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On generalized Robertson–Walker spacetimes satisfying some curvature condition

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Abstract: We give necessary and sufficient conditions for warped product manifolds (M, g), of dimension $\geqslant 4$, with 1-dimensional base, and in particular, for generalized Robertson–Walker spacetimes, to satisfy some generalized Einstein metric condition. Namely, the difference tensor $R \cdot C - C \cdot R$, formed from the curvature tensor R and the Weyl conformal curvature tensor C, is expressed by the Tachibana tensor $Q(S, R)$ formed from the Ricci tensor S and R. We also construct suitable examples of such manifolds. They are quasi-Einstein, i.e. at every point of M $\operatorname{rank}(S - \alpha\, g) \leqslant 1$, for some $\alpha \in \mathbb{R}$, or non-quasi-Einstein.

Key words: Warped product, generalized Robertson–Walker spacetime, Einstein manifold, quasi-Einstein manifold, essentially conformally symmetric manifold, Tachibana tensor, generalized Einstein metric condition, pseudosymmetry type curvature condition, Ricci-pseudosymmetric hypersurface

1. Introduction

A semi-Riemannian manifold (M, g), $n = \dim M \geqslant 3$, is said to be an *Einstein manifold* if at every point its Ricci tensor S is proportional to the metric tensor g, i.e. on M we have

$$S = \frac{\kappa}{n}\, g, \tag{1}$$

where κ is the scalar curvature of (M, g). In particular, if S vanishes identically on M then (M, g) is called a *Ricci flat manifold*. If at every point of M its Ricci tensor satisfies $\operatorname{rank} S \leqslant 1$ then (M, g) is called a *Ricci-simple manifold* (see, e.g., [31, 41]).

Let (M, g), $n \geqslant 3$, be a semi-Riemannian manifold and let U_S be the set of all points of M at which $S \neq \frac{\kappa}{n} g$. The manifold (M, g), $n \geqslant 3$, is said to be *quasi-Einstein* (see, e.g., [56] and [22] and references therein) if at every point of $U_S \subset M$ we have $\operatorname{rank}(S - \alpha\, g) = 1$, for some $\alpha \in \mathbb{R}$.

For the curvature tensor R and the Weyl conformal curvature tensor C of (M, g), $n \geqslant 4$, we can define on M the $(0, 6)$-tensors $R \cdot C$ and $C \cdot R$. For precise definition of the symbols used, we refer to Sections 2 and 3 of this paper, as well as to [5, 22, 24, 32]. It is obvious that for any Ricci flat, as well as conformally flat, semi-Riemannian manifold (M, g), $n \geqslant 4$, we have $R \cdot C - C \cdot R = 0$. For non-Ricci flat Einstein manifolds the

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tensor $R \cdot C - C \cdot R$ is nonzero. Namely, any Einstein manifold (M, g), $n \geqslant 4$, satisfies ([32], Theorem 3.1)

$$R \cdot C - C \cdot R \;=\; \frac{\kappa}{(n-1)n}\, Q(g, R) \;=\; \frac{\kappa}{(n-1)n}\, Q(g, C)\,, \tag{2}$$

i.e. at every point of M the difference tensor $R \cdot C - C \cdot R$ and the Tachibana tensor $Q(g, R)$, or $Q(g, C)$, are linearly dependent. We also mention that for any semi-Riemannian manifold (M, g), $n \geqslant 4$, we have some identity (see Eq. (32)) that expresses the tensor $R \cdot C - C \cdot R$ by some $(0, 6)$-tensors. In particular, by making use of that identity we can express the difference tensor of some hypersurfaces in space forms by a linear combination of the Tachibana tensors $Q(g, R)$ and $Q(S, R)$ ([24], Theorem 3.2; see also our Theorem 6.1(ii) and Proposition 4.1).

We can also investigate semi-Riemannian manifolds (M, g), $n \geqslant 4$, for which the difference tensor $R \cdot C - C \cdot R$ is expressed by one of the following Tachibana tensors: $Q(g, R)$, $Q(g, C)$, $Q(S, R)$, or $Q(S, C)$. In this way we obtain 4 curvature conditions. The first results related to those conditions are given in [32]. We refer to [22] for a survey on this subject. Since these conditions are satisfied on any semi-Riemannian Einstein manifold, they can be named *generalized Einstein metric conditions* (cf. [6], Chapter 16). In particular, (2) is also a condition of this kind and in Section 2 we present results on manifolds satisfying

$$R \cdot C - C \cdot R \;=\; L_1\, Q(g, R)\,. \tag{3}$$

In this paper we restrict our investigations to non-Einstein and nonconformally flat semi-Riemannian manifolds (M, g), $n \geqslant 4$, satisfying on $U = \{x \in M \,:\, Q(S, R) \neq 0 \text{ at } x\}$ the condition

$$R \cdot C - C \cdot R \;=\; L\, Q(S, R)\,, \tag{4}$$

where L is some function on this set. We recall that at all points of a semi-Riemannian manifold (M, g), $n \geqslant 3$, at which its Ricci tensor S is nonzero and $Q(S, R) = 0$, we have ([9], Theorem 4.1)

$$R \cdot R \;=\; 0\,, \tag{5}$$

i.e. such a manifold is *semisymmetric*. We also recall that if

$$R \cdot S \;=\; 0 \tag{6}$$

holds on a semi-Riemannian manifold then it is called *Ricci-semisymmetric*. The condition (4), under some additional assumptions, was considered in [45] (see Theorem 2.2 of this paper).

Our main results are related to warped products $\overline{M} \times_F \widetilde{N}$ with 1-dimensional base manifold $(\overline{M}, \bar{g})$ satisfying (4). Evidently, generalized Robertson–Walker spacetimes are warped products of this kind. We investigate separately 2 cases where the fiber $(\widetilde{N}, \tilde{g})$ is either non-Einstein or Einstein manifold. In the first case, in Section 4, we prove that the associated function L satisfies $L = \frac{1}{n-2}$ and we show that the warping function F is a polynomial of the second degree. Moreover, $(\widetilde{N}, \tilde{g})$ satisfies some curvature condition presented by (46) (see Theorems 4.1 and 4.2). Furthermore, in the second case (see Section 5), i.e. when $\overline{M} \times_F \widetilde{N}$ is a quasi-Einstein manifold, we show that also $L = \frac{1}{n-2}$ and F is a polynomial of the second degree (see Theorem 5.1). Based on these results, in Section 6 we give examples of warped products satisfying (4). In particular, we construct an example of a warped product manifold satisfying (4) having non-Einstein fiber realizing (46). Finally, we mention that recently hypersurfaces in space forms having the tensor $R\cdot C - C\cdot R$ expressed by some Tachibana tensors were investigated in [26].

2. Preliminaries

Throughout this paper all manifolds are assumed to be connected paracompact manifolds of class C^∞. Let (M,g) be an n-dimensional semi-Riemannian manifold and let ∇ be its Levi-Civita connection and $\mathfrak{X}(M)$ the Lie algebra of vector fields on M. We define on M the endomorphisms $X \wedge_A Y$ and $\mathcal{R}(X,Y)$ of $\mathfrak{X}(M)$ by

$$\begin{aligned} (X\wedge_A Y)Z &= A(Y,Z)X - A(X,Z)Y\,, \\ \mathcal{R}(X,Y)Z &= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z\,, \end{aligned}$$

respectively, where A is a symmetric $(0,2)$-tensor on M and $X,Y,Z \in \mathfrak{X}(M)$. The Ricci tensor S, the Ricci operator $\mathcal{S}$, and the scalar curvature κ of (M,g) are defined by $S(X,Y) = tr\{Z \to \mathcal{R}(Z,X)Y\}$, $g(\mathcal{S}X,Y) = S(X,Y)$, and $\kappa = tr\,\mathcal{S}$, respectively. The endomorphism $\mathcal{C}(X,Y)$ is defined by

$$\mathcal{C}(X,Y)Z = \mathcal{R}(X,Y)Z - \frac{1}{n-2}(X\wedge_g \mathcal{S}Y + \mathcal{S}X \wedge_g Y - \frac{\kappa}{n-1}X\wedge_g Y)Z\,,$$

assuming that $n \geqslant 3$. Now the $(0,4)$-tensor G, the Riemann-Christoffel curvature tensor R, and the Weyl conformal curvature tensor C of (M,g) are defined by

$$\begin{aligned} G(X_1,X_2,X_3,X_4) &= g((X_1\wedge_g X_2)X_3,X_4)\,, \\ R(X_1,X_2,X_3,X_4) &= g(\mathcal{R}(X_1,X_2)X_3,X_4)\,, \\ C(X_1,X_2,X_3,X_4) &= g(\mathcal{C}(X_1,X_2)X_3,X_4)\,, \end{aligned}$$

respectively, where $X_1, X_2, \ldots \in \mathfrak{X}(M)$. Furthermore, we define the following sets: $U_R = \{x \in M \,:\, R \neq \frac{\kappa}{(n-1)n}G \text{ at } x\}$, $U_S = \{x\in M \,:\, S \neq \frac{\kappa}{n}g \text{ at } x\}$, and $U_C = \{x \in M \,:\, C \neq 0 \text{ at } x\}$. It is easy to see that $U_S \cup U_C = U_R$.

