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Pullback diagram of H^*-algebras

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Abstract: In this paper we obtain some properties for the pullback diagram of H^*-algebras. More precisely, we prove that the commutative diagram of H^*-algebras and morphisms

$$\begin{array}{ccc} A_1 & \xrightarrow{\varphi_1} & B_1 \\ \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \psi_2} \\ A_2 & \xrightarrow{\varphi_2} & B_2 \end{array}$$

is pullback and ψ_1 is an injection if and only if ψ_1 is a surjection, ψ_2 is an injection, and $\ker\varphi_1 \cap \ker\psi_1 = \{0\}$.

Key words: H^*-algebra, morphisms, pullback diagram, trace-class

1. Introduction and preliminaries

Pedersen [5] introduced the notion of a pullback diagram in the category of C^*-algebras and investigated some properties of these diagrams. The authors of [2] generalized the construction of a pullback diagram in the category of Hilbert C^*-modules. Some properties of pullback diagrams are stable under H^*-algebras [2, 5]. In this paper we use these properties to discover new ones for pullback diagrams of H^*-algebras.

An H^*-algebra, introduced by Ambrose [1] in the associative case, is a Banach algebra A, satisfying the following conditions:

(i) A is itself a Hilbert space under an inner product $\langle .,. \rangle$;

(ii) For each a in A there is an element a^* in A, the so-called adjoint of a, such that we have both $\langle ab, c\rangle = \langle b, a^*c\rangle$ and $\langle ab, c\rangle = \langle a, cb^*\rangle$ for all $b, c \in A$.

Example 1.1 *The Hilbert space* $l^2 = \{(a_n) : \quad a_n \in \mathbb{C}, \quad \sum_{n=1}^{\infty} |a_n|^2 < \infty\}$ *is an* H^**-algebra, where for each* (a_n) *and* (b_n) *in* l^2, $(a_n)(b_n) = (a_n b_n)$ *and* $(a_n)^* = (\overline{a_n})$.

Example 1.2 *Any Hilbert space is an* H^**-algebra, where the product of each pair of elements is to be zero. Of course, in this case, the adjoint* a^* *of* a *need not be unique; in fact, every element is an adjoint of every element.*

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Using the axiom of choice we can assign to each element a of an H^*-algebra A a unique element $a^* \in A$. From now on any H^*-algebra is assumed to be equipped with a choice function, and so we have a well-defined function $a \mapsto a^*$ on any H^*-algebra. If A and B are H^*-algebras, then a continuous $*$-homomorphism $\varphi : A \to B$ is called a morphism.
Recall that $A_0 = \{a \in A : \ aA = \{0\}\} = \{a \in A : \ Aa = \{0\}\}$ is called the annihilator ideal of A. A proper H^*-algebra is an H^*-algebra with zero annihilator ideal. Ambrose [1] proved that an H^*-algebra is proper if and only if every element has a unique adjoint and showed that every H^*-algebra A can be presented as a direct sum of the form $A_0 \oplus A_0^{\perp}$, where $A_0^{\perp} = \{a \in A : \langle a, b \rangle = 0, \quad \text{for all} \quad b \in A_0\}$.

The trace-class $\tau(A)$ of A is defined by the set $\tau(A) = \{ab : \quad a, b \in A\}$. The trace functional tr on $\tau(A)$ is defined by $tr(ab) = \langle b, a^* \rangle = \langle a, b^* \rangle = tr(ba)$ for each $a, b \in A$, in particular $tr(aa^*) = \langle a, a \rangle = \| a \|^2$, for all $a \in A$. Since Ambrose [1] up to now, there are many mathematicians who have worked on H^*-algebras and developed it in several directions; see [4], [3], [6], and the references cited therein. In this paper, we generalize the concept of the pullback diagram in the framework of H^*-algebras and investigate some conditions under which a diagram of H^*-algebras is pullback.

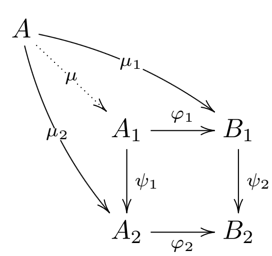
2. Pullback constructions in H^*-algebras

In this section we introduce a pullback diagram of H^*-algebras and investigate some of its properties. For this, we need the following definition.

