A}^2(\Omega\backslash K)$. So, there exist $S_{\Omega\backslash K}(x, \cdot) \in \mathcal{A}^2(\Omega\backslash K)$ such that $u(x) = \langle u, S_{\Omega\backslash K}(x, \cdot)\rangle_{\mathcal{A}^2(\Omega\backslash K)}$, for every $u \in \mathcal{A}^2(\Omega\backslash K)$.
As $S_{\Omega\backslash K}(x, \cdot) \in \mathcal{A}^2(\Omega\backslash K)$, there exist unique $V_\Omega(x, \cdot) \in b^2(\Omega)$ and $W_{\mathbb{R}^n\backslash K}(x, \cdot) \in b^2(\mathbb{R}^n\backslash K)$, such that $S_{\Omega\backslash K}(x, \cdot) = V_\Omega(x, \cdot) + W_{\mathbb{R}^n\backslash K}(x, \cdot)$ on $\Omega\backslash K$. Let $x \in \Omega\backslash K$ be arbitrarily chosen. It follows that

$$u(x) = \int_\Omega v(y)\overline{V_\Omega(x, y)}dy + \int_{\mathbb{R}^n\backslash K} w(y)\overline{W_{\mathbb{R}^n\backslash K}(x, y)}dy,$$

where $u = v + w$ is a decomposition of u in $\mathcal{A}^2(\Omega\backslash K)$. By the fact that $b^2(\Omega) \subseteq \mathcal{A}^2(\Omega\backslash K)$ and $b^2(\mathbb{R}^n\backslash K) \subseteq \mathcal{A}^2(\Omega\backslash K)$, it follows that $V_\Omega(x, \cdot) = R_\Omega(x, \cdot)$ on Ω and $W_{\mathbb{R}^n\backslash K}(x, \cdot) = R_{\mathbb{R}^n\backslash K}(x, \cdot)$ on $\mathbb{R}^n\backslash K$, where R_Ω and $R_{\mathbb{R}^n\backslash K}$ are reproducing kernels on Ω and $\mathbb{R}^n\backslash K$, respectively. We proved the following theorem.

Theorem 8 *It holds that*

$$S_{\Omega\backslash K}(x, \cdot) = R_\Omega(x, \cdot) + R_{\mathbb{R}^n\backslash K}(x, \cdot)$$

on $\Omega\backslash K$.

Some useful properties of the reproducing kernel $S_{\Omega\backslash K}$ on $\Omega\backslash K$ are given by the following theorem.

Theorem 9 *The reproducing kernel $S_{\Omega\backslash K}$ on $\Omega\backslash K$ has the following properties:*
1. $S_{\Omega\backslash K}$ is real-valued.

2. If (u_m) *is an orthonormal basis of* $\mathcal{A}^2(\Omega\backslash K)$*, then*

$$S_{\Omega\backslash K}(x,y) = \sum_{m=1}^{\infty} \overline{u_m(x)} u_m(y)$$

for all $x,y \in \Omega\backslash K$.

3. $S_{\Omega\backslash K}(x,y) = S_{\Omega\backslash K}(y,x)$ *for all* $x,y \in \Omega\backslash K$.

4. $\|S_{\Omega\backslash K}(x,\cdot)\|^2_{\mathcal{A}^2(\Omega\backslash K)} = S_{\Omega\backslash K}(x,x)$ *for all* $x \in \Omega\backslash K$.

Proof 1. and 3. follow immediately from Theorem 8 and from Proposition 8.4 in [2].