Let $\mathcal{B}(X,Y)$ be a skew-symmetric endomorphism of $\mathfrak{X}(M)$ and let B be a $(0,4)$-tensor associated with $\mathcal{B}(X,Y)$ by

$$B(X_1,X_2,X_3,X_4) = g(\mathcal{B}(X_1,X_2)X_3,X_4)\,. \tag{7}$$

The tensor B is said to be a *generalized curvature tensor* if

$$\begin{aligned} B(X_1,X_2,X_3,X_4) + B(X_2,X_3,X_1,X_4) + B(X_3,X_1,X_2,X_4) &= 0\,, \\ B(X_1,X_2,X_3,X_4) &= B(X_3,X_4,X_1,X_2)\,. \end{aligned}$$

Let $\mathcal{B}(X,Y)$ be a skew-symmetric endomorphism of $\mathfrak{X}(M)$ and let B be the tensor defined by (7). We extend the endomorphism $\mathcal{B}(X,Y)$ to derivation $\mathcal{B}(X,Y)\cdot$ of the algebra of tensor fields on M, assuming that it commutes with contractions and $\mathcal{B}(X,Y)\cdot f = 0$, for any smooth function f on M. Now for a $(0,k)$-tensor field T, $k \geqslant 1$, we can define the $(0,k+2)$-tensor $B\cdot T$ by

$$\begin{aligned} (B\cdot T)(X_1,\ldots,X_k;X,Y) &= (\mathcal{B}(X,Y)\cdot T)(X_1,\ldots,X_k) \\ &= -T(\mathcal{B}(X,Y)X_1,X_2,\ldots,X_k) - \cdots - T(X_1,\ldots,X_{k-1},\mathcal{B}(X,Y)X_k)\,. \end{aligned}$$

In addition, if A is a symmetric $(0,2)$-tensor then we define the $(0,k+2)$-tensor $Q(A,T)$, named a *Tachibana tensor* ([26]), by

$$\begin{aligned} Q(A,T)(X_1,\ldots,X_k;X,Y) &= (X\wedge_A Y\cdot T)(X_1,\ldots,X_k) \\ = \; -T((X\wedge_A Y)X_1,X_2,\ldots,X_k) &- \cdots - T(X_1,\ldots,X_{k-1},(X\wedge_A Y)X_k)\,. \end{aligned}$$

In this manner we obtain the $(0,6)$-tensors $B\cdot B$ and $Q(A,B)$. Setting in the above formulas $\mathcal{B}=\mathcal{R}$ or $\mathcal{B}=\mathcal{C}$, $T=R$ or $T=C$ or $T=S$, $A=g$ or $A=S$, we get the tensors $R\cdot R$, $R\cdot C$, $C\cdot R$, $R\cdot S$, $Q(g,R)$, $Q(S,R)$, $Q(g,C)$, and $Q(g,S)$.

Let B_{hijk}, T_{hijk}, and A_{ij} be the local components of generalized curvature tensors B and T and a symmetric $(0,2)$-tensor A on M, respectively, where $h,i,j,k,l,m,p,q\in\{1,2,\ldots,n\}$. The local components $(B\cdot T)_{hijklm}$ and $Q(A,T)_{hijklm}$ of the tensors $B\cdot T$ and $Q(A,T)$ are the following:

$$(B\cdot T)_{hijklm} = g^{pq}(T_{pijk}B_{qhlm}+T_{hpjk}B_{qilm}+T_{hipk}B_{qjlm}+T_{hijp}B_{qklm})\,,$$

$$\begin{aligned} Q(A,T)_{hijklm} &= A_{hl}T_{mijk}+A_{il}T_{hmjk}+A_{jl}T_{himk}+A_{kl}T_{hijm} \\ &- A_{hm}T_{lijk}-A_{im}T_{hljk}-A_{jm}T_{hilk}-A_{km}T_{hijl}\,. \end{aligned} \tag{8}$$

For a symmetric $(0,2)$-tensor E and a $(0,k)$-tensor T, $k\geqslant 2$, we define their Kulkarni–Nomizu product $E\wedge T$ by (see, e.g., [21])

$$\begin{aligned} &(E\wedge T)(X_1,X_2,X_3,X_4;Y_3,\ldots,Y_k) \\ = \;& E(X_1,X_4)T(X_2,X_3,Y_3,\ldots,Y_k)+E(X_2,X_3)T(X_1,X_4,Y_3,\ldots,Y_k) \\ &-E(X_1,X_3)T(X_2,X_4,Y_3,\ldots,Y_k)-E(X_2,X_4)T(X_1,X_3,Y_3,\ldots,Y_k)\,. \end{aligned}$$

According to [26], the tensor $E\wedge T$ is called a *Kulkarni–Nomizu tensor*. Clearly, the tensors R, C, G, and $E\wedge F$, where E and F are symmetric $(0,2)$-tensors, are generalized curvature tensors.

A semi-Riemannian manifold (M,g), $n\geqslant 3$, is said to be *locally symmetric* if $\nabla R=0$ holds on M. It is obvious that the last condition leads immediately to the integrability condition (5). Manifolds satisfying (5) are called *semisymmetric* ([58, 59]). Riemannian semisymmetric manifolds were classified in [58] and [59]. Non-Riemannian semi-Riemannian manifolds (M,g), $n\geq 4$, with parallel Weyl tensor ($\nabla C=0$), which are in addition nonlocally symmetric ($\nabla R\neq 0$) and nonconformally flat ($C\neq 0$) are called *essentially conformally symmetric manifolds*, e.c.s. manifolds, in short ([13]–[18]). E.c.s. manifolds are semisymmetric (see, e.g., [13, 14]). In Remark 6.1(v) we present more details related to those manifolds. Another important subclass of semisymmetric manifolds, investigated recently, form *second-order symmetric manifolds* [7, 55], i.e. semi-Riemannian manifolds satisfying $\nabla\nabla R=0$. As a weaker condition than (5) there is

$$R\cdot R = L_R Q(g,R)\,, \tag{9}$$

which is considered on $U_R\subset M$, and hence L_R is a function uniquely determined on this set. On $M\setminus U_R$ we have $R\cdot R=Q(g,R)=0$. We note that $Q(g,R)=0$ at a point if and only if $R=\frac{\kappa}{(n-1)n}G$ at this point. A semi-Riemannian manifold (M,g), $n\geqslant 3$, is said to be *pseudosymmetric* if (9) holds on $U_R\subset M$ [5, 19, 28, 43].

A semi-Riemannian manifold (M, g), $n \geqslant 3$, is said to be *Ricci-symmetric* if $\nabla S = 0$ holds on M. It is obvious that the last condition leads immediately to the integrability condition (6). Manifolds satisfying (6) are called *Ricci-semisymmetric*. As a weaker condition than (6) there is

$$R \cdot S = L_S Q(g, S)\,, \tag{10}$$

which is considered on $U_S \subset M$, and hence L_S is a function uniquely determined on this set. On $M \setminus U_S$ we have $R \cdot S = Q(g, S) = 0$. We note that $Q(g, S) = 0$ at a point if and only if (1) holds at this point (cf. [9], Lemma 2.1 (i)). A semi-Riemannian manifold (M, g), $n \geqslant 3$, is said to be *Ricci-pseudosymmetric* if (10) holds on $U_S \subset M$ [5, 19, 28, 46].

Every locally symmetric, resp. semisymmetric and pseudosymmetric, manifold is Ricci-symmetric, resp. Ricci-semisymmetric and Ricci-pseudosymmetric. In all cases, the converse statements are not true. We refer to [5, 22, 28, 44] for a wider presentation of results related to these classes of manifolds.

A geometric interpretation of (9), resp. (10), is given in [43], resp. in [46]. Semi-Riemannian manifolds for which their curvature tensor R is expressed by a linear combination of the Kulkarni–Nomizu tensors $S \wedge S$, $g \wedge S$, and G are called Roter-type manifolds; see, e.g., [36] and references therein. Precisely, a semi-Riemannian manifold (M, g), $n \geqslant 4$, is said to be a *Roter-type manifold* if

$$R \;=\; \frac{\phi}{2}\, S \wedge S + \mu\, g \wedge S + \eta\, G \tag{11}$$

holds on the set U_1 of all points of $U_S \cap U_C \subset M$ at which $\operatorname{rank}(S - \alpha\, g) \geqslant 2$, for every $\alpha \in \mathbb{R}$. It is easy to prove that the functions ϕ, μ, and η are uniquely determined on U_1. Using (11) and suitable definitions we can verify that on U_1 the condition (9) is satisfied (e.g., see [36], Eqs. (7) and (8); [22], Theorem 6.7) with $L_R = \phi^{-1}((n-2)(\mu^2 - \phi\eta) - \mu)$, and that the difference tensor $R \cdot C - C \cdot R$ is expressed on U_1 by a linear combination of the tensors $Q(S, R)$, $Q(g, R)$, and $Q(S, G)$ ([24], Eq. (47)), or, equivalently, by a linear combination of the tensors $Q(g, R)$ and $Q(S, G)$ ([24], Eq. (48)).

