

## Osserman lightlike hypersurfaces of indefinite $\mathcal{S}$ -manifolds

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**Abstract:** We mainly deal with the problem of admissibility for screen distributions on a lightlike hypersurface of both a semi-Riemannian manifold and an indefinite  $\mathcal{S}$ -manifold. In the latter case, we first show that a characteristic screen distribution is never admissible, and then we provide a characterization for admissible screen distributions on proper totally umbilical lightlike hypersurfaces. Finally, in studying Osserman conditions, we characterize Osserman totally umbilical hypersurfaces of a semi-Riemannian manifold, obtaining explicit results on the eigenvalues of the pseudo-Jacobi operators in the case of lightlike hypersurfaces with Lorentzian screen leaves.

**Key words:** Lightlike hypersurface, semi-Riemannian manifold, admissible screen distribution, pseudo-Jacobi operator, Osserman condition, indefinite  $\mathcal{S}$ -manifold

### 1. Introduction

The study of sectional curvature has always been one of the most interesting topics, since it can provide information about the properties of manifolds. The study of Osserman conditions perfectly falls within this area of interest. In Riemannian geometry, the original problem involving the Osserman conditions is known as the *Osserman Conjecture*. Namely, let  $(M, g)$  be a Riemannian manifold, with curvature tensor  $R$ , and  $X$  a unit vector of  $T_pM$ ,  $p \in M$ . It is then possible to define the symmetric endomorphism  $R_X : X^\perp \rightarrow X^\perp$  such that  $R_X = R_p(\cdot, X)X$ , called the *Jacobi operator* with respect to  $X$  at  $p$ . A Riemannian manifold  $(M, g)$  whose Jacobi operators have eigenvalues independent of  $X \in T_pM$  and  $p \in M$  is said to be an *Osserman manifold*. In [27], Osserman made the following conjecture.

**OSSERMAN CONJECTURE.** Any Osserman manifold is either a locally flat space or a locally rank-one symmetric space.

During the last years, the interest for this subject has been growing, and many authors have looked for a complete answer to the conjecture. In [20] the reader can find the developments of this research in Riemannian, Lorentzian, and semi-Riemannian contexts. For the Lorentzian context, a complete, positive answer to the conjecture was obtained in a sequence of papers [9, 18, 19]. In the Riemannian context, the conjecture was solved by Chi [12, 13, 14] for the manifolds with dimensions different from  $4m$ ,  $m \geq 1$ , and it still remained open for  $4m$ -dimensional manifolds. Recently Nikolayevsky provided results [23, 24, 25] for the manifolds whose dimensions were left out by Chi.

Recently a generalization of the Osserman conditions to the context of degenerate (or lightlike) geometry

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was proposed (see, for example, [1, 3, 4, 5]). To this end it has been necessary to introduce a suitable new type of Jacobi operator [3] to work with, since this context is characterized by some unusual features as being not trivial of the kernel of the metric tensor.

It is worth investigating what happens in the degenerate context allowing the presence of an additional geometrical structure on the ambient manifold. Hence, in this paper we find conditions for which a lightlike hypersurface of an indefinite  $\mathcal{S}$ -manifold is an Osserman manifold.

In Sections 2 and 3 we recall some standard facts on  $g.f.f$ -manifolds and give the basic tools on lightlike geometry needed in the sequel.

In Section 4 we briefly discuss a key point of our study. Namely, since most of the geometrical objects in lightlike geometry depend on the choice of a screen distribution, we give a detailed investigation of the problem of admissibility of a screen distribution, in the sense of [3]. In particular, we deal with this problem for the remarkable subclass of the so-called *characteristic screen distributions*, very important in the study of lightlike hypersurfaces of indefinite  $\mathcal{S}$ -manifolds. Even if sometimes such screen distributions have been used as admissible, we show that characteristic screen distributions are not admissible.

In Section 5 we continue the study of admissible screen distributions considering the case of lightlike hypersurfaces of an indefinite  $\mathcal{S}$ -space form. Looking at the results in [3] and weakening its hypotheses, we show that any admissible screen distribution on a proper totally umbilical lightlike hypersurface is integrable. Then, extending Theorem 4.2 in [3], we obtain a characterization for admissible screen distributions in the case of lightlike hypersurfaces of an indefinite  $\mathcal{S}$ -space form.

Finally, in Section 6 we deal with Osserman conditions on lightlike hypersurfaces. At first, following [1, 3], we briefly recall the definition of a pseudo-Jacobi operator and that of an Osserman lightlike hypersurface. Then, after a few remarks that slightly simplify the study of the pseudo-Jacobi operator, we give a characterization of Osserman totally umbilical lightlike hypersurfaces involving the behaviors of the leaves of its screen distributions. There we apply the result in the case of lightlike hypersurfaces with Lorentzian screen leaves, obtaining a characterization that involves explicit results on the eigenvalues of the pseudo-Jacobi operators.

All manifolds, tensor fields, and maps are assumed to be smooth, and we assume that all manifolds are connected and paracompact. We use the Einstein convention, omitting the sum symbol for repeated indexes. Following the notation of Kobayashi and Nomizu [22] for the curvature tensors on a (semi)-Riemannian manifold  $(M, g)$ , we put  $R(X, Y, Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z$ , and  $R(X, Y, Z, W) = g(R(Z, W, Y), X)$ , for any  $X, Y, Z, W \in \Gamma(TM)$ .