Definition 2.1 *A commutative diagram of H^*-algebras and morphisms*

$$\begin{array}{ccc} A_1 & \xrightarrow{\varphi_1} & B_1 \\ \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \psi_2} \\ A_2 & \xrightarrow{\varphi_2} & B_2 \end{array}$$

is pullback if $\ker(\varphi_1) \cap \ker(\psi_1) = \{0\}$ *and for every other pair of morphisms $\mu_1 : A \to B_1$ and $\mu_2 : A \to A_2$ from an H^*-algebra A that satisfies condition $\psi_2\mu_1 = \varphi_2\mu_2$, there is a unique morphism $\mu : A \to A_1$ such that $\mu_1 = \varphi_1\mu$ and $\mu_2 = \psi_1\mu$.*



The following theorem is proven in the framework of C^*-algebras. It is easy to show this theorem in the category of H^*-algebras.

Theorem 2.2 (see [5, proposition 3.1]) A commutative diagram of H^*-algebras and morphisms

$$\begin{array}{ccc} A_1 & \xrightarrow{\varphi_1} & B_1 \\ \downarrow{\psi_1} & & \downarrow{\psi_2} \\ A_2 & \xrightarrow{\varphi_2} & B_2 \end{array} \tag{1}$$

is pullback if and only if the following conditions hold:
$(i)\ker(\varphi_1)\cap\ker(\psi_1)=\{0\}$,
$(ii)\psi_1(\ker(\varphi_1))=\ker(\varphi_2)$,
$(iii){\psi_2}^{-1}(\varphi_2(A_2))=\varphi_1(A_1)$.

Note that we can replace (i) by $\ker\varphi_1\subseteq(\ker\psi_1)^\perp$ in the category of H^*-algebras. Recall that $\ker\varphi_1$ and $\ker\psi_1$ are the ideals of A. For each $a\in\ker\varphi_1$ and $b\in\ker\psi_1$, we have $ab^*\in\ker\varphi_1\ker\psi_1=\ker\varphi_1\cap\ker\psi_1=\{0\}$. So by [6], $\langle a,b\rangle=tr(b^*a)=tr(ab^*)=0$, and then $\ker\varphi_1$ is orthogonal to $\ker\psi_1$.

Theorem 2.3 *Let (1) be a pullback diagram of H^*-algebras. Then ψ_1 is surjective and ${\psi_1}^{-1}(\ker\varphi_2)=\ker\varphi_1\oplus\ker\psi_1$. Also, ψ_1 is injective if and only if ψ_2 is injective.*

Proof Since the diagram is pullback, the conditions of Theorem 2.2 hold.

First we show that ψ_1 is surjective. On the contrary, suppose that there exists $a_2\in A_2$ such that $a_2\neq\psi_1(a_1)$ for each $a_1\in A_1$. So, by (ii), $a_2\notin\ker\varphi_2$. By (iii), there exists $a_1'\in A_1$ such that $\psi_2\varphi_1(a_1')=\varphi_2(a_2)$. By commutativity of the diagram $\varphi_2(a_2)=\varphi_2(\psi_1(a_1'))$ and $a_2-\psi_1(a_1')\in\ker\varphi_2$. By (ii), there exists $a_1''\in\ker\varphi_1$ such that $a_2-\psi_1(a_1')=\psi_1(a_1'')$. Thus, $a_2=\psi_1(a_1'+a_1'')$. This contradicts the initial assumption. Hence, ψ_1 is surjective.

For the second part, let $a_1+b_1\in\ker\varphi_1\oplus\ker\psi_1$. By (ii), we have $\psi_1(a_1+b_1)=\psi_1(a_1)\in\ker\varphi_2$. Then $\ker\varphi_1\oplus\ker\psi_1\subseteq{\psi_1}^{-1}(\ker\varphi_2)$.

Conversely, let $a_1\in{\psi_1}^{-1}(\ker\varphi_2)$ be arbitrary. Then there exists $a_2\in\ker\varphi_2$ such that $\psi_1(a_1)=a_2$. We also have $A_1=(\ker\psi_1)^\perp\oplus\ker\psi_1$, since $\ker\psi_1$ is a closed ideal of H^*-algebra A_1. We can write $a_1=b_1+b_1'$, for some $b_1\in(\ker\psi_1)^\perp$ and $b_1'\in\ker\psi_1$. Hence, $\psi_1(a_1)=\psi_1(b_1')+\psi_1(b_1)=\psi_1(b_1)$ and by (ii), there exists $b_1''\in\ker\varphi_1\subseteq(\ker\psi_1)^\perp$ such that $\psi_1(b_1'')=a_2$. By the above discussion it is enough to show that $b_1\in\ker\varphi_1$, but $b_1''-b_1\in\ker\psi_1$, since $\psi_1(b_1''-b_1)=a_2-a_2=0$. On the other hand, $b_1''-b_1\in(\ker\psi_1)^\perp$. Thus, $b_1=b_1''\in\ker\varphi_1$.