Semi-Riemannian manifolds (M, g), $n \geqslant 4$, satisfying (3) on $U_S \cap U_C \subset M$ were investigated in [31]. Among other results in that paper it was proven that: (i) $R \cdot C = C \cdot R = 0$ and $\operatorname{rank} S = 1$ hold on $U_S \cap U_C$, provided that (M, g) is a quasi-Einstein manifold; and (ii) (11), with some special coefficients ϕ, μ, η such that $R \cdot R = 0$, and $C \cdot R = -L_1\, Q(g, R)$ hold on $U_S \cap U_C$, provided that (M, g) is a non-quasi-Einstein manifold. We also mention that manifolds satisfying

$$C \cdot R \;=\; L\, Q(g, R) \tag{12}$$

were investigated in [50]. Furthermore, we have

Theorem 2.1 ([32], Theorem 4.1 and Corollary 4.1) *Let (M, g), $n \geq 4$, be a semi-Riemannian manifold. If $R \cdot C - C \cdot R = L\, Q(g, C)$ holds on $U_S \cap U_C \subset M$, for some function L, then $R \cdot R = L\, Q(g, R)$ and $C \cdot R = 0$ on this set. In particular, if $R \cdot C = C \cdot R$ holds on $U_S \cap U_C$ then $R \cdot R = R \cdot C = C \cdot R = 0$ on this set.*

As we remarked in Section 1, manifolds satisfying (4) were investigated in [45]. We have

Theorem 2.2 ([45], Theorem 3.1 and Proposition 3.1) *(i) Let* (M,g), $n \geqslant 4$, *be a nonconformally flat and non-Einstein Ricci-semisymmetric manifold satisfying (4). Then on the set consisting of all points of* M *at which* L *is nonzero we have* $L = \frac{1}{n-2}$ *and*

$$R(\mathcal{S}X,Y,Z,W) \; = \; \frac{\kappa}{n-1}\, R(X,Y,Z,W)\,. \tag{13}$$

(ii) Let (M,g), $n \geqslant 4$, *be a semi-Riemannian manifold satisfying (13). Then* (M,g) *is a Ricci-semisymmetric manifold fulfilling (4) with* $L = \frac{1}{n-2}$.

We mention that hypersurfaces isometrically immersed in spaces of constant curvature having the tensor $R \cdot C$ expressed by a linear combination of the Tachibana tensors $Q(g,R)$, $Q(S,R)$, and $Q(S,G)$ were investigated in [40].

Quasi-Einstein hypersurfaces isometrically immersed in spaces of constant curvature were investigated in [33, 41]; see also the references therein. In particular, in [33] an example of a quasi-Einstein hypersurface in a semi-Riemannian space of constant curvature was found. More precisely, in that paper it was shown that some warped product $\overline{M} \times_F \widetilde{N}$, with $\dim \overline{M} = 1$ and $\dim \widetilde{N} \geqslant 4$, can be locally realized as a nonpseudosymmetric Ricci-pseudosymmetric quasi-Einstein hypersurface in a semi-Riemannian space of constant curvature. The difference tensor of that hypersurface is expressed by a linear combination of the tensors $Q(g,R)$ and $Q(S,R)$.

3. Warped product manifolds

Warped products play an important role in Riemannian geometry (see, e.g., [49, 51]) as well as in general relativity theory (see, e.g., [3, 4, 34, 51]). Many well-known spacetimes of this theory, i.e. solutions of the Einstein field equations, are warped products, e.g., the Schwarzschild, Kottler, Reissner–Nordström, Reissner–Nordström–de Sitter, and Vaidya, as well as Robertson–Walker, spacetimes. We recall that a warped product $\overline{M} \times_F \widetilde{N}$ of a 1-dimensional manifold $(\overline{M}, \bar{g})$, $\bar{g}_{11} = -1$, and a 3-dimensional Riemannian space of constant curvature $(\widetilde{N}, \tilde{g})$, with a warping function F, is said to be a *Robertson–Walker spacetime* (see, e.g., [3, 4, 51, 57]). It is well-known that the Robertson–Walker spacetimes are conformally flat quasi-Einstein manifolds. More generally, one also considers warped products $\overline{M} \times_F \widetilde{N}$ of $(\overline{M}, \bar{g})$, $\dim \overline{M} = 1$, $\bar{g}_{11} = -1$, with a warping function F and $(n-1)$-dimensional Riemannian manifold $(\widetilde{N}, \tilde{g})$, $n \geqslant 4$. Such warped products are called *generalized Robertson–Walker spacetimes* [1, 2, 38, 53, 54]. Curvature conditions of pseudosymmetry type on such spacetimes have been considered among others in [8, 9, 10, 12, 35, 36, 37, 42, 48]. We also mention that Einstein generalized Robertson–Walker spacetimes were classified in [2]. From (1) and (2) we immediately get the following:

Theorem 3.1 *On any Einstein manifold* (M,g), $n \geqslant 4$ *we have*

$$R \cdot C - C \cdot R \; = \; \frac{1}{n-1}\, Q(S,R)\,. \tag{14}$$

In particular, (14) holds on any Einstein generalized Robertson–Walker spacetime.

Let now $(\overline{M}, \bar{g})$ and $(\widetilde{N}, \tilde{g})$, $\dim \overline{M} = p$, $\dim \widetilde{N} = n - p$, $1 \leqslant p < n$, be semi-Riemannian manifolds. Let $F : \overline{M} \to \mathbb{R}^+$ be a positive smooth function on $\overline{M}$. The warped product manifold, or in short warped product, $\overline{M} \times_F \widetilde{N}$ of $(\overline{M}, \bar{g})$ and $(\widetilde{N}, \tilde{g})$ is the product manifold $\overline{M} \times \widetilde{N}$ with the metric $g = \bar{g} \times_F \tilde{g}$ defined by $\bar{g} \times_F \tilde{g} = \pi_1^* \bar{g} + (F \circ \pi_1)\, \pi_2^* \tilde{g}$, where $\pi_1 : \overline{M} \times \widetilde{N} \longrightarrow \overline{M}$ and $\pi_2 : \overline{M} \times \widetilde{N} \longrightarrow \widetilde{N}$ are the natural projections on $\overline{M}$ and $\widetilde{N}$, respectively. With respect to Corollary 3.1, in this paper we consider non-Einstein warped products $\overline{M} \times_F \widetilde{N}$ with 1-dimensional base manifold $(\overline{M}, \bar{g})$ and an $(n-1)$-dimensional fiber $(\widetilde{N}, \tilde{g})$, $n \geqslant 4$.

Let $\{\overline{U} \times \widetilde{V}; x^1, x^2 = y^1, \dots, x^n = y^{n-1}\}$ be a product chart for $\overline{M} \times \widetilde{N}$, where $\{\overline{U}; x^1\}$ and $\{\widetilde{V}; y^\alpha\}$ are systems of charts on $(\overline{M}, \bar{g})$ and $(\widetilde{N}, \tilde{g})$, respectively. The local components of the metric $g = \bar{g} \times_F \tilde{g}$ with respect to this chart are the following: $g_{11} = \bar{g}_{11} = \varepsilon = \pm 1$, $g_{hk} = F\, \tilde{g}_{\alpha\beta}$ if $h = \alpha$ and $k = \beta$, and $g_{hk} = 0$ otherwise, $\alpha, \beta, \gamma, \dots \in \{2, \dots, n\}$ and $h, i, j, k \dots \in \{1, 2, \dots, n\}$. We will denote by bars (resp., by tildes) tensors formed from $\bar{g}$ (resp., $\tilde{g}$). It is known that the local components Γ^h_{ij} of the Levi-Civita connection ∇ of $\overline{M} \times_F \widetilde{N}$ are the following (see, e.g., [49, 35]):

$$\begin{aligned} \Gamma^1_{11} &= 0\,, \quad \Gamma^\alpha_{\beta\gamma} = \widetilde{\Gamma}^\alpha_{\beta\gamma}\,, \quad \Gamma^1_{\alpha\beta} = -\frac{\varepsilon}{2} F' \tilde{g}_{\alpha\beta}\,, \\ \Gamma^\alpha_{a\beta} &= \frac{1}{2F} F' \delta^\alpha_\beta\,, \quad \Gamma^1_{\alpha 1} = \Gamma^\alpha_{11} = 0\,, \quad F' = \partial_1 F = \frac{\partial F}{\partial x^1}\,. \end{aligned} \tag{15}$$

The local components R_{hijk} of the curvature tensor R and the local components S_{hk} of the Ricci tensor S of $\overline{M} \times_F \widetilde{N}$, which may not vanish identically, are the following (see, e.g., [20, 35]):

$$R_{\alpha 11\beta} = -\frac{1}{2} T_{11}\, \tilde{g}_{\alpha\beta} = -\frac{\operatorname{tr} T}{2}\, g_{11} \tilde{g}_{\alpha\beta}\,, \quad R_{\alpha\beta\gamma\delta} = F(\widetilde{R}_{\alpha\beta\gamma\delta} - \frac{\Delta_1 F}{4F}\, \widetilde{G}_{\alpha\beta\gamma\delta})\,, \tag{16}$$

$$S_{11} = -\frac{n-1}{2\,F}\, T_{11}\,, \quad S_{\alpha\beta} = \widetilde{S}_{\alpha\beta} - (\frac{\operatorname{tr} T}{2} + (n-2)\, \frac{\Delta_1 F}{4\,F})\, \tilde{g}_{\alpha\beta}\,, \tag{17}$$

$$\begin{aligned} & T_{11} = F'' - \frac{(F')^2}{2\,F}\,, \quad \operatorname{tr} T = \bar{g}^{11} T_{11} = \varepsilon\big(F'' - \frac{(F')^2}{2\,F}\big)\,, \\ & \Delta_1 F = \Delta_{1\bar{g}} F = \bar{g}^{11} (F')^2 = \varepsilon (F')^2\,. \end{aligned} \tag{18}$$

The scalar curvature κ of $\overline{M} \times_F \widetilde{N}$ satisfies the following relation:

$$\kappa = \frac{1}{F}\Big(\tilde{\kappa} - (n-1)\big(\operatorname{tr} T + (n-2)\, \frac{\Delta_1 F}{4\,F}\big)\Big). \tag{19}$$