If  $p \in M$ , for any nondegenerate 2-plane  $\pi = \text{span}\{X, Y\}$  in  $T_p M$ , that is a 2-plane such that  $\Delta(\pi) = g(X, X)g(Y, Y) - g(X, Y)^2 \neq 0$ , we denote by  $K_p(X, Y)$  the *sectional curvature* of  $M$  with respect to  $\pi$  at  $p$ , defined by

$$K_p(X, Y) = \frac{R_p(X, Y, X, Y)}{\Delta(\pi)} = \frac{g_p(R_p(X, Y, Y), X)}{\Delta(\pi)} = \frac{g_p(R_Y(X), X)}{\Delta(\pi)}.$$

## 2. Preliminaries

A *globally framed  $f$ -manifold* (briefly  $g.f.f$ -manifold) is a manifold  $M^{2n+s}$  endowed with a nowhere vanishing  $(1, 1)$ -tensor field  $\varphi$  of constant rank with a parallelizable kernel such that  $\varphi^3 + \varphi = 0$ . This means that  $\varphi$  is an  $f$ -structure on  $M$  and there exist  $s$  global vector fields  $\xi_i$  and  $s$  1-forms  $\eta^i$ ,  $i \in \{1, \dots, s\}$ , satisfying  $\varphi^2 = -I + \eta^i \otimes \xi_i$  and  $\eta^i(\xi_j) = \delta_j^i$ . A  $g.f.f$ -manifold  $(M^{2n+s}, \varphi, \xi_i, \eta^i)$ ,  $i \in \{1, \dots, s\}$ , is said to be an *indefinite  $g.f.f$ -*

manifold if it is endowed with a semi-Riemannian metric  $g$  satisfying  $g(\varphi X, \varphi Y) = g(X, Y) - \varepsilon_i \eta^i(X) \eta^i(Y)$ , for any tangent vector fields  $X$  and  $Y$ , where  $\varepsilon_i = \pm 1$ , according to whether  $\xi_i$  is spacelike or timelike.

An *indefinite  $\mathcal{S}$ -manifold* is, by definition, an indefinite *g.f.f.*-manifold that is *normal*, that is,  $N = N_\varphi + 2d\eta^i \otimes \xi_i = 0$ , where  $N_\varphi$  is the Nijenhuis torsion of  $\varphi$ , and satisfies the condition  $d\eta^i(X, Y) = \Phi(X, Y) = g(X, \varphi Y)$  for any  $i \in \{1, \dots, s\}$  and  $X, Y \in \Gamma(TM)$ .

In an indefinite  $\mathcal{S}$ -manifold one has  $(\nabla_X \varphi)Y = g(\varphi X, \varphi Y) \bar{\xi} + \bar{\eta}(Y) \varphi^2 X$ , where  $\bar{\eta} = \sum_{i=1}^s \varepsilon_i \eta^i$  and  $\bar{\xi} = \sum_{i=1}^s \xi_i$ , which implies that  $\nabla_X \xi_i = -\varepsilon_i \varphi X$  and that  $\ker(\varphi)$  is an integrable flat distribution. We remark that an indefinite  $\mathcal{S}$ -manifold is never flat since  $K(X, \xi_i) = \varepsilon_i$  for any  $X \in \mathcal{D}_p$ , where  $\mathcal{D}$  denotes the distribution  $\text{Im}(\varphi)$  [10].

The reader can find more details about these structures in the Riemannian case, for example in [7, 8, 21] and in [17], where the notion of almost  $\mathcal{S}$ -manifolds was introduced, while one can principally find in [10] the extension of this theory to the semi-Riemannian context.

Every  $\mathcal{S}$ -manifold is subject to the following topological condition: it has to be either noncompact or compact with a vanishing Euler characteristic, since it admits never vanishing vector fields. This implies that such a manifold always admits semi-Riemannian metrics, and in particular Lorentz metrics.

Let  $(M^{2n+s}, \varphi, \xi_\alpha, \eta^\alpha, g)$ ,  $\alpha \in \{1, \dots, s\}$ , be an indefinite  $\mathcal{S}$ -manifold. The curvature tensor  $R$  satisfies the following formulas, for any  $X, Y \in \mathcal{D}$  and any  $\alpha, \beta, \gamma, \delta \in \{1, \dots, s\}$  (see [10]):

$$\begin{aligned} R(X, \xi_\alpha, X, Y) &= 0, \quad R(\xi_\alpha, X, \xi_\beta, Y) = \varepsilon_\alpha \varepsilon_\beta g(X, Y), \\ R(\xi_\alpha, X, \xi_\beta, \xi_\gamma) &= 0, \quad R(\xi_\alpha, \xi_\delta, \xi_\beta, \xi_\gamma) = 0. \end{aligned} \tag{2.1}$$

If  $p \in M$ , for any non-lightlike vector  $X \in \mathcal{D}_p$  the 2-plane  $\pi = \text{span}\{X, \varphi X\}$  is said to be a  $\varphi$ -plane, and its sectional curvature  $H_p(X)$  is called the  $\varphi$ -sectional curvature of  $M$  at  $p$  with respect to  $X$ . If the  $\varphi$ -sectional curvature  $H_p(X)$  is independent of both the non-lightlike vector  $X$  and the point  $p$ , then  $M$  is said to be an *indefinite  $\mathcal{S}$ -space form*, denoted by  $M(c)$ ,  $c \in \mathbb{R}$  being the constant value of the  $\varphi$ -sectional curvature. In [10] it was proven that an indefinite  $\mathcal{S}$ -manifold is an indefinite  $\mathcal{S}$ -space form  $M(c)$  if, and only if, the Riemannian  $(0, 4)$ -type curvature tensor field  $R$  is given by

$$\begin{aligned} R(X, Y, Z, W) &= -\frac{c+3\varepsilon}{4} \{g(\varphi Y, \varphi Z)g(\varphi X, \varphi W) - g(\varphi X, \varphi Z)g(\varphi Y, \varphi W)\} \\ &\quad - \frac{c-\varepsilon}{4} \{\Phi(W, X)\Phi(Z, Y) - \Phi(Z, X)\Phi(W, Y) \\ &\quad + 2\Phi(X, Y)\Phi(W, Z)\} - \{\bar{\eta}(W)\bar{\eta}(X)g(\varphi Z, \varphi Y) \\ &\quad - \bar{\eta}(W)\bar{\eta}(Y)g(\varphi Z, \varphi X) + \bar{\eta}(Y)\bar{\eta}(Z)g(\varphi W, \varphi X) \\ &\quad - \bar{\eta}(Z)\bar{\eta}(X)g(\varphi W, \varphi Y)\}, \end{aligned} \tag{2.2}$$

for any  $X, Y, Z, W \in \Gamma(TM)$ , where  $\bar{\eta} = \sum_{\alpha=1}^s \varepsilon_\alpha \eta^\alpha$  and  $\varepsilon = \sum_{\alpha=1}^s \varepsilon_\alpha$ .

### 3. Lightlike hypersurfaces

We briefly recall some basic elements of lightlike geometry. Standard references are the books [15] and [16].