Now suppose that ψ_1 is injective. By (ii), $\ker\varphi_1={\psi_1}^{-1}(\ker\varphi_2)$. By (iii), if $b_1\in\ker\psi_2=\psi_2^{-1}(\{0\})\subseteq\psi_2^{-1}(\varphi_2(A_2))=\varphi_1(A_1)$, there exists $a_1\in A_1$ such that $b_1=\varphi_1(a_1)$. Thus, $\varphi_2\psi_1(a_1)=\psi_2\varphi_1(a_1)=\psi_2(b_1)=0$. This forces that $\psi_1(a_1)\in\ker\varphi_2$ or $a_1\in{\psi_1}^{-1}(\ker\varphi_2)=\ker\varphi_1$. Then $b_1=\varphi_1(a_1)=0$ and ψ_2 is injective.

Suppose that ψ_2 is injective. To prove injectivity of ψ_1, we show that $\ker\psi_1=0$ or $\psi_1^{-1}(\ker\varphi_2)=\ker\varphi_1\oplus 0$. By (ii), $\ker\varphi_1\subseteq{\psi_1}^{-1}(\ker\varphi_2)$. It is enough to show that ${\psi_1}^{-1}(\ker\varphi_2)\subseteq\ker\varphi_1$. Let $a_1\in{\psi_1}^{-1}(\ker\varphi_2)$. Then there exists $a_2\in\ker\varphi_2$ such that $\psi_1(a_1)=a_2$, but $0=\varphi_2(a_2)=\varphi_2\psi_1(a_1)=\psi_2\varphi_1(a_1)$. Since ψ_2 is injective, $\varphi_1(a_1)=0$. Hence, $a_1\in\ker\varphi_1$. □

In the next theorem we introduce some properties under which a diagram of H^*-algebras is pullback.

Theorem 2.4 *Suppose*

$$\begin{array}{ccc} A_1 & \xrightarrow{\varphi_1} & B_1 \\ \downarrow{\psi_1} & & \downarrow{\psi_2} \\ A_2 & \xrightarrow{\varphi_2} & B_2 \end{array}$$

is a commutative diagram of H^*-algebras and morphisms. Then this diagram is pullback and ψ_1 is an injection if and only if ψ_1 is a surjection, ψ_2 is an injection, and $\ker\varphi_1 \cap \ker\psi_1 = \{0\}$.

Proof Suppose that the diagram is pullback and ψ_1 is an injection. Then by Theorem 2.3, ψ_1 is surjective, ψ_2 is injective, and $\ker\varphi_1 \cap \ker\psi_1 = \{0\}$.

Conversely, let ψ_1 be a surjection, ψ_2 be an injection, and $\ker\varphi_1 \cap \ker\psi_1 = \{0\}$. We are going to show that $\psi_1(\ker\varphi_1) = \ker\varphi_2$ and ${\psi_2}^{-1}(\varphi_2(A_2)) = \varphi_1(A_1)$.

Let $a_2 \in \psi_1(\ker\varphi_1)$. Then there exists $a_1 \in \ker\varphi_1$ such that $a_2 = \psi_1(a_1)$. By commutativity of the diagram and injectivity of ψ_2, we have $\varphi_2(a_2) = \varphi_2\psi_1(a_1) = \psi_2\varphi_1(a_1) = 0$. Hence, $a_2 \in \ker\varphi_2$ and $\psi_1(\ker\varphi_1) \subseteq \ker\varphi_2$.

Conversely, let $a_2 \in \ker\varphi_2$. Then $\psi_1(a_1) = a_2$ for some $a_1 \in A_1$, since ψ_1 is onto. But $\varphi_2(a_2) = \varphi_2\psi_1(a_1) = \psi_2\varphi_1(a_1) = 0$. Hence, $\varphi_1(a_1) = 0$, since ψ_2 is an injection. It implies that $a_1 \in \ker\varphi_1$ and $a_2 = \psi_1(a_1) \in \psi_1(\ker\varphi_1)$. Hence, $\ker\varphi_2 \subseteq \psi_1(\ker\varphi_1)$.