Using (8), (16), and (17) we can check that the local components $Q(g,R)_{hijklm}$ and $Q(S,R)_{hijklm}$ of the Tachibana tensors $Q(g,R)$ and $Q(S,R)$, which may not vanish identically, are the following:

$$Q(g,R)_{1\beta\gamma\delta 1\mu} = F g_{11} \big(\widetilde{R}_{\mu\beta\gamma\delta} + (\frac{\operatorname{tr} T}{2} - \frac{\Delta_1 F}{4F}) \widetilde{G}_{\mu\beta\gamma\delta}\big)\,, \tag{20}$$

$$Q(g,R)_{\alpha\beta\gamma\delta\lambda\mu} = F^2\, Q(\tilde{g}, \widetilde{R})_{\alpha\beta\gamma\delta\lambda\mu}\,, \tag{21}$$

$$
\begin{aligned}
Q(S,R)_{1\beta\gamma\delta 1\mu} &= -\frac{\operatorname{tr} T}{2} g_{11}\big((n-1)\widetilde{R}_{\mu\beta\gamma\delta} - \tilde{g}_{\beta\gamma}\widetilde{S}_{\delta\mu} + \tilde{g}_{\beta\delta}\widetilde{S}_{\gamma\mu} \\
&+ \big(\frac{\operatorname{tr} T}{2} - \frac{\Delta_1 F}{4F}\big)\widetilde{G}_{\mu\beta\gamma\delta}\big)\,, \qquad (22)\\
Q(S,R)_{1\beta 1\delta\lambda\mu} &= -\frac{\operatorname{tr} T}{2} g_{11}\, Q(\tilde{g},\widetilde{S})_{\beta\delta\lambda\mu}\,, \qquad (23)\\
Q(S,R)_{\alpha\beta\gamma\delta\lambda\mu} &= F\, Q(\widetilde{S},\widetilde{R})_{\alpha\beta\gamma\delta\lambda\mu} - \frac{\Delta_1 F}{4}\, Q(\widetilde{S},\widetilde{G})_{\alpha\beta\gamma\delta\lambda\mu} \\
&- F\big(\frac{\operatorname{tr} T}{2} + \frac{(n-2)\Delta_1 F}{4F}\big) Q(\tilde{g},\widetilde{R})_{\alpha\beta\gamma\delta\lambda\mu}\,. \qquad (24)
\end{aligned}
$$

Let V be the $(0,4)$-tensor with the local components

$$
V_{hijk} = g^{lm} S_{hl} R_{mijk} = S_h{}^l R_{lijk}\,. \qquad (25)
$$

Using (16) and (17) we can verify that the only nonzero components of this tensor are the following:

$$
\begin{aligned}
V_{1\beta\gamma 1} &= \frac{n-1}{4F}(\operatorname{tr} T)\, T_{11}\tilde{g}_{\beta\gamma} = \frac{n-1}{4F}(\operatorname{tr} T)^2\, g_{11}\tilde{g}_{\beta\gamma}\,,\\
V_{\alpha 11\delta} &= -\frac{\operatorname{tr} T}{2F}\, g_{11}\Big(\widetilde{S}_{\alpha\delta} - \big(\frac{\operatorname{tr} T}{2} + \frac{(n-2)\Delta_1 F}{4F}\big)\tilde{g}_{\alpha\delta}\Big)\,,\\
V_{\alpha\beta\gamma\delta} &= \widetilde{S}_\alpha{}^\epsilon \widetilde{R}_{\epsilon\beta\gamma\delta} - \big(\frac{\operatorname{tr} T}{2} + \frac{(n-2)\Delta_1 F}{4F}\big)\widetilde{R}_{\alpha\beta\gamma\delta} - \frac{\Delta_1 F}{4F}(\tilde{g}_{\beta\gamma}\widetilde{S}_{\alpha\delta} - \tilde{g}_{\beta\delta}\widetilde{S}_{\alpha\gamma})\\
&+ \big(\frac{\operatorname{tr} T}{2} + \frac{(n-2)\Delta_1 F}{4F}\big)\frac{\Delta_1 F}{4F}\,\widetilde{G}_{\alpha\beta\gamma\delta}\,. \qquad (26)
\end{aligned}
$$

The last equality yields

$$
\begin{aligned}
V_{\alpha\beta\gamma\delta} + V_{\beta\alpha\gamma\delta} &= \widetilde{S}_\alpha{}^\epsilon \widetilde{R}_{\epsilon\beta\gamma\delta} + \widetilde{S}_\beta{}^\epsilon \widetilde{R}_{\epsilon\alpha\gamma\delta}\\
&\quad -\frac{\Delta_1 F}{4F}(\tilde{g}_{\beta\gamma}\widetilde{S}_{\alpha\delta} - \tilde{g}_{\beta\delta}\widetilde{S}_{\alpha\gamma} + \tilde{g}_{\alpha\gamma}\widetilde{S}_{\beta\delta} - \tilde{g}_{\alpha\delta}\widetilde{S}_{\beta\gamma})\\
&= (\widetilde{R}\cdot\widetilde{S})_{\alpha\beta\gamma\delta} - \frac{\Delta_1 F}{4F}\, Q(\tilde{g},\widetilde{S})_{\alpha\beta\gamma\delta}\,. \qquad (27)
\end{aligned}
$$

Let P be a $(0,6)$-tensor with local components

$$
\begin{aligned}
P_{hijklm} &= g_{hl}V_{mijk} - g_{hm}V_{lijk} - g_{il}V_{mhjk} + g_{im}V_{lhjk} + g_{jl}V_{mkhi} - g_{jm}V_{lkhi}\\
&- g_{kl}V_{mjhi} + g_{km}V_{ljhi} - g_{ij}(V_{hklm} + V_{khlm}) - g_{hk}(V_{ijlm} + V_{jilm})\\
&+ g_{ik}(V_{hjlm} + V_{jhlm}) + g_{hj}(V_{iklm} + V_{kilm})\,. \qquad (28)
\end{aligned}
$$

The local components of P, which may not vanish identically, are the following:

$$P_{1\beta 1\delta\lambda\mu} = g_{11}\big((\widetilde{R}\cdot\widetilde{S})_{\beta\delta\lambda\mu} + (\frac{\mathrm{tr}\,T}{2} - \frac{\Delta_1 F}{4F})Q(\tilde{g},\widetilde{S})_{\beta\delta\lambda\mu}\big)\,, \tag{29}$$

$$\begin{aligned} P_{1\beta\gamma\delta 1\mu} &= g_{11}\Big(\widetilde{S}_{\mu}{}^{\epsilon}\widetilde{R}_{\epsilon\beta\gamma\delta} - (\frac{\mathrm{tr}\,T}{2} + \frac{(n-2)\Delta_1 F}{4F})\widetilde{R}_{\mu\beta\gamma\delta} \\ &+ (\frac{\mathrm{tr}\,T}{2} - \frac{\Delta_1 F}{4F})(\tilde{g}_{\beta\gamma}\widetilde{S}_{\delta\mu} - \tilde{g}_{\beta\delta}\widetilde{S}_{\gamma\mu}) \\ &+ \big((n-2)(\frac{\Delta_1 F}{4F})^2 - \frac{(\mathrm{tr}\,T)^2}{4} - \frac{(n-3)\mathrm{tr}\,T}{2}\frac{\Delta_1 F}{4F})\widetilde{G}_{\mu\beta\gamma\delta}\Big)\,, \end{aligned} \tag{30}$$

$$\begin{aligned} P_{\alpha\beta\gamma\delta\lambda\mu} &= F\big(\tilde{g}_{\alpha\lambda}V_{\mu\beta\gamma\delta} - \tilde{g}_{\alpha\mu}V_{\lambda\beta\gamma\delta} - \tilde{g}_{\beta\lambda}V_{\mu\alpha\gamma\delta} + \tilde{g}_{\beta\mu}V_{\lambda\alpha\gamma\delta} + \tilde{g}_{\gamma\lambda}V_{\mu\delta\alpha\beta} \\ &- \tilde{g}_{\gamma\mu}V_{\lambda\delta\alpha\beta} - \tilde{g}_{\delta\lambda}V_{\mu\gamma\alpha\beta} + \tilde{g}_{\delta\mu}V_{\lambda\gamma\alpha\beta} - \tilde{g}_{\beta\gamma}(V_{\alpha\delta\lambda\mu} + V_{\delta\alpha\lambda\mu}) \\ &- \tilde{g}_{\alpha\delta}(V_{\beta\gamma\lambda\mu} + V_{\gamma\beta\lambda\mu}) + \tilde{g}_{\beta\delta}(V_{\alpha\gamma\lambda\mu} + V_{\gamma\alpha\lambda\mu}) + \tilde{g}_{\alpha\gamma}(V_{\beta\delta\lambda\mu} + V_{\delta\beta\lambda\mu})\big)\,. \end{aligned} \tag{31}$$

4. Warped products with non-Einsteinian fiber

Since we investigate non-Einstein and nonconformally flat manifolds satisfying (4), we restrict our considerations to the set $\mathcal{U} = U \cap U_S \cap U_C$.