Let  $(\bar{M}^{n+2}, \bar{g})$  be a semi-Riemannian manifold and  $M$  a (connected) hypersurface of  $\bar{M}$ . Rather differently from the Riemannian context, the induced metric  $g = \bar{g}|_M$  on  $M$  can be degenerate. To point

out these particular features, for any  $p \in M$ , one considers the *radical* (or *null*) space of  $M$  at  $p$  (see [26, page 53]), that is

$$\text{Rad } T_p M = T_p M^\perp \cap T_p M = \{V \in T_p M \mid g_p(V, W) = 0 \text{ for all } W \in T_p M\}.$$

The metric  $g$  is nondegenerate if and only if  $\text{Rad } T_p M = \{0\}$ , for any  $p \in M$ , and, on the contrary, we say that  $M$  is a *lightlike hypersurface* if  $\text{Rad } T_p M \neq \{0\}$  at any  $p \in M$ . It is easy then to prove that  $\text{Rad } TM : p \in M \mapsto \text{Rad } T_p M$  is a rank-one distribution over  $M$ , called the *radical distribution*. More precisely, for a lightlike hypersurface, it is  $\text{Rad } TM = TM^\perp$ .

Let  $M$  be a lightlike hypersurface of  $(\bar{M}^{n+2}, \bar{g})$ . Any complementary vector space  $S(T_p M)$  of the radical space  $\text{Rad } T_p M$  in  $T_p M$  is nondegenerate ([15, page 5], [16, page 5]) and, supposing  $M$  to be paracompact, it is possible to choose a nondegenerate distribution  $S(TM) : p \in M \mapsto S(T_p M)$  of rank  $n$  on  $M$ , called a *screen distribution* of  $M$ . We remark that the choice of a screen distribution is not unique, but it is always possible to find one and after making the choice of the screen distribution the lightlike hypersurface will be denoted by the triple  $(M, g, S(TM))$ . Thus, one has the decomposition

$$TM = \text{Rad } TM \perp S(TM) = TM^\perp \perp S(TM),$$

where the symbol  $\perp$  used instead of  $\oplus$  stands for an orthogonal direct sum. A very important tool in the theory of lightlike geometry is provided by the following fundamental theorem ([15, page 79], [16, page 44]).

**Theorem 3.1** *Let  $(M, g, S(TM))$  be a lightlike hypersurface of a semi-Riemannian manifold  $(\bar{M}, \bar{g})$ . There then exists a unique rank-one vector subbundle  $\text{ltr}(M)$  of  $T\bar{M}$ , with base space  $M$ , such that for any nonzero section  $E$  of  $TM^\perp$  on a coordinate neighborhood  $\mathcal{U} \subset M$ , there exists a unique section  $N$  of  $\text{ltr}(M)$  on  $\mathcal{U}$  satisfying  $\bar{g}(N, E) = 1$ ,  $\bar{g}(N, N) = 0$ , and  $\bar{g}(N, W) = 0$  for any  $W \in \Gamma(S(TM)|_{\mathcal{U}})$ .*

The vector bundle  $\text{ltr}(M)$  is called the *lightlike transversal vector bundle* of  $M$  with respect to  $S(TM)$ . As a consequence we have the following decomposition:

$$T\bar{M}|_M = S(TM) \perp \{\text{Rad}(TM) \oplus \text{ltr}(M)\} = TM \oplus \text{ltr}(M). \quad (3.1)$$

Let  $\{E, N\}$  be a pair of local sections on  $\mathcal{U}$  as in Theorem 3.1. We can write the local form of the Gauss and Weingarten equations for the lightlike hypersurface  $(M, g, S(TM))$  as follows: for any  $X, Y \in \Gamma(TM|_{\mathcal{U}})$ ,

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \quad \bar{\nabla}_X N = -A_N X + \tau(X)N, \quad (3.2)$$

where  $\bar{\nabla}$  is the Levi-Civita connection of  $\bar{M}$ ;  $\nabla$  is a torsion-free linear connection on  $M$ , called the *induced connection* on  $M$ ;  $B$  is a symmetric  $(0, 2)$ -type tensor field on  $M$ , called the *local lightlike second fundamental form* of  $M$ ;  $A_N$  is a  $(1, 1)$ -type tensor field, called the *shape operator* of  $M$  in  $\bar{M}$ ; and  $\tau$  is a 1-form on  $M$ .

**Remark 3.2** *It is important to point out that, in general, the induced connection  $\nabla$  is not metric, and that the local lightlike second fundamental form  $B$  is independent of the choice of the screen distribution, since  $B(X, Y) = \bar{g}(\bar{\nabla}_X Y, E)$ . Moreover,  $B(X, E) = 0$ , for any  $X \in \Gamma(TM)$ .*

A lightlike hypersurface  $(M, g, S(TM))$  is said to be *totally umbilical* if, for any coordinate neighborhood  $\mathcal{U} \subset M$ , there exists a local function  $\rho \in \mathfrak{F}(\mathcal{U})$  such that  $B(X, Y) = \rho g(X, Y)$ , for any  $X, Y \in \Gamma(TM|_{\mathcal{U}})$ , and

*totally geodesic* when  $\rho = 0$ , which means  $B = 0$  ([15, 16]). A totally umbilical hypersurface that is not totally geodesic ( $\rho \neq 0$ ) is called *proper totally umbilical*. Clearly, these definitions are independent of the choice of the screen distribution.

Considering the decomposition  $TM = \text{Rad } TM \perp S(TM)$ , we can also write the local form of the Gauss and Weingarten equations with respect to the screen distribution  $S(TM)$ . Indeed, if  $P : \Gamma(TM) \rightarrow \Gamma(S(TM))$  is the projection morphism, we have

$$\nabla_X PY = \nabla_X^* PY + C(X, PY)E, \quad \nabla_X E = -A_E^* X - \tau(X)E, \quad (3.3)$$

for any  $X, Y \in \Gamma(TM|_{\mathcal{U}})$ . Here the  $(0, 2)$ -type tensor field  $C$  on  $M$  is called the *local screen fundamental form* for  $S(TM)$ .