For the next part, let $b_1 \in \varphi_1(A_1)$. Then $b_1 = \varphi_1(a_1)$ for some $a_1 \in A_1$. Put $a_2 = \psi_1(a_1)$. Hence, $\psi_2^{-1}(\varphi_2(a_2)) = \psi_2^{-1}(\varphi_2(\psi_1(a_1))) = \psi_2^{-1}(\psi_2(\varphi_1(a_1))) = \varphi_1(a_1) = b_1$. Therefore, $b_1 \in \psi_2^{-1}(\varphi_2(A_2))$. Conversely, suppose that $b_1 \in \psi_2^{-1}(\varphi_2(A_2))$. Then there exists $a_2 \in A_2$ such that $b_1 = \psi_2^{-1}\varphi_2(a_2)$. By surjectivity of ψ_1 there exists $a_1 \in A_1$ such that $\psi_1(a_1) = a_2$. Thus, $b_1 = \psi_2^{-1}(\varphi_2(\psi_1(a_1))) = \psi_2^{-1}(\psi_2(\varphi_1(a_1))) = \varphi_1(a_1)$, since ψ_2 is injective. So $b_1 \in \varphi_1(A_1)$. Since ψ_2 is injective, ψ_1 is too by Theorem 2.3. □

Now we want to show that the pullback diagram of H^*-algebras have some further properties beyond corresponding to the case of the pullback diagram of C^*-algebras.

Theorem 2.5 *Suppose that the diagram (1) is pullback, and then the following statements are true.*

(i) $\psi_1((\ker\varphi_1)^\perp) = (\ker\varphi_2)^\perp$,

(ii) ${\psi_1}^*((\ker\varphi_2)^\perp) \subseteq (\ker\varphi_1)^\perp$,

(iii) $\psi_2\varphi_1((\ker\varphi_1)^\perp) = \varphi_2\psi_1((\ker\varphi_1)^\perp)$,

(iv) ${\psi_2}^{-1}(\varphi_2((\ker\varphi_2)^\perp)) = \varphi_1((\ker\varphi_1)^\perp)$,

(v) $\ker\varphi_1 \cap \ker\psi_1 = \{0\}$.

Conversely, if the diagram (1) is commutative, it satisfies in the conditions (iv) and (v), and ψ_1 is surjective, then it is a pullback diagram.

Proof

(i) Suppose that the diagram is pullback. Then the conditions of 2.2 hold and by Theorem 2.3 ψ_1 is surjective. First we show that $\psi_1((\ker\varphi_1)^\perp) \subseteq (\psi_1(\ker\varphi_1))^\perp = (\ker\varphi_2)^\perp$. Let $\psi_1(a_1)$ and $\psi_1(a_1')$ be elements in $\psi_1((\ker\varphi_1)^\perp)$ and $\psi_1(\ker\varphi_1)$, respectively. Then $\langle\psi_1(a_1), \psi_1(a_1')\rangle = tr(\psi_1({a_1}^*a_1')) = tr(0) = 0$ (since

$(\ker\varphi_1)^{\perp}$ is a closed ideal of A_1 and $a_1{}^*a_1' \in (\ker\varphi_1)^{\perp}(\ker\varphi_1) = (\ker\varphi_1)^{\perp} \cap (\ker\varphi_1) = \{0\})$.
Conversely, let $a_2'' \in (\ker\varphi_2)^{\perp}$ and $a_2' \in (\ker\varphi_2)$ be arbitrary. Then $a_2' + a_2'' = a_2 \in A_2$. By surjectivity of ψ_1, there exists $a_1 \in A_1$ such that $a_2 = \psi_1(a_1)$. On the other hand, $A_1 = \ker\varphi_1 \oplus (\ker\varphi_1)^{\perp}$, so $a_1 = a_1' + a_1''$ for some $a_1' \in \ker\varphi_1$ and $a_1'' \in (\ker\varphi_1)^{\perp}$, and by linearity of ψ_1, we can write $a_2 = \psi_1(a_1') + \psi_1(a_1'')$.

Therefore, $a_2' - \psi_1(a_1') = \psi_1(a_1'') - a_2''$. The left side of equality is in $\ker\varphi_2$ and by the above inclusion the right side belongs to $(\ker\varphi_2)^{\perp}$. This implies that $\psi_1(a_1'') = a_2''$. Hence, $(\ker\varphi_2)^{\perp} \subseteq \psi_1((\ker\varphi_1)^{\perp})$.