We assume that the warped product $\overline{M}\times_F \widetilde{N}$ satisfies (4) and the fiber $(\widetilde{N},\tilde{g})$ is not Einsteinian. Now we shall use the following identity, which holds on any semi-Riemannian manifold ([24], Section 4):

$$(n-2)(R\cdot C - C\cdot R)_{hijklm} = Q(S,R)_{hijklm} - \frac{\kappa}{n-1}Q(g,R)_{hijklm} + P_{hijklm}\,. \tag{32}$$

Thus, in view of (32) and the definition of the tensor P, condition (4) can be written in the following form:

$$\big((n-2)L - 1\big)\,Q(S,R)_{hijklm} = P_{hijklm} - \frac{\kappa}{n-1}Q(g,R)_{hijklm}\,. \tag{33}$$

For $h=1,\ i=\beta,\ j=1,\ k=\delta,\ l=\lambda,\ m=\mu$, in view of (20)–(24), (33) yields

$$\big((n-2)L - 1\big)\,Q(S,R)_{1\beta 1\delta\lambda\mu} = P_{1\beta 1\delta\lambda\mu}\,. \tag{34}$$

Substituting (23) and (29) into (34) we obtain

$$(\widetilde{R}\cdot\widetilde{S})_{\beta\delta\lambda\mu} = \big(\frac{\Delta_1 F}{4F} - \frac{(n-2)L}{2}\,\mathrm{tr}\,T\big)\,Q(\tilde{g},\widetilde{S})_{\beta\delta\lambda\mu}\,. \tag{35}$$

On the other hand, (4) implies

$$\begin{aligned} &(R\cdot C - C\cdot R)(X_1,X_2,X_3,X_4;X,Y) + (R\cdot C - C\cdot R)(X,Y,X_1,X_2;X_3,X_4) \\ +\ &(R\cdot C - C\cdot R)(X_3,X_4,X,Y;X_1,X_2) \ = \ 0\,, \end{aligned}$$

which in virtue of Proposition 4.1 of [24] is equivalent to

$$\begin{aligned} &(R\cdot C)(X_1,X_2,X_3,X_4;X,Y) + (R\cdot C)(X,Y,X_1,X_2;X_3,X_4) \\ +\ &(R\cdot C)(X_3,X_4,X,Y;X_1,X_2) \ = \ 0\,. \end{aligned}$$

Furthermore, we have ([8], Section 3, eq. (3.19)):

$$(\widetilde{R}\cdot\widetilde{S})_{\beta\delta\lambda\mu} = (\frac{\Delta_1 F}{4F} - \frac{\mathrm{tr}\,T}{2})\,Q(\tilde{g},\widetilde{S})_{\beta\delta\lambda\mu}\,. \tag{36}$$

Since $(\widetilde{N},\tilde{g})$, $\dim\widetilde{N}\geqslant 3$, is not Einsteinian, the tensor $Q(\tilde{g},\widetilde{S})$ is a nonzero tensor. Let $Q(\tilde{g},\widetilde{S})\neq 0$ at $x\in\mathcal{U}$. Thus, on a coordinate neighborhood $V_1\subset\mathcal{U}$ of x, in virtue of (35) and (36), we get

$$(L-\frac{1}{n-2})\,\mathrm{tr}\,T = 0\,.$$

We assert that $\mathrm{tr}\,T=0$. Supposing that $\mathrm{tr}\,T\neq 0$ at $y\in V_1$ we have $L=\frac{1}{n-2}$ on some neighborhood $U_1\subset V_1$ of y. Therefore, (33) reduces on U_1 to

$$P = \frac{\kappa}{n-1}\,Q(g,R)\,. \tag{37}$$

Evidently, on U_1 we also have

$$\frac{\Delta_1 F}{4F} - \frac{\mathrm{tr}\,T}{2} = const. \tag{38}$$

Now (37) gives

$$P_{1\beta\gamma\delta 1\mu} = \frac{\kappa}{n-1}\,Q(g,R)_{1\beta\gamma\delta 1\mu}\,. \tag{39}$$

Substituting into this equality (19), (22), and (30) we obtain

$$\begin{aligned}\widetilde{S}_{\mu}{}^{\epsilon}\widetilde{R}_{\epsilon\beta\gamma\delta} &= (\frac{\tilde{\kappa}}{n-1} - \frac{\mathrm{tr}\,T}{2})\,\widetilde{R}_{\mu\beta\gamma\delta} + (\frac{\Delta_1 F}{4F} - \frac{\mathrm{tr}\,T}{2})(\tilde{g}_{\beta\gamma}\widetilde{S}_{\mu\delta} - \tilde{g}_{\beta\delta}\widetilde{S}_{\mu\gamma})\\ &- (\frac{\tilde{\kappa}}{n-1} - \frac{\mathrm{tr}\,T}{2})(\frac{\Delta_1 F}{4F} - \frac{\mathrm{tr}\,T}{2})\widetilde{G}_{\mu\beta\gamma\delta}\,.\end{aligned} \tag{40}$$

Using now (38) and (40) we see that $\mathrm{tr}\,T = const.$ and consequently also $\frac{\Delta_1 F}{4F} = const.$ on U_1. Whence, after standard calculations, we deduce that F must be of the form

$$F(x^1) = (ax^1+b)^2\,,\quad a,b\in\mathbb{R}\,. \tag{41}$$

For such F we have $\mathrm{tr}\,T=0$ on U_1, a contradiction. Therefore

$$\mathrm{tr}\,T = 0 \tag{42}$$

on V_1. Thus, (38) reduces on V_1 to

$$\frac{\Delta_1 F}{4F} = const. = c_1\,. \tag{43}$$

Note that (42) and (43), in the same manner as above, imply (41) and we have

$$\mathrm{tr}\,T = 0\,,\quad \frac{\Delta_1 F}{4F} = c_1 = \varepsilon a^2\,. \tag{44}$$

We prove now that

$$L = \frac{1}{n-2} \tag{45}$$

on V_1. Applying (19), (20), (21), (30), (42), and (43) to

$$\big((n-2)L-1\big)\, Q(S,R)_{1\beta\gamma\delta 1\mu} = P_{1\beta\gamma\delta 1\mu} - \frac{\kappa}{n-1}\, Q(g,R)_{1\beta\gamma\delta 1\mu}$$

we get

$$\widetilde{S}_{\mu}{}^{\epsilon}\widetilde{R}_{\epsilon\beta\gamma\delta} = \frac{\tilde{\kappa}}{n-1}\,\widetilde{R}_{\mu\beta\gamma\delta} + \varepsilon a^2(\tilde{g}_{\beta\gamma}\widetilde{S}_{\mu\delta} - \tilde{g}_{\beta\delta}\widetilde{S}_{\mu\gamma}) - \frac{\varepsilon\tilde{\kappa}a^2}{n-1}\,\widetilde{G}_{\mu\beta\gamma\delta}\,. \tag{46}$$

On the other hand, (26), by (42) and (43), gives

$$V_{\alpha\beta\gamma\delta} = \widetilde{S}_{\alpha}{}^{\epsilon}\widetilde{R}_{\epsilon\beta\gamma\delta} - (n-2)c_1\widetilde{R}_{\alpha\beta\gamma\delta} - c_1(\tilde{g}_{\beta\gamma}\widetilde{S}_{\alpha\delta} - \tilde{g}_{\beta\delta}\widetilde{S}_{\alpha\gamma}) + (n-2)c_1^2\widetilde{G}_{\alpha\beta\gamma\delta}\,,$$

which by (46) turns into

$$V_{\alpha\beta\gamma\delta} = \big(\frac{\tilde{\kappa}}{n-1} - (n-2)c_1\big)\widetilde{R}_{\alpha\beta\gamma\delta} + (n-2)c_1^2\widetilde{G}_{\alpha\beta\gamma\delta}\,.$$

Substituting this into (31) we obtain

$$P_{\alpha\beta\gamma\delta\lambda\mu} = F\big(\frac{\tilde{\kappa}}{n-1} - (n-2)c_1\big)\, Q(\tilde{g},\widetilde{R})_{\alpha\beta\gamma\delta\lambda\mu}\,. \tag{47}$$

Now

$$\big((n-2)L-1\big)Q(S,R)_{\alpha\beta\gamma\delta\lambda\mu} = P_{\alpha\beta\gamma\delta\lambda\mu} - \frac{\kappa}{n-1}\, Q(g,R)_{\alpha\beta\gamma\delta\lambda\mu}\,,$$

by making use of (21), (47), and

$$\frac{\kappa}{n-1} = \frac{1}{F}\,\big(\frac{\tilde{\kappa}}{n-1} - (n-2)c_1\big)\,,$$

turns into

$$\big((n-2)L-1\big)Q(S,R)_{\alpha\beta\gamma\delta\lambda\mu} = 0\,. \tag{48}$$

Since $V_1 \subset \mathcal{U}$ and in virtue of (22), (23), and (42) we have

$$Q(S,R)_{1\beta\gamma\delta 1\mu} = Q(S,R)_{1\beta 1\delta\lambda\mu} = 0\,,$$

at least one of the local components of $Q(S,R)_{\alpha\beta\gamma\delta\lambda\mu}$ must be nonzero. Therefore, (48) implies (45). Thus we have proven:

Theorem 4.1 *Let $\overline{M}\times_F\widetilde{N}$ be a warped product manifold with 1-dimensional base manifold $(\overline{M},\bar{g})$ and non-Einstein $(n-1)$-dimensional fiber $(\widetilde{N},\tilde{g})$, $n\geqslant 4$. If (4) is satisfied on $\overline{M}\times_F\widetilde{N}$ then on the set $\mathcal{U}$ we have (46) and*

$$L = \frac{1}{n-2}\,,\quad F(x^1) = (ax^1+b)^2\,,\ a,b\in\mathbb{R}\,. \tag{49}$$