**Remark 3.3** *In general, the local screen fundamental form  $C$  is not symmetric. It is symmetric on  $\Gamma(S(TM))$  if and only if the screen distribution is integrable.*

A screen distribution  $S(TM)$  is said to be *totally umbilical* if, for any coordinate neighborhood  $\mathcal{U} \subset M$ , there exists a local function  $\lambda \in \mathfrak{F}(\mathcal{U})$  such that  $C(X, PY) = \lambda g(X, Y)$ , for any  $X, Y \in \Gamma(TM|_{\mathcal{U}})$ , and *totally geodesic* when  $\lambda = 0$ , that is  $C = 0$  [15, 16].

From (3.2) and (3.3), one can easily deduce the following useful formulas:

$$B(X, Y) = g(A_E^* X, Y), \quad C(X, PY) = g(A_N X, PY), \quad (3.4)$$

for any  $X, Y \in \Gamma(TM|_{\mathcal{U}})$ .

Denote by  $\bar{R}$  and  $R$  the Riemann curvature tensors of  $\bar{\nabla}$  and  $\nabla$ , respectively. Using the local Gauss and Weingarten equations of  $M$  and  $S(TM)$ , with respect to a pair of local sections  $\{E, N\}$  as in Theorem 3.1, one gets the local Gauss–Codazzi equations ([15, page 94], [16, page 66]). We quote only 3 of them that will be very useful later:

$$\begin{aligned} \bar{g}(\bar{R}(X, Y)Z, E) &= (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) + \tau(X)B(Y, Z) \\ &\quad - \tau(Y)B(X, Z), \end{aligned} \quad (3.5)$$

$$\begin{aligned} \bar{g}(\bar{R}(X, Y)Z, PW) &= g(R(X, Y)Z, PW) + B(X, Z)C(Y, PW) \\ &\quad - B(Y, Z)C(X, PW), \end{aligned} \quad (3.6)$$

$$\begin{aligned} \bar{g}(\bar{R}(X, Y)PZ, N) &= (\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) - \tau(X)C(Y, PZ) \\ &\quad + \tau(Y)C(X, PZ), \end{aligned} \quad (3.7)$$

where we put  $(\nabla_X C)(Y, PZ) = X(C(Y, PZ)) - C(\nabla_X Y, PZ) - C(Y, \nabla_X^* PZ)$ , for any  $X, Y, Z, W \in \Gamma(TM|_{\mathcal{U}})$ .

Each of the above formulas, as most of the geometry of a lightlike hypersurface, is related to the choice of a screen distribution. Due to these features, which make the geometry of a lightlike hypersurface very different from the classical case of a nondegenerate hypersurface, it has been necessary to resolve some problems to deal with the Osserman conditions in this context. More precisely, problems arise about finding a suitable definition of a Jacobi operator related to the presence of a degenerate metric and regarding the lack of Riemannian curvature symmetry properties for the screen distribution. In [3] the authors solved these issues.

#### 4. Admissible and characteristic screen distributions in indefinite $\mathcal{S}$ -manifolds

From (3.5), (3.6), and (3.7), it is clear that the induced Riemannian curvature tensor on a non-totally geodesic lightlike hypersurface may not be an algebraic curvature tensor, i.e. it does not satisfy the usual symmetry properties

$$\begin{aligned} R(X, Y, Z, W) &= R(Z, W, X, Y), \\ R(X, Y, Z, W) &= -R(Y, X, Z, W) = -R(X, Y, W, Z), \end{aligned} \quad (4.1)$$

which enable us to have a Jacobi operator with good properties. Nevertheless, since each geometrical object on a lightlike hypersurface, except for the local lightlike second fundamental form  $B$ , depends on the choice of the screen distribution, it is possible to find some suitable screen distributions such that these symmetry properties hold, and these screen distributions are called *admissible*. In [3] some fundamental results about conditions to have an admissible screen distribution were provided (see also [16, pages 146, 147]).

As in [11], we give the following definition.

**Definition 4.1** *Let  $(\bar{M}^{2n+s}, \bar{\varphi}, \bar{\xi}_\alpha, \bar{\eta}^\alpha, \bar{g})$ ,  $\alpha \in \{1, \dots, s\}$ , be an indefinite g.g.f.-manifold, and  $M$  a lightlike hypersurface of  $\bar{M}$  with  $\ker(\bar{\varphi}) \subset TM$ . On such a lightlike hypersurface, a screen distribution  $S(TM)$  is called characteristic if*

- (i)  $\ker(\bar{\varphi}) \subset S(TM)$ ,
- (ii)  $\bar{\varphi}(E) \in \Gamma(S(TM)|_{\mathcal{U}})$ ,

on any coordinate neighborhood  $\mathcal{U} \subset M$ . A lightlike hypersurface  $(M, g, S(TM))$  is said to be characteristic if  $\ker(\bar{\varphi}) \subset TM$  and the chosen screen distribution  $S(TM)$  is characteristic.

These particular lightlike hypersurfaces have often been used in the related literature, and in fact the above conditions (i) and (ii) together with (3.1) yield very special formulas and properties of the lightlike hypersurfaces. Despite their importance, the characteristic screen distributions are not admissible, as we are going to see. Thus, one does not have to use them when the properties in (4.1) are needed and, in particular, they can not be used in the study of Osserman conditions in lightlike geometry.

Preliminarily, let us compute the local lightlike second fundamental form  $B$  and the local screen fundamental form  $C$  of a characteristic screen distribution on suitable pairs of elements.

**Lemma 4.2** *Let  $(\bar{M}^{2n+s}, \bar{\varphi}, \bar{\xi}_\alpha, \bar{\eta}^\alpha, \bar{g})$ ,  $\alpha \in \{1, \dots, s\}$ , be an indefinite  $\mathcal{S}$ -manifold, and  $(M, g, S(TM))$  a characteristic lightlike hypersurface. Then, for any  $\alpha, \beta \in \{1, \dots, s\}$ , one has*

$$\begin{aligned} B(\bar{\varphi}E, \bar{\xi}_\alpha) &= 0, & C(\bar{\varphi}E, \bar{\xi}_\alpha) &= \varepsilon_\alpha, & B(\bar{\varphi}N, \bar{\xi}_\alpha) &= \varepsilon_\alpha \\ C(\bar{\xi}_\alpha, \bar{\xi}_\beta) &= 0, & B(\bar{\xi}_\alpha, \bar{\xi}_\beta) &= 0. \end{aligned} \quad (4.2)$$

**Proof** Since  $\bar{M}$  is an indefinite  $\mathcal{S}$ -manifold, then  $\bar{\nabla}_{\bar{\varphi}E}\bar{\xi}_\alpha = \varepsilon_\alpha E$ . On the other hand, by (3.2), one gets  $\bar{\nabla}_{\bar{\varphi}E}\bar{\xi}_\alpha = \nabla_{\bar{\varphi}E}\bar{\xi}_\alpha + B(\bar{\varphi}E, \bar{\xi}_\alpha)N$ , and hence  $B(\bar{\varphi}E, \bar{\xi}_\alpha) = 0$ . In the same way, using (3.3), one gets the other formulas.  $\square$

**Remark 4.3** From (4.2) it follows that characteristic lightlike hypersurfaces can be neither totally umbilical nor totally geodesic.