2. If $u_m = v_m + w_m$ is a decomposition of $\mathcal{A}^2(\Omega\backslash K)$, then we have

$$\begin{aligned} S_{\Omega\backslash K}(x,y) &= \sum_{m=1}^{\infty} \langle S_{\Omega\backslash K}(x,\cdot), u_m\rangle_{\mathcal{A}^2(\Omega\backslash K)} u_m(y) \\ &= \sum_{m=1}^{\infty} \left(\langle R_\Omega(x,\cdot), v_m\rangle_{b^2(\Omega)} + \langle R_{\mathbb{R}^n\backslash K}(x,\cdot) w_m\rangle\right) u_m(y) \\ &= \sum_{m=1}^{\infty} \left(\overline{v_m(x)} + \overline{w_m(x)}\right) u_m(y) \end{aligned}$$

and claim 2. follows.

4.

$$\begin{aligned} \|S_{\Omega\backslash K}(x,\cdot)\|^2_{\mathcal{A}^2(\Omega\backslash K)} &= \|R_\Omega(x,\cdot)\|^2_{b^2(\Omega)} + \|R_{\mathbb{R}^n\backslash K}\|^2_{b^2(\mathbb{R}^n\backslash K)} \\ &= R_\Omega(x,x) + R_{\mathbb{R}^n\backslash K}(x,x) \\ &= S_{\Omega\backslash K}(x,x) \end{aligned}$$

□

The following lemma can be found in [2].

Lemma 5 *Suppose* $\Omega_1 \subseteq \Omega_2 \subseteq \ldots$ *is an increasing sequence of open subsets of* $\mathbb{R}^n$ *and* $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$*. Then* $R_\Omega(x,y) = \lim_{k\to\infty} R_{\Omega_k}(x,y)$*, for all* $x,y \in \Omega$.

Theorem 10 *Supppose* $\Omega_1 \subseteq \Omega_2 \subseteq \ldots$ *is an increasing sequence of open subsets of* $\mathbb{R}^n$*, and* $K_1 \supseteq K_2 \supseteq \ldots$ *is a decreasing sequence of compact sets,* $K_1 \subset \Omega_1$*. If we denote* $\Omega'_j = \Omega_j\backslash K_j, \Omega = \bigcup_{j=1}^{\infty} \Omega_j, K = \bigcap_{j=1}^{\infty} K_j$ *then* $S_{\Omega\backslash K}(x,y) = \lim_{j\to\infty} S_{\Omega'_j}(x,y)$ *for all* $x,y \in \Omega\backslash K$.

Proof By Lemma 5 we have

$$\begin{aligned} S_{\Omega\backslash K}(x,y) &= R_\Omega(x,y) + R_{\mathbb{R}^n\backslash K}(x,y) \\ &= \lim_{j\to\infty} R_{\Omega_j}(x,y) + \lim_{j\to\infty} R_{\mathbb{R}^n\backslash K_j}(x,y) \\ &= \lim_{j\to\infty} S_{\Omega_j\backslash K_j}(x,y) = \lim_{j\to\infty} S_{\Omega'_j}(x,y) \end{aligned}$$

for every $x,y \in \Omega\backslash K$ and the theorem is proven. □

Remark 3 *If* $\mathcal{A}^2(\Omega\backslash K) = b^2(\Omega\backslash K)$*, then for any* $x \in \Omega\backslash K$*, we have* $R_{\Omega\backslash K}(x,\cdot) \in \mathcal{A}^2(\Omega\backslash K)$*, so it would be interesting to see relations between* $S_{\Omega\backslash K}(x,\cdot)$ *and* $R_{\Omega\backslash K}(x,\cdot)$ *and their components in* $\mathcal{A}^2(\Omega\backslash K)$*. We can easily show by using Theorem 7 that* $S_{\Omega\backslash(a)}(x,\cdot) = R_{\Omega\backslash\{a\}}(x,\cdot)$ *holds for* $n \geq 4$ *and every* $a \in \Omega$*. The hardness of a problem to find a compact set for which equality of reproducing kernels does not hold (if there is any) is influenced by not having an explicit formula for reproducing kernels in harmonic Bergman space on domains outside compact sets.*

Remark 4 *It would be interesting to consider the same ideas in the case of solutions of parabolic equations, because the decomposition theorem holds in that case also (see [8]).*

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