Corollary 4.1 *Let $\overline{M}\times_F\widetilde{N}$ be a generalized Robertson–Walker spacetime with non-Einstein fiber $(\widetilde{N},\tilde{g})$, $n\geqslant 4$. If (4) is satisfied on $\overline{M}\times_F\widetilde{N}$ then (46) and (49) hold on $\mathcal{U}$.*

Proposition 4.1 *Under assumptions of Theorem 4.1 the fiber manifold* $(\widetilde{N}, \tilde{g})$ *is a Ricci-pseudosymmetric manifold of constant type (see, e.g., [41]), precisely*

$$\widetilde{R} \cdot \widetilde{S} = \varepsilon a^2 \, Q(\tilde{g}, \widetilde{S})\,. \tag{50}$$

Moreover, if $n \geqslant 5$ *then the difference tensor* $\widetilde{R} \cdot \widetilde{C} - \widetilde{C} \cdot \widetilde{R}$ *of the fiber is expressed by the Tachibana tensors* $Q(\widetilde{S}, \widetilde{R})$ *and* $Q(\tilde{g}, \widetilde{R})$*; precisely, we have*

$$(n-3)(\widetilde{R} \cdot \widetilde{C} - \widetilde{C} \cdot \widetilde{R}) = Q(\widetilde{S}, \widetilde{R}) - \frac{\tilde{\kappa}}{(n-1)(n-2)}\, Q(\tilde{g}, \widetilde{R})\,. \tag{51}$$

Proof First we observe that (46) implies (50). Applying the identity (32) to $(\widetilde{N}, \tilde{g})$ we have

$$(n-3)(\widetilde{R} \cdot \widetilde{C} - \widetilde{C} \cdot \widetilde{R}) = Q(\widetilde{S}, \widetilde{R}) + \widetilde{P} - \frac{\tilde{\kappa}}{n-2}\, Q(\tilde{g}, \widetilde{R})\,. \tag{52}$$

Using now (46) we get

$$\widetilde{V}_{\alpha\beta\gamma\delta} + \widetilde{V}_{\beta\alpha\gamma\delta} = \varepsilon a^2 Q(\tilde{g}, \widetilde{S})_{\alpha\beta\gamma\delta}$$

and

$$\widetilde{P} = \frac{\tilde{\kappa}}{n-1}\, Q(\tilde{g}, \widetilde{R}) - \varepsilon a^2 Q(\widetilde{S}, \widetilde{G}) - \varepsilon a^2 \, \tilde{g} \wedge Q(\tilde{g}, \widetilde{S})\,,$$

which by making use of $\tilde{g} \wedge Q(\tilde{g}, \widetilde{S}) = -Q(\widetilde{S}, \widetilde{G})$ (see (28) of [24]) reduces to $\widetilde{P} = \frac{\tilde{\kappa}}{n-1}\, Q(\tilde{g}, \widetilde{R})$. Substituting this equality into (52) we obtain (51). □

We have also the converse statement to Theorem 4.1.

Theorem 4.2 *Let* $(\overline{M}, \bar{g})$*,* $\bar{g}_{11} = \varepsilon$*, be a* 1*-dimensional manifold and let* $(\widetilde{N}, \tilde{g})$ *be an* $(n-1)$*-dimensional non-Einstein manifold,* $n \geqslant 4$*, satisfying (46). If* $F(x^1) = (ax^1 + b)^2$*, then the warped product* $\overline{M} \times_F \widetilde{N}$ *fulfills (4) with* $L = \frac{1}{n-2}$*.*

Proof As we have seen (cf. (33)), (4) for $L = \frac{1}{n-2}$ takes the form

$$P = \frac{\kappa}{n-1}\, Q(g, R)\,. \tag{53}$$

Using now (27), (44), and (50) we have

$$V_{\alpha\beta\gamma\delta} + V_{\beta\alpha\gamma\delta} = 0\,. \tag{54}$$

Taking into account (26), (44), and (46), we obtain

$$V_{\alpha\beta\gamma\delta} = \phi \widetilde{R}_{\alpha\beta\gamma\delta} + \psi\, \widetilde{G}_{\alpha\beta\gamma\delta}$$

for some ψ and $\phi = \frac{\tilde{\kappa}}{n-1} - \varepsilon(n-2)a^2$. Substituting the above equality and (54) into (31) we get

$$P_{\alpha\beta\gamma\delta\lambda\mu} = F\phi\, Q(\tilde{g}, \widetilde{R})_{\alpha\beta\gamma\delta\lambda\mu}\,,$$

which in view of (19) and (44) takes the form

$$P_{\alpha\beta\gamma\delta\lambda\mu} = \frac{\kappa}{n-1}\, Q(g,R)_{\alpha\beta\gamma\delta\lambda\mu}\,.$$

Using (29), (44), and (50), we have $P_{1\beta 1\delta\lambda\mu} = 0$, which means that (34) is satisfied. Finally, in the same manner we obtain (39). Thus we see that (53) is satisfied for all systems of indices. □

Corollary 4.2 *The equality (46) is satisfied on every Einstein manifold* $(\widetilde{N}, \tilde{g})$. *Thus every warped product* $\overline{M} \times_F \widetilde{N}$ *with* 1*-dimensional base* $(\overline{M}, \bar{g})$, *Einsteinian fiber* $(\widetilde{N}, \tilde{g})$, $\dim \widetilde{N} \geqslant 4$, *which is not a space of constant curvature, and the warping function* $F(x^1) = (ax^1 + b)^2$, $\varepsilon a^2 \neq \frac{\tilde{\kappa}}{(n-2)(n-1)}$, *satisfies (4) with* $L = \frac{1}{n-2}$.

Remark 4.1. Let $\overline{M} \times_F \widetilde{N}$ be the warped product manifold with 1-dimensional base $(\overline{M}, \bar{g})$, Einsteinian fiber $(\widetilde{N}, \tilde{g})$, $\dim \widetilde{N} \geqslant 4$, and the warping function $F(x^1) = (ax^1 + b)^2$, $\varepsilon a^2 = \frac{\tilde{\kappa}}{(n-2)(n-1)}$. Now (17) yields $S = 0$, i.e. $\overline{M} \times_F \widetilde{N}$ is a Ricci-flat manifold.

5. Warped products with Einsteinian fiber

In this section we consider non-Einstein warped products $\overline{M} \times_F \widetilde{N}$, $\dim \overline{M} = 1$, assuming that a fiber $(\widetilde{N}, \tilde{g})$ is an Einstein manifold, i.e.

$$\widetilde{S}_{\alpha\beta} = \frac{\tilde{\kappa}}{n-1}\, \tilde{g}_{\alpha\beta}\,. \tag{55}$$

Using (17) we can easily show that such warped product is a quasi-Einstein manifold. It is worth noticing that $\widetilde{R} \neq \frac{\tilde{\kappa}}{(n-1)(n-2)} \widetilde{G}$ on $\mathcal{U}$. Using (23), (29), and (55), we get

$$Q(S,R)_{1\beta 1\delta\lambda\mu} = P_{1\beta 1\delta\lambda\mu} = 0\,. \tag{56}$$

Analogously, in view of (22), (30), and (55), we have

$$Q(S,R)_{1\alpha\beta\gamma 1\delta} = \frac{\operatorname{tr} T}{2}\, g_{11} \left(-(n-1)\widetilde{R}_{\delta\alpha\beta\gamma} + \Big(\frac{\tilde{\kappa}}{n-1} - \big(\frac{\operatorname{tr} T}{2} - \frac{\Delta_1 F}{4F}\big)\Big) \widetilde{R}_{\delta\alpha\beta\gamma} \right), \tag{57}$$

$$\begin{aligned} P_{1\beta\gamma\delta 1\mu} &= g_{11} \Big(\eta \widetilde{R}_{\mu\beta\gamma\delta} \\ &+ \Big(\big(\frac{\operatorname{tr} T}{2} - \frac{\Delta_1 F}{4F}\big) \frac{\tilde{\kappa}}{n-1} + (n-2)\big(\frac{\Delta_1 F}{4F}\big)^2 - \frac{(\operatorname{tr} T)^2}{4} - (n-3)\frac{\operatorname{tr} T}{2}\, \frac{\Delta_1 F}{4F}\Big) \widetilde{G}_{\mu\beta\gamma\delta} \Big) \end{aligned} \tag{58}$$

where

$$\eta = \frac{\tilde{\kappa}}{n-1} - \frac{\operatorname{tr} T}{2} - \frac{(n-2)\Delta_1 F}{4F}\,.$$

Finally, making use of (26) and (55), we obtain

$$V_{\alpha\beta\gamma\delta} = \eta\, (\widetilde{R}_{\alpha\beta\gamma\delta} - \frac{\Delta_1 F}{4F}\, \widetilde{G}_{\alpha\beta\gamma\delta}),$$

and next, in virtue of (24) and (31), also

$$Q(S,R)_{\alpha\beta\gamma\delta\lambda\mu} = F\eta\, Q(\tilde{g},\widetilde{R})_{\alpha\beta\gamma\delta\lambda\mu}\,, \tag{59}$$

$$P_{\alpha\beta\gamma\delta\lambda\mu} = F\eta\, Q(\tilde{g},\widetilde{R})_{\alpha\beta\gamma\delta\lambda\mu}\,. \tag{60}$$

Thus, taking into account (20), (21), and (56)–(60), we see that (33) is equivalent to the following 2 equalities:

$$\begin{aligned} &\big((n-2)L-1\big)\,\frac{\mathrm{tr}\,T}{2}\,\Big(-(n-1)\widetilde{R}+\big(\frac{\tilde{\kappa}}{n-1}-\big(\frac{\mathrm{tr}\,T}{2}-\frac{\Delta_1 F}{4F}\big)\big)\,\widetilde{G}\Big)\\ =\ &\frac{\mathrm{tr}\,T}{2}\,\big(\widetilde{R}+\big(\frac{\mathrm{tr}\,T}{2}-\frac{\Delta_1 F}{4F}\big)\,\widetilde{G}\big)\,, \end{aligned} \tag{61}$$

$$\big((n-2)L-1\big)\,\eta\, Q(\tilde{g},\widetilde{R}) = \frac{\mathrm{tr}\,T}{2}\,Q(\tilde{g},\widetilde{R})\,. \tag{62}$$

We consider 2 cases: (i) $\mathrm{tr}\,T=0$ and (ii) $\mathrm{tr}\,T\neq 0$.