(ii) Let $a_1 \in \psi_1{}^*((\ker\varphi_2)^{\perp})$. Then there exists $a_2' \in (\ker\varphi_2)^{\perp}$ such that $a_1 = \psi_1{}^*(a_2')$. For each $a_1{}' \in \ker\varphi_1$, we have $\langle a_1, a_1{}'\rangle = \langle \psi_1{}^*(a_2'), a_1'\rangle = \langle a_2', \psi_1(a_1')\rangle = 0$. Then $a_1 \in (\ker\varphi_1)^{\perp}$.
(Recall that by pullbackness of diagram $\psi_1(\ker\varphi_1) = \ker\varphi_2$ and $\psi_1{}^*$ is the adjoint of ψ_1 in $B(A_1, A_2)$.)

Parts (iii) and (v) are obvious and for proving (iv),

$$\begin{aligned}
\psi_2{}^{-1}(\varphi_2(\ker\varphi_2)^{\perp}) &= \psi_2{}^{-1}(\varphi_2(\ker\varphi_2 \oplus (\ker\varphi_2)^{\perp}))\\
&= \psi_2{}^{-1}(\varphi_2(A_2))\\
&= \varphi_1(A_1)\\
&= \varphi_1(\ker\varphi_1 \oplus (\ker\varphi_1)^{\perp})\\
&= \varphi_1((\ker\varphi_1)^{\perp}).
\end{aligned}$$

Conversely, by surjectivity of ψ_1 and commutativity of the diagram, we have $\psi_1(\ker\varphi_1) \subseteq \ker\varphi_2$. If $a_2' \in \ker\varphi_2$ and $a_2'' \in (\ker\varphi_1)^{\perp}$, then $a_2' + a_2'' \in \ker\varphi_2 \oplus (\ker\varphi_2)^{\perp} = A_2 = \psi_1(A_1) = \psi_1(\ker\varphi_1 \oplus (\ker\varphi_1)^{\perp})$. Therefore, there exists $a_1' + a_1'' \in \ker\varphi_1 \oplus (\ker\varphi_1)^{\perp}$, such that $a_2' + a_2'' = \psi_1(a_1' + a_1'') = \psi_1(a_1') + \psi_1(a_1'')$. Then $a_2' - \psi_1(a_1') = \psi_1(a_1'') - a_2''$. The left side of equality is in $\ker\varphi_2$ and the right side belongs to $(\ker\varphi_2)^{\perp}$. It implies that $a_2' = \psi_1(a_1{}')$. Then $\ker\varphi_2 \subseteq \psi_1(\ker\varphi_1)$. Now by (iv) we have,

$$\begin{aligned}
\psi_2{}^{-1}(\varphi_2(A_2)) &= \psi_2{}^{-1}(\varphi_2(\ker\varphi_2 \oplus (\ker\varphi_2)^{\perp}))\\
&= \psi_2{}^{-1}(\varphi_2(\ker\varphi_2)^{\perp})\\
&= \varphi_1((\ker\varphi_1)^{\perp})\\
&= \varphi_1(\ker\varphi_1 \oplus (\ker\varphi_1)^{\perp})\\
&= \varphi_1(A_1).
\end{aligned}$$

By (v) all conditions of 2.2 holds. Then the diagram is pullback. □

We can verify that Theorem 2.2 holds for diagrams of trace classes of H^*-algebras.

Theorem 2.6 *Consider the following left commutative diagram of H^*-algebras and morphisms, in which ψ_2 is injective.*

$$\begin{array}{ccc}
A_1 & \xrightarrow{\varphi_1} & B_1\\
\downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \psi_2}\\
A_2 & \xrightarrow{\varphi_2} & B_2
\end{array}
\qquad\qquad
\begin{array}{ccc}
\tau(A_1) & \xrightarrow{\varphi_1{}'} & \tau(B_1)\\
\downarrow{\scriptstyle \psi_1'} & & \downarrow{\scriptstyle \psi_2{}'}\\
\tau(A_2) & \xrightarrow{\varphi_2'} & \tau(B_2)
\end{array}$$

If the left diagram is pullback, then the right diagram is pullback. Conversely, if the right commutative diagram is pullback, $\psi_1(A_1)$ is closed, and A_2 is a proper H^*-algebra, then the left diagram is pullback.

Proof Suppose that the left diagram is pullback. It thus satisfies in the conditions of Theorem 2.2. We prove that the right diagram satisfies in the conditions of Theorem 2.2. Recall that for $i = 1, 2$ $\varphi'_i = \varphi_i|_{\tau(A_i)}$ and $\psi'_1 = \psi_1|_{\tau(A_1)}$, $\psi'_2 = \psi_2|_{\tau(A_2)}$.