**Proposition 4.4** Let  $(\bar{M}^{2n+s}, \bar{\varphi}, \bar{\xi}_\alpha, \bar{\eta}^\alpha, \bar{g})$ ,  $\alpha \in \{1, \dots, s\}$ , be an indefinite  $\mathcal{S}$ -manifold, and  $(M, g, S(TM))$  a characteristic lightlike hypersurface. Then  $S(TM)$  is not an admissible screen distribution.

**Proof** It is easy to see that for a characteristic screen distribution  $\bar{\varphi}N \in \Gamma(S(TM))$  [11]. We compute separately  $R(\bar{\xi}_\alpha, \bar{\varphi}E, \bar{\xi}_\beta, \bar{\varphi}N)$  and  $R(\bar{\xi}_\alpha, \bar{\varphi}E, \bar{\varphi}N, \bar{\xi}_\beta)$ . From (3.6) and (2.1), using (4.2), we have

$$\begin{aligned} R(\bar{\xi}_\alpha, \bar{\varphi}E, \bar{\xi}_\beta, \bar{\varphi}N) &= \bar{R}(\bar{\xi}_\alpha, \bar{\varphi}E, \bar{\xi}_\beta, \bar{\varphi}N) + B(\bar{\xi}_\alpha, \bar{\xi}_\beta)C(\bar{\varphi}E, \bar{\varphi}N) \\ &\quad - B(\bar{\varphi}E, \bar{\xi}_\beta)C(\bar{\xi}_\alpha, \bar{\varphi}N) \\ &= \bar{R}(\bar{\xi}_\alpha, \bar{\varphi}E, \bar{\xi}_\beta, \bar{\varphi}N) = \varepsilon_\alpha \varepsilon_\beta \\ R(\bar{\xi}_\alpha, \bar{\varphi}E, \bar{\varphi}N, \bar{\xi}_\beta) &= \bar{R}(\bar{\xi}_\alpha, \bar{\varphi}E, \bar{\varphi}N, \bar{\xi}_\beta) + B(\bar{\xi}_\alpha, \bar{\varphi}N)C(\bar{\varphi}E, \bar{\xi}_\beta) \\ &\quad - B(\bar{\varphi}E, \bar{\varphi}N)C(\bar{\xi}_\alpha, \bar{\xi}_\beta) \\ &= -\varepsilon_\alpha \varepsilon_\beta + \varepsilon_\beta \varepsilon_\alpha = 0. \end{aligned}$$

Then our statement follows. □

### 5. Admissible screen distributions on totally umbilical lightlike hypersurfaces

First, we provide some necessary conditions to have admissible screen distributions on a (noncharacteristic) lightlike hypersurface of an indefinite  $\mathcal{S}$ -space form.

**Proposition 5.1** Let  $(\bar{M}(c)^{2n+s}, \bar{\varphi}, \bar{\xi}_\alpha, \bar{\eta}^\alpha, \bar{g})$ ,  $\alpha \in \{1, \dots, s\}$ , be an indefinite  $\mathcal{S}$ -space form, and  $(M, g, S(TM))$  a lightlike hypersurface with  $\ker(\bar{\varphi}) \subset \Gamma(TM)$ . If  $S(TM)$  is an admissible screen distribution, then either  $c = \varepsilon = \sum_{\alpha=1}^s \varepsilon_\alpha$  or the hypersurface is  $\bar{\varphi}$ -invariant, i.e.  $\bar{\varphi}(TM) \subseteq TM$ .

**Proof** From (3.6), using Remark 3.2, and from (2.2), for any  $X \in \Gamma(TM)$  one gets

$$\begin{aligned} R(E, X, E, PX) &= \bar{R}(E, X, E, PX) \\ &= 3 \frac{c - \varepsilon}{4} \bar{g}(PX, \bar{\varphi}E)g(E, \bar{\varphi}X) = -3 \frac{c - \varepsilon}{4} \bar{g}(E, \bar{\varphi}X)^2. \end{aligned}$$

On the other hand, being  $S(TM)$  admissible,  $R(E, X, E, PX) = -R(E, X, PX, E) = g(R(E, X, PX), E) = 0$ , since  $E \in \Gamma(\text{Rad}(TM))$ . Thus, either  $c = \varepsilon$  or  $\bar{g}(\bar{\varphi}X, E) = 0$ , that is the component of  $\bar{\varphi}X$  along  $\text{ltr}(M)$ , and hence  $\bar{\varphi}X \in \Gamma(TM)$ . □

In [3, Th. 4.3], [16, Th. 3.6.24, page 152] one finds a characterization for admissible screen distributions on proper totally umbilical lightlike hypersurfaces of a semi-Riemannian space form. Here we give a proof of the necessary condition, i.e. we prove that an admissible screen distribution is totally umbilical. The proof of this condition is different from that provided in [3, 16] and it generalizes Theorem 4.3 in [3] to the case of semi-Riemannian manifolds without constant sectional curvature. We then use it to extend the characterization to proper totally umbilical lightlike hypersurfaces of indefinite  $\mathcal{S}$ -space forms.