(i) $\mathrm{tr}\,T=0$. Since $Q(\tilde{g},\widetilde{R})\neq 0$ on $\mathcal{U}$, (62) leads to

$$\big((n-2)L-1\big)\big(\frac{\tilde{\kappa}}{n-1}-\frac{(n-2)\Delta_1 F}{4F}\big)=0\,.$$

Supposing that $\frac{\tilde{\kappa}}{(n-2)(n-1)}=\frac{\Delta_1 F}{4F}$ and using (17) we obtain $S=0$, a contradiction. Thus we get $\frac{\tilde{\kappa}}{(n-2)(n-1)}\neq\frac{\Delta_1 F}{4F}$ and $L=\frac{1}{n-2}$. Moreover, solving the differential equation $\mathrm{tr}\,T=0$, one can see that the warping function F must be of the form (41). Thus, we have the situation described in Corollary 4.2.

(ii) $\mathrm{tr}\,T\neq 0$. Now (61) leads to

$$(n-2)\big((n-1)L-1\big)\,\widetilde{R}=\Big((n-2)L\big(\frac{\tilde{\kappa}}{n-1}-\big(\frac{\mathrm{tr}\,T}{2}-\frac{\Delta_1 F}{4F}\big)\big)-\frac{\tilde{\kappa}}{n-1}\Big)\widetilde{G}\,,$$

whence $L=\frac{1}{n-1}$ and

$$\frac{\Delta_1 F}{4F}-\frac{\mathrm{tr}\,T}{2} = \frac{\tilde{\kappa}}{(n-1)(n-2)}\,. \tag{63}$$

It is worth noticing that under the above equalities (62) also holds. Applying (63) to (17), after standard calculations, we get (1), a contradiction. Thus we have proven:

Theorem 5.1 *Let $\overline{M}\times_F\widetilde{N}$ be a non-Einstein warped product manifold with 1-dimensional base manifold $(\overline{M},\bar{g})$ and Einsteinian $(n-1)$-dimensional fiber $(\widetilde{N},\tilde{g})$. If (4) is satisfied on $\overline{M}\times_F\widetilde{N}$, then on the set $\mathcal{U}$ we have:*

$$L=\frac{1}{n-2}\,,\quad F(x^1)=(ax^1+b)^2\,,\ a,b\in\mathbb{R}\,,\quad \varepsilon a^2\neq\frac{\widetilde{\kappa}}{(n-2)(n-1)}\,.$$

Remark 5.1. Let $\overline{M}\times_F\widetilde{N}$ be the warped product manifold with 1-dimensional base $(\overline{M},\bar{g})$, Einsteinian fiber $(\widetilde{N},\tilde{g})$, $\dim\widetilde{N}\geqslant 4$, and the warping function F.

(i) We assume that F satisfies (63). As we stated above, now $\overline{M} \times_F \widetilde{N}$ is an Einstein manifold. Furthermore, it is easy to check that (63), in view of (18), takes the form

$$F\,F'' - (F')^2 + 2\varepsilon C_1 F \;=\; 0\,, \quad C_1 \;=\; \frac{\tilde{\kappa}}{(n-1)(n-2)}\,. \tag{64}$$

This is exactly equation (29) of [36]. We can check that the following functions are solutions of (64) (cf. [36], Lemma 3.1):

$$\begin{aligned}
F(x^1) &= \varepsilon C_1\,(x^1 + \frac{\varepsilon c}{C_1})^2\,, \qquad \varepsilon C_1 > 0\,,\\
F(x^1) &= \frac{c}{2}\big(\exp(\pm\frac{b}{2}x^1) - \frac{2\varepsilon C_1}{b^2 c}\exp(\mp\frac{b}{2}x^1)\big)^2\,, \qquad c > 0\,, \qquad b \neq 0\,,\\
F(x^1) &= \frac{2\varepsilon C_1}{c}\,\big(1 + \sin(cx^1 + b)\big)\,, \qquad \frac{\varepsilon C_1}{c} > 0\,,
\end{aligned}$$

where b and c are constants and x^1 belongs to a suitable nonempty open interval of $\mathbb{R}$.
(ii) As was mentioned in Section 3, Einstein generalized Robertson–Walker spacetimes were classified in [2]. As was stated in [2], if $\overline{M} \times_f \widetilde{N}$ is a generalized Robertson–Walker spacetime, with $\varepsilon = -1$, the warping function f, and the Einsteinian fiber $(\widetilde{N}, \tilde{g})$ and $\varepsilon = -1$ is an Einstein manifold, then the differential equations given in (3) of [2] must be satisfied. Those equations, adopted to our denotations, takes the form

$$f f'' \;=\; \frac{\kappa}{(n-1)n} f^2\,, \qquad \frac{\kappa}{(n-1)n} f^2 \;=\; \frac{\widetilde{\kappa}}{(n-2)(n-1)} + (f')^2\,,$$

respectively. If now we set $f = \sqrt{F}$ then the equations presented above lead to (64).

6. Examples

Corollary 4.2 and Theorem 5.1 give rise to examples of warped products satisfying (4) with Einstein fiber. The problem of finding of a warped product satisfying (4) with non-Einstein fiber reduces, via Theorem 4.2, to the problem of finding an example of a semi-Riemannian manifold $(\widetilde{N}, \widetilde{g})$, $\dim \widetilde{N} = n - 1 \geqslant 3$, fulfilling (46). To obtain a suitable example we will use results of [21, 24, 30]. First of all, we adopt results contained in Theorem 3.1 of [21] and in Theorem 3.2 of [24]. Those results we can present in the following:

Theorem 6.1 *Let $(\widetilde{N}, \widetilde{g})$ be a hypersurface isometrically immersed in a semi-Riemannian space of constant curvature $N^n_s(c)$, $n \geqslant 4$, with signature $(s, n-s)$, where $c = \frac{\tau}{(n-1)n}$, τ is the scalar curvature of the ambient space and $\widetilde{g}$ is the metric tensor induced on $\widetilde{N}$. Moreover, let the second fundamental tensor H of $\widetilde{N}$ satisfy on some nonempty connected set $\widetilde{U} \subset \widetilde{N}$ the equation*

$$H^3 \;=\; tr(H)\,H^2 + \lambda\,H, \tag{65}$$

where λ is some function on $\widetilde{U}$, and let the constant $\varepsilon = \pm 1$ be defined by the Gauss equation of $\widetilde{N}$ in $N^n_s(c)$, i.e. by

$$\widetilde{R} \;=\; \frac{\varepsilon}{2}\,H \wedge H + \frac{\tau}{(n-1)n}\,\widetilde{G}\,.$$

(i) (cf. [21], Theorem 3.1) On $\widetilde{U}$ *we have*

$$\widetilde{S}_{\mu}{}^{\epsilon}\widetilde{R}_{\epsilon\beta\gamma\delta} \;=\; \mu\,(\widetilde{R}_{\mu\beta\gamma\delta} - \frac{\tau}{(n-1)n}\,\widetilde{G}_{\mu\beta\gamma\delta}) + \frac{\tau}{(n-1)n}\,(\widetilde{g}_{\beta\gamma}\widetilde{S}_{\mu\delta} - \widetilde{g}_{\beta\delta}\widetilde{S}_{\mu\gamma}), \tag{66}$$

where $\mu = \frac{(n-2)\tau}{(n-1)n} - \varepsilon\,\lambda$.

(ii) (cf. [24], Theorem 3.2) If $n \geqslant 5$ *then on* $\widetilde{U}$ *we have*

$$(n-3)\,(\widetilde{R}\cdot\widetilde{C} - \widetilde{C}\cdot\widetilde{R}) \;=\; Q(\widetilde{S},\widetilde{R}) + (\frac{(n-2)\tau}{(n-1)n} - \varepsilon\,\lambda - \frac{\widetilde{\kappa}}{n-2})Q(\widetilde{g},\widetilde{R})\,, \tag{67}$$

where $\widetilde{\kappa}$ *is the scalar curvature of* $\widetilde{N}$.