(i) $\ker\varphi'_1 \cap \ker\psi'_1 \subseteq \ker\varphi_1 \cap \ker\psi_1 = \{0\}$.

(ii) We will show that $\psi'_1(\ker\varphi'_1) = \ker\varphi'_2$. Let $a_2a'_2 \in \ker\varphi'_2 \subseteq \ker\varphi_2$. By Theorem 2.3, ψ_1 is a surjection, so $\psi_1(a_1) = a_2$ and $\psi_1(a'_1) = a'_2$ for some a_1 and a'_1 in A_1. By commutativity of diagram $\psi_2\varphi_1(a_1a'_1) = \varphi_2(\psi_1(a_1)\psi_1(a'_1)) = \varphi_2(a_2a'_2) = 0$. Since ψ_2 is an injection, $a_1\acute{a_1} \in \ker{\varphi_1}'$. Also, $\psi'_1(a_1a'_1) = \psi_1(a_1a'_1) = a_2a'_2$. Hence, $a_2a'_2 \in \psi'_1(\ker\varphi'_1)$.

Conversely, let $a_1a'_1 \in \ker\varphi'_1 \subseteq \ker\varphi_1$. Then $\varphi_2\psi_1(a_1a'_1) = 0$, since $\psi_1(\ker\varphi_1) = \ker\varphi_2$. This implies that $\psi'_1(a_1)\psi'_1(a'_1) = \psi'_1(a_1a'_1) \in \ker\varphi'_2$.

(iii) Finally, we prove that $\acute{\psi_2}^{-1}\varphi'_2(\tau(A_2)) = \varphi'_1(\tau(A_1))$. Let $a_2\acute{a_2} \in \tau(A_2)$. By surjectivity of ψ_1 and commutativity of the diagram, we have

$$\begin{aligned}
\acute{\psi_2}^{-1}\varphi'_2(a_2a'_2) &= {\psi_2}^{-1}\varphi_2(a_2a'_2) \\
&= {\psi_2}^{-1}\varphi_2(\psi_1(a_1)\psi_1(a'_1)) && \text{(for some } a_1, a'_1 \in A_1) \\
&= \psi_2^{-1}\psi_2\varphi_1(a_1a'_1) && (\psi_2 \text{ is injective}) \\
&= \varphi_1(a_1a'_1) \\
&= \varphi'_1(a_1a'_1).
\end{aligned}$$

Conversely, by Theorem 2.4, we will show that ψ_1 is a surjection, ψ_2 is an injection, and $\ker\varphi_1 \cap \ker \psi_1 = \{0\}$.

We know that ψ'_1 is surjective. Now if on the contrary $\psi_1(A_1) \neq A_2$, then there exists a nonzero element e_2 in A_2 such that $e_2 \perp \psi_1(A_1)$ (recall that A_2 is a Hilbert space and $\psi_1(A_1)$ is closed). Since A_2 is proper, there exists $a_0 \in A_2$ such that $e_2a_0 \neq 0$, and by [1, Lemma 2.2], $e_2a_0a_0^*e_2^* \neq 0$. Since ψ'_1 is surjective, there is $a_1a'_1 \in \tau(A_1)$ such that $\psi'_1(a_1a'_1) = e_2a_0a_0^*e_2^*$. On the other hand, we have,

$$\langle\psi'_1(a_1a'_1)e_2a_0a_0^* \ , \ e_2\rangle = \langle\psi'_1(a_1a'_1) \ , \ e_2a_0a_0^*e_2^*\rangle = \| e_2a_0a_0^*e_2^* \|^2 \neq 0. \quad (*)$$

Also, $e_2a_0a_0^* = \psi'_1(b_1b_2)$ for some $b_1b_2 \in \tau(A_1)$, so $\psi'_1(a_1a'_1)e_2a_0a_0^* = \psi'_1(a_1a'_1b_1b_2) \in \psi_1(A_1)$. Hence, $\langle\psi'_1(a_1a'_1)e_2a_0a_0^* , e_2\rangle = 0$. That is in contradiction with relation $(*)$. Then ψ_1 is a surjection.

By assumption ψ_2 is an injection. If $a_1 \in \ker\varphi_1 \cap \ker\psi_1$, then $a_1a_1^* \in \ker\varphi'_1 \cap \ker\psi'_1 = \{0\}$. So $\langle a_1, a_1\rangle = tr(a_1a_1^*) = 0$ forces $a_1 = 0$. Hence, $\ker\varphi_1 \cap \ker\psi_1 = \{0\}$ □

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