**Lemma 5.2** *Let  $(\bar{M}, \bar{g})$  be a  $(n+2)$ -dimensional semi-Riemannian manifold, with  $n > 1$ , and  $(M, g, S(TM))$  a proper totally umbilical lightlike hypersurface of  $\bar{M}$ . If  $S(TM)$  is an admissible screen distribution, then it is integrable.*

**Proof** Put  $B(X, Y) = \rho g(X, Y)$ , for any  $X, Y \in \Gamma(TM|_{\mathcal{U}})$ , where  $\rho$  is a nonvanishing smooth function on a coordinate neighborhood  $\mathcal{U} \subset M$ . From (3.6), for any  $X, Y, Z, W \in \Gamma(S(TM)|_{\mathcal{U}})$ , we have

$$R(X, Y, Z, W) = \bar{R}(X, Y, Z, W) + \rho g(X, Z)C(Y, W) - \rho g(Y, Z)C(X, W),$$

from which, putting  $Z = X$  and  $W = X$ , by the admissibility of the screen distribution, it follows that

$$0 = \rho g(X, X)C(Y, X) - \rho g(Y, X)C(X, X).$$

Choosing  $X, Y \in \Gamma(S(TM)|_{\mathcal{U}})$  both non-lightlike and  $X \perp Y$  we get  $C(Y, X) = 0$ , since  $\rho \neq 0$ . Now, let  $(W_i)_{1 \leq i \leq n}$  be a local orthonormal frame for  $S(TM)|_{\mathcal{U}}$ . Let  $U, V \in \Gamma(S(TM)|_{\mathcal{U}})$ , and writing  $U = u^i W_i$  and  $V = v^i W_i$ , with  $u^i, v^i \in \mathfrak{F}(\mathcal{U})$ , we get

$$C(U, V) = u^i v^j C(W_i, W_j) = v^j u^i C(W_j, W_i) = C(V, U);$$

thus,  $C$  is symmetric on  $\Gamma(S(TM))$  and, by Remark 3.3,  $S(TM)$  is integrable.  $\square$

Let us recall the following definition ([2, 16]).

**Definition 5.3** *A lightlike hypersurface  $(M, g, S(TM))$  of a semi-Riemannian manifold is said to be locally screen conformal if the shape operators  $A_N$  and  $A_E^*$  of  $M$  and  $S(TM)$ , respectively, are related by  $A_N = f A_E^*$ , where  $f$  is a nonvanishing smooth function on a coordinate neighborhood  $\mathcal{U} \subset M$ .*

**Remark 5.4** *From (3.4) one easily deduces that the above definition is equivalent to the condition  $C(X, PY) = fB(X, Y)$ , for any  $X, Y \in \Gamma(TM)$ . It follows that any locally screen conformal hypersurface has integrable screen distribution and that the geometry of  $M$  is equivalent to the geometry of the leaves of  $S(TM)$ . More precisely,  $M$  is totally umbilical (resp.: totally geodesic) if and only if  $S(TM)$  is totally umbilical (res.: totally geodesic) (see also [2, Th. 2], [16, Th. 2.2.9, page 56]).*

In [3, Th. 3.2], [16, Th. 3.6.17, page 147] the following fundamental result, which links the admissibility of a screen distribution to the fact of being locally screen conformal, was provided.

**Theorem 5.5** *Let  $(\bar{M}, \bar{g})$  be a semi-Riemannian manifold and  $(M, g, S(TM))$  a lightlike hypersurface of  $\bar{M}$  with non-totally geodesic, integrable screen distribution. Then  $S(TM)$  is admissible if, and only if, at least one of the following conditions holds:*

- (i)  $M$  is totally geodesic;
- (ii)  $M$  is locally screen conformal, and  $\bar{R}(X, PY)(\text{Rad}(TM)) \subset \text{Rad}(TM)$ , for any  $X, Y \in \Gamma(TM)$  (ambient holonomy condition).

By Lemma 5.2, Theorem 5.5, and Remark 5.4, one easily gets the following result.



**Proposition 5.6** *Let  $(\bar{M}, \bar{g})$  be a  $(n+2)$ -dimensional semi-Riemannian manifold and  $(M, g, S(TM))$  a proper totally umbilical lightlike hypersurface of  $\bar{M}$ , with non-totally geodesic screen distribution. If  $S(TM)$  is admissible, then it is totally umbilical.*

Now we can prove the following characterization for admissible screen distributions of proper totally umbilical lightlike hypersurfaces of an indefinite  $\mathcal{S}$ -space form, which generalizes that in [3, 16].

**Theorem 5.7** *Let  $(\bar{M}(c)^{2n+s}, \bar{\varphi}, \bar{\xi}_\alpha, \bar{\eta}^\alpha, \bar{g})$ ,  $\alpha \in \{1, \dots, s\}$ , be an indefinite  $\mathcal{S}$ -space form, and  $(M, g, S(TM))$  a proper totally umbilical lightlike hypersurface with  $\ker(\bar{\varphi}) \subset \Gamma(TM)$ . Assume either that  $c = \varepsilon = \sum_{\alpha=1}^s \varepsilon_\alpha$  or that  $M$  is  $\bar{\varphi}$ -invariant. Then a screen distribution is admissible if and only if it is totally umbilical.*

**Proof** By Proposition 5.6, we have that an admissible screen distribution is totally umbilical.

Conversely, put  $C(X, PY) = \lambda g(X, PY)$  and  $B(X, Y) = \rho g(X, Y)$  for any  $X, Y \in \Gamma(TM)$ , where  $\lambda$  and  $\rho$  are smooth functions on a coordinate neighborhood  $\mathcal{U} \subset M$ . We are going to show that the induced Riemannian curvature tensor  $R$  on  $M$  is an algebraic curvature tensor, i.e. it satisfies (4.1) on  $M$ . For any  $X, Y, Z, W \in \Gamma(TM)$ , since  $W = PW + \bar{g}(W, N)E$ , from (3.6) we have

$$R(X, Y, Z, W) = \bar{R}(X, Y, Z, PW) + \rho\lambda\{g(X, Z)g(Y, PW) - g(Y, Z)g(X, PW)\}.$$

Since either  $M$  is a  $\bar{\varphi}$ -invariant hypersurface or  $\bar{M}$  has  $\bar{\varphi}$ -sectional curvature  $c = \varepsilon$ , by (2.2) a straightforward calculation shows that  $\bar{R}(X, Y, Z, E) = 0$  for any  $X, Y, Z \in \Gamma(TM)$ . Moreover, since  $g(X, PY) = g(X, Y)$  for any  $X, Y \in \Gamma(TM)$ , we have

$$R(X, Y, Z, W) = \bar{R}(X, Y, Z, W) + \rho\lambda\{g(X, Z)g(Y, W) - g(Y, Z)g(X, W)\},$$

for any  $X, Y, Z, W \in \Gamma(TM)$ . The right side of the above identity is clearly an algebraic curvature tensor on  $M$ , i.e. it satisfies (4.1), and this finishes the proof.  $\square$