We note that (66) implies immediately that $\widetilde{R}\cdot\widetilde{S} = \frac{\tau}{(n-1)n}\,Q(\tilde{g},\widetilde{S})$. In addition, if we assume that on $\widetilde{U}$ we have

$$\lambda = 0 \quad \text{and} \quad (n-2)\tau = n\widetilde{\kappa}, \tag{68}$$

then (46) holds on $\widetilde{U}$. The last remark suggests a solution of our problem. Namely, the last 2 conditions are realized on the hypersurface presented in Example 5.1 of [30]. Let (M,g) be the manifold defined in Example 5.1 of [30]. We denote it by $(\widetilde{N},\widetilde{g})$. Clearly $(\widetilde{N},\widetilde{g})$ is a manifold of dimension $\geqslant 4$. However, it is easy to verify that if we repeat the construction of $(\widetilde{N},\widetilde{g})$ for the 3-dimensional case then all curvature properties remain true, excluding, of course, properties expressed by its Weyl conformal curvature tensor. Thus, without loss of generality, we can assume that $\dim \widetilde{N} = n-1 \geqslant 3$. In Example 5.1 of [30], among other things, it was shown that $(\widetilde{N},\widetilde{g})$ is locally isometric to a hypersurface in a semi-Riemannian space of nonzero constant curvature. Since our considerations are local, we can assume that $(\widetilde{N},\widetilde{g})$ is a hypersurface isometrically immersed in that space. Since $(\widetilde{N},\widetilde{g})$ fulfils (68), Theorem 4.2 finishes our construction. We note that by making use of (67) and (68), we obtain (51).

Remark 6.1. (i) The Roter-type warped products $\overline{M}\times_F\widetilde{N}$ with 1-dimensional base manifold $(\overline{M},\overline{g})$ and non-Einstein $(n-1)$-dimensional fiber $(\widetilde{N},\widetilde{g})$, $n \geqslant 4$, were investigated in [36]. Among other results it was proven that the curvature tensor $\widetilde{R}$ of the fiber $(\widetilde{N},\widetilde{g})$ is expressed by the Kulkarni–Nomizu tensors $\widetilde{S}\wedge\widetilde{S}$, $\widetilde{g}\wedge\widetilde{S}$, and $\widetilde{G}$, i.e. the fiber also is a Roter-type manifold, provided that $n \geqslant 5$. Therefore, if we assume that the fiber manifold $(\widetilde{N},\widetilde{g})$ considered in Theorem 6.1 is a nonpseudosymmetric Ricci-pseudosymmetric hypersurface (for instance, the Cartan hypersurfaces of dimension 6, 12, or 24 have this property (see, e.g., [41])), then fibers of both constructions are nonisometric.
(ii) From (12), by a suitable contraction, we get

$$C\cdot S \;=\; L\,Q(g,S)\,. \tag{69}$$

We refer to [47] and [50] for examples of warped products satisfying (69). The condition (69) holds on some hypersurfaces in semi-Riemannian space forms, and, in particular, on the Cartan hypersurfaces ([21], Theorems 3.1 and 4.3). Hypersurfaces in semi-Euclidean space satisfying (69) were investigated in [52].

(iii) We also can investigate semi-Riemannian manifolds (M,g), $n \geqslant 4$, satisfying on $U_C \subset M$ the following condition of pseudosymmetric type (see, e.g., [11, 29]):

$$R \cdot R - Q(S,R) \quad = \quad L\, Q(g,C)\,, \tag{70}$$

where L is some function on this set. Warped products satisfying (70) were investigated in [11]. Among other results, in [11] it was shown that this condition is satisfied on every 4-dimensional warped product $\overline{M} \times_F \widetilde{N}$ with 1-dimensional base. Thus, in particular, every 4-dimensional generalized Robertson–Walker spacetime satisfies (70). We mention that (70) holds on every hypersurface in a semi-Riemannian space of constant curvature (see, e.g., [22], eq. (4.4)).

(iv) In [23] (Example 4.1) a warped product $\overline{M} \times_F \widetilde{N}$ of an $(n-1)$-dimensional base $(\overline{M}, \overline{g})$, $n \geqslant 4$, and an 1-dimensional fiber $(\widetilde{N}, \widetilde{g})$ satisfying $\operatorname{rank} S = 1$, $\kappa = 0$, $R \cdot R = 0$, and $C \cdot S = 0$ was constructed. In addition, we can easily check that (4) with $L = \frac{1}{n-2}$ and $Q(S,C) = Q(S,R)$ hold on $\overline{M} \times_F \widetilde{N}$ ([25]). Therefore, on $\overline{M} \times_F \widetilde{N}$ we also have $(n-2)\,(R \cdot C - C \cdot R) = Q(S,C)$. Semi-Riemannian manifolds satisfying $R \cdot C - C \cdot R = L\, Q(S,C)$, for some function L, were investigated in [25]. An example of a quasi-Einstein non-Ricci-simple manifold satisfying the last condition was given in Section 6 of [22].

(v) We recall that non-Riemannian semi-Riemannian manifolds (M,g), $n \geq 4$, with parallel Weyl tensor $(\nabla C = 0)$, which are in addition nonlocally symmetric $(\nabla R \neq 0)$ and nonconformally flat $(C \neq 0)$, are called essentially conformally symmetric manifolds, or e.c.s. manifolds, in short (see, e.g., [13, 14]). E.c.s. manifolds are semisymmetric manifolds satisfying ([13], Theorems 7, 8 and 9): $Q(S,C) = 0$, $C(\mathcal{S}X,Y,Z,W) = 0$, $S(\mathcal{S}X,Y) = 0$, $\kappa = 0$. In addition, on every e.c.s. manifold (M,g) we have [14]: $\operatorname{rank} S \leqslant 2$ and $F\, C = (1/2)\, S \wedge S$, where F is some function on M, called the fundamental function. The local structure of e.c.s. manifolds is determined [15, 17]. Certain e.c.s. metrics are realized on compact manifolds [16, 18]. Let now (M,g), $n \geq 4$, be an e.c.s. manifold satisfying $\operatorname{rank} S \leqslant 1$. Now it is easy to check that at all points of M at which $\operatorname{rank} S = 1$ the conditions $Q(S,C) = 0$, $C(\mathcal{S}X,Y,Z,W) = 0$ turn into $Q(S,R) = 0$, $R(\mathcal{S}X,Y,Z,W) = 0$, respectively. The last equality means that the tensor V, defined by (25), vanishes. Therefore, (28) reduces to $P = 0$. Now we see that the identity (32) turns into $R \cdot C - C \cdot R = 0$, and, in consequence, at all points of M at which $\operatorname{rank} S = 1$ we have $R \cdot C - C \cdot R = Q(S,R) = 0$. Thus, we can state that the last condition holds on any Ricci-simple e.c.s. manifold. Finally, we also remark that the last result is an immediate consequence of Theorem 2.2.

7. Conclusions

Let $\overline{M} \times_F \widetilde{N}$ be the warped product of an 1-dimensional manifold $(\overline{M}, \overline{g})$, $\overline{g}_{11} = \varepsilon = \pm 1$, the warping function $F : \overline{M} \to \mathbb{R}^+$, and an $(n-1)$-dimensional, $n \geqslant 4$, semi-Riemannian manifold $(\widetilde{N}, \widetilde{g})$.

If $(\widetilde{N}, \widetilde{g})$ is a semi-Riemannian space of constant curvature then $\overline{M} \times_F \widetilde{N}$ is a quasi-Eintein conformally flat pseudosymmetric manifold. Evidently, the Friedmann–Lemaître–Robertson–Walker spacetimes belong to this class of manifolds. Furthermore, if the fiber $(\widetilde{N}, \widetilde{g})$, $n \geqslant 5$, is an Einstein manifold, which is not of constant curvature, then $\overline{M} \times_F \widetilde{N}$ is a quasi-Einstein nonconformally flat nonpseudosymmetric Ricci-pseudosymmetric manifold. In this case the difference tensor $R \cdot C - C \cdot R$ is expressed by a linear combination of the Tachibana tensors $Q(g,R)$ and $Q(S,R)$ ([8]).

If the fibre $(\widetilde{N},\widetilde{g})$, $n \geqslant 4$, is a conformally flat Ricci simple manifold such that its scalar curvature $\widetilde{\kappa}$ vanishes then $\overline{M} \times_F \widetilde{N}$ is a non-conformally flat pseudosymmetric manifold, provided that $F = F(x^1) = \exp x^1$ ([10], Proposition 4.2 and Example 4.1). In addition we have $(n-1)\,(R\cdot C - C\cdot R) = Q(S,C)$ [25].

If the fiber $(\widetilde{N},\widetilde{g})$, $n \geqslant 4$, is some Roter-type manifold and the warping function F satisfies (64), then $\overline{M} \times_F \widetilde{N}$ is a Roter-type manifold and in consequence a nonconformally flat pseudosymmetric manifold ([36], Theorem 5.1). As was mentioned in Section 2, the tensor $R\cdot C - C\cdot R$ is expressed by a linear combination of some Tachibana tensors.

The above presented facts show that under some conditions imposed on the fiber or the fiber and the warping function of a generalized Robertson–Walker spacetime, such spacetime is a pseudosymmetric or Ricci-pseudosymmetric manifold and its difference tensor $R\cdot C - C\cdot R$ is expressed by a linear combination of some Tachibana tensors. In this paper we consider an inverse problem. Namely, if the tensors $R\cdot C - C\cdot R$ and $Q(S,R)$ are linearly dependent on a generalized Robertson–Walker spacetime then we determine the warping function, as well as curvature properties of the fiber of such spacetime. In the case where the considered generalized Robertson–Walker spacetimes are 4-dimensional manifolds, it is possible to apply the algebraic classification of spacetimes satisfying some conditions of the pseudosymmetry type given in [27]; see also [39, 42].

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