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Monomial ideals of linear type

Monica LA BARBIERA*, **Paola Lea STAGLIANÒ**
Department of Mathematics, University of Messina, Messina, Italy

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Abstract: Let $S = K[x_1, \dots, x_n; y_1, \dots, y_m]$ be the polynomial ring in 2 sets of variables over a field K. We investigate some classes of monomial ideals of S in order to classify ideals of the linear type.

Key words: Mixed products ideals, Veronese bi-type ideals, Rees algebra, ideals of linear type

1. Introduction

Let $R = K[x_1, \dots, x_n]$ be the polynomial ring over a field K. The monomial ideals of R are ideals generated by monomials and they have been intensively studied. Some problems arise when we would study good properties of monomial ideals and the same properties for some algebras related to them. The most important of such algebras is the Rees algebra $\Re(I) = \bigoplus_{i \geq 0} I^i t^i$ ([1], §1.5, §4.5). In this paper we investigate the ideal of presentation N of the Rees algebra associated to monomial ideals. If N is generated by linear relations, namely R-homogeneous elements of degree 1, then the ideal is said to be of linear type. Our aim is to study monomial ideals of linear type.

Let I be an ideal of R generated by polynomials $f_1, \dots, f_s$. Consider the presentation $\varphi : R[T_1, \dots, T_s] \to \Re(I) = R[f_1 t, \dots, f_s t]$ of the Rees algebra $\Re(I)$ of I, defined by setting $\varphi(T_i) = f_i t$, $i = 1, \dots, s$. Let N denote the kernel of φ and it is R-homogeneous. I is said to be of linear type if and only if N is generated by R-homogeneous elements of degree 1. In other words, I is of linear type if and only if the canonical map $\psi : Sym_R(I) \to \Re(I)$, from the symmetric algebra of I to the Rees algebra of I, is an isomorphism. Several classes of ideals of R of linear type are known. For instance, ideals generated by d-sequences and M-sequences are of linear type [4], [12].

Set $S = K[x_1, \dots, x_n; y_1, \dots, y_m]$, the polynomial ring in 2 sets of variables over a field K. Recently monomial ideals of S have been introduced and some properties have been studied [13], [10]. In this paper we consider the ideals of mixed products $L = I_k J_r + I_s J_t$, where $k + r = s + t$ and I_k (resp. J_r) is the monomial ideal of S generated by all square-free monomials of degree k (resp. r) in the variables $x_1, \dots, x_n$ (resp. $y_1, \dots, y_m$) [10].

Moreover, we consider