## 6. Osserman lightlike hypersurfaces

It is known that in a semi-Riemannian manifold  $(\bar{M}, \bar{g})$ , using the  $(1, 3)$ -curvature tensor field  $\bar{R}$ , one can define the Jacobi operator  $\bar{J}_z$  w.r.t. an element  $z$  of the unit spacelike (resp.: timelike) tangent sphere  $S_p^-(\bar{M}) = \{z \in T_p\bar{M} \mid \bar{g}(z, z) = 1\}$  (resp.  $S_p^+(\bar{M}) = \{z \in T_p\bar{M} \mid \bar{g}(z, z) = -1\}$ ) at a point  $p \in \bar{M}$ . It is, by definition, the endomorphism  $\bar{J}_z : z^\perp \rightarrow z^\perp$  such that

$$\bar{J}_z(x) = \bar{R}_p(x, z)z. \quad (6.1)$$

It is evident that  $\bar{J}_z(x)$  is the only tangent vector in  $T_p\bar{M}$  such that, for any  $y \in z^\perp$ ,  $\bar{g}(\bar{J}_z(x), y) = -\bar{R}(x, z, z, y)$ . In other words, using the natural musical isomorphisms related to the metric  $\bar{g}$ , (6.1) is equivalent to

$$\bar{J}_z(x) = -\bar{R}_p(x, z, z, \cdot)^\sharp. \quad (6.2)$$

When the metric is degenerate, due to the above equivalence, (6.1) completely fails, since we cannot use the musical isomorphisms. Nevertheless, the equivalence between (6.1) and (6.2) suggests one way to introduce a new kind of Jacobi operator, defining a new kind of musical isomorphism adapted to degenerate metrics. This

was the way followed in [1] and [3], where this new kind of Jacobi operator was introduced and studied [4]. The construction was generalized to the case of lightlike submanifolds in [16, 6].

Namely, let  $(M, g, S(TM))$  be a lightlike hypersurface of a semi-Riemannian manifold  $(\bar{M}, \bar{g})$  and  $\{E, N\}$  a pair of local sections on a coordinate neighborhood  $\mathcal{U} \subset M$ , as in Theorem 3.1. One may consider the 1-form  $n$  on  $M$  defined by

$$n(X) = \bar{g}(X, N) \quad \forall X \in \Gamma(TM).$$

A new metric  $\tilde{g}$  on  $M$  is then defined putting

$$\tilde{g}(X, Y) = g(X, Y) + n(X)n(Y),$$

for any  $X, Y \in \Gamma(TM)$ . It is easy to verify that  $\tilde{g}$  is a nondegenerate semi-Riemannian metric on  $M$  such that

$$\tilde{g}(E, E) = 1, \quad \tilde{g}(E, W) = 0, \quad \tilde{g}(X, W) = g(X, W),$$

for any  $X \in \Gamma(TM)$  and  $W \in \Gamma(S(TM))$ . Having a nondegenerate metric allows us to consider the musical isomorphisms  $\flat$  and  $\sharp$  naturally associated with it, and pseudo-Jacobi operators can be introduced on the lightlike hypersurface as in (6.2).

Namely, according to the notation of [1, 3, 6, 16, 20], let  $S_p^-(M)$ ,  $S_p^+(M)$  and  $S_p(M)$  denote the sets of unit timelike, unit spacelike, and nonnull vectors in  $T_pM$ ,  $p \in M$ , respectively:

$$\begin{aligned} S_p^-(M) &= \{X \in T_pM \mid g_p(X, X) = -1\}, \\ S_p^+(M) &= \{X \in T_pM \mid g_p(X, X) = 1\}, \\ S_p(M) &= S_p^-(M) \cup S_p^+(M). \end{aligned}$$

**Definition 6.1** ([3, 6, 16]) *Let  $(M, g, S(TM))$  be a lightlike hypersurface of a semi-Riemannian manifold  $(\bar{M}, \bar{g})$  such that  $S(TM)$  is an admissible screen distribution. If  $p \in M$  and  $X \in S_p(M)$ , the self-adjoint endomorphism  $J_R(X) : X^\perp \rightarrow X^\perp$  such that  $J_R(X)Y = -R_p(Y, X, X, \cdot)^\sharp$  is called the pseudo-Jacobi operator with respect to  $X$ , where  $R$  is the induced Riemannian curvature tensor on  $M$ , and the orthogonal space  $X^\perp$  is computed with respect to the metric  $g$ .*

**Remark 6.2** *By definition,  $J_R(X)Y$  is the unique tangent vector to  $M$  at  $p$  such that  $\tilde{g}_p(J_R(X)Y, Z) = R_p(X, Y, X, Z)$ , for any  $Y, Z \in X^\perp$ . It follows that  $J_R(X)E = 0$ , and hence  $J_R(X)|_{\Gamma(\text{Rad}(TM))} = 0$  and  $J_R(X)(X^\perp) \subset \Gamma(S(T_pM))$ , for any  $X \in S_p(M)$  ([3]). Moreover, for any  $Z \in \Gamma(S(TM))$  we have*

$$g(J_R(X)Y, Z) = \tilde{g}(J_R(X)Y, Z) = R(X, Y, X, Z) = g(R(Y, X)X, Z).$$

Hence,

$$J_R(X)Y = P(R(Y, X)X), \tag{6.3}$$

for any  $X \in S_p(M)$  and any  $Y \in X^\perp$ , where  $P : \Gamma(TM) \rightarrow \Gamma(S(TM))$  is the canonical projection morphism.

Now, we set up the Osserman condition for lightlike hypersurfaces.

**Definition 6.3** ([3, 16]) *A lightlike hypersurface  $M$  of a semi-Riemannian manifold  $(\bar{M}, \bar{g})$  is called timelike (resp.: spacelike) Osserman at a point  $p \in M$  if, for each admissible screen distribution  $S(TM)$ , the characteristic polynomial of  $J_R(X)$  is independent of  $X \in S_p^-(M)$  (resp.:  $X \in S_p^+(M)$ ). Moreover, if this condition holds at each  $p \in M$ , then  $M$  is called pointwise timelike (resp.: spacelike) Osserman.*

In [3] the authors showed that  $M$  being a timelike Osserman lightlike hypersurface at a point  $p \in M$  is equivalent to  $M$  being spacelike Osserman at  $p \in M$ , so that it can be simply said ‘‘Osserman lightlike hypersurface’’ at a point. Moreover, in the same paper it was pointed out that the above definition is independent of the choice of the admissible screen distribution.

**Remark 6.4** *Taking  $X \in S_p(M)$ ,  $p \in M$ , we write  $X^\perp = \text{Rad}(TM) \perp PX^{\perp S(TM)}$ , and since  $g(PX, PX) = g(X, X)$ , we can consider another Jacobi-type operator, namely the operator  $J_R^*(PX) : PX^{\perp S(TM)} \rightarrow PX^{\perp S(TM)}$  such that, for any  $Y \in PX^{\perp S(TM)}$ ,  $J_R^*(PX)(Y) = P(R(Y, PX)PX)$ , which we call the screen Jacobi operator associated with  $J_R(X)$ . Due to the admissibility of the screen distribution, we get  $P(R(X, Y)Z) = 0$  anytime one of the arguments is  $E$ . Thus, by (6.3), we have*

$$J_R(X)(Y) = J_R^*(PX)PY, \quad (6.4)$$

for any  $X \in S_p(M)$  and any  $Y \perp X$ . This clearly implies that  $M$  is Osserman at  $p$  if and only if the characteristic polynomial of  $J_R^*(PX)$  is independent of  $X \in S_p^+(M)$  (or  $X \in S_p^-(M)$ ).

**Theorem 6.5** *Let  $(M, g, S(TM))$  be a totally umbilical lightlike hypersurface of a semi-Riemannian manifold  $(\bar{M}, \bar{g})$  with admissible screen distribution. Then  $M$  is Osserman at  $p$  if and only if the leaves of  $S(TM)$  are Osserman at this point.*

**Proof** Let  $R$  be the Riemannian curvature tensor induced on  $M$ , and  $R^*$  the Riemannian curvature tensor on the leaves of  $S(TM)$ , defined by the Levi-Civita connection  $\nabla^*$ . Straightaway, for any  $X, Y, Z \in \Gamma(S(TM))$ , by (3.3), we have

$$\begin{aligned} R(X, Y)Z &= R^*(X, Y)Z + [C(X, Z)A_E^*Y - C(Y, Z)A_E^*X] \\ &\quad + [(\nabla_X C)(Y, Z) - (\nabla_Y C)(X, Z)]E \\ &\quad + [\tau(Y)C(X, Z) - \tau(X)C(Y, Z)]E. \end{aligned} \quad (6.5)$$

Since  $M$  is totally umbilical, then  $S(TM)$  is also totally umbilical, and hence  $B(X, Y) = \rho g(X, Y)$  and  $C(X, PY) = \lambda g(X, Y)$ , being  $\rho, \lambda$  smooth local functions. Using (3.4), from (6.5) it follows

$$P(R(X, Z)Z) = R^*(X, Z)Z + \rho\lambda[g(X, Z)Z - g(Z, Z)X], \quad (6.6)$$

for any  $X, Y, Z \in S(TM)$ . When  $p \in M$ , it is plain that  $Z \in S_p(M)$  if and only if  $PZ \in S_p(M^*)$  where  $M^*$  is the leaf of  $S(TM)$  through  $p$  and, by (6.6), (6.4) yields

$$J_R(Z)(X) = R_{PZ}^*(PX) - \varepsilon_Z \rho \lambda PX, \quad (6.7)$$

for any  $X \in Z^\perp$ , where  $R_{PZ}^*$  is the Jacobi operator of the leaf  $M^*$  at  $p$ , and  $\varepsilon_Z = g(Z, Z) = \pm 1$ . This, by Remark 6.4, concludes the proof.  $\square$

An interesting consequence of the above result is obtained recalling that the Osserman Conjecture is true in Lorentzian manifolds, as shown in Theorem 3.1.2 [20, pg. 42], which we quote here for the convenience of the reader.

**Theorem 6.6** ([20], Th. 3.1.2, page 42) *“If  $(M, g)$  is a connected  $(n \geq 3)$ -dimensional Lorentzian pointwise Osserman manifold, then  $(M, g)$  is a real space form, i.e.  $(M, g)$  is of constant sectional curvature.”*

Using this, we get the following characterization.

**Theorem 6.7** *Under the hypotheses of Theorem 6.5, if  $\dim(\bar{M}) = n + 2$ , with  $n \geq 3$ , and the index of  $\bar{g}$  is 2, then the lightlike hypersurface  $(M, g, S(TM))$  is pointwise Osserman if and only if each leaf of  $S(TM)$  is a real space form.*

Moreover, for any  $p \in M$  and any  $Z \in S_p^-(M)$ , the pseudo-Jacobi operator  $J_R(Z)$  is diagonalizable, with exactly 2 eigenvalues,  $k_0 = 0$  with multiplicity 1, and  $k_1 = \kappa_p + \lambda(p)\rho(p)$  with multiplicity  $n - 2$ , where  $\kappa_p$  is the sectional curvature of the leaf  $M^*$  at the point  $p \in M$ , and  $\lambda, \rho$  are local smooth functions on a coordinate neighborhood  $\mathcal{U}$  of  $p$ .

**Proof** By (3.1) on a coordinate neighborhood  $\mathcal{U}$  of  $p$  we have the decomposition  $T\bar{M}|_{\mathcal{U}} = S(TM) \perp \{\text{Rad}(TM) \oplus \text{ltr}(M)\}$  and, by Theorem 3.1, the restriction of  $\bar{g}$  on  $\text{Rad}(TM) \oplus \text{ltr}(M)$  has index 1. Thus, the index of  $g$  on  $S(TM)$  is 1, i.e. its leaves are Lorentzian manifolds. Now our statement follows from Theorem 6.5, Theorem 6.6, and (6.7), considering that, if  $Z \in S_p^-(M)$ , then  $R_{PZ}^*$  is diagonalizable, since it is a self-adjoint endomorphism of the space  $PZ^{\perp_{S(TM)}}$ , on which the metric is positive defined.  $\square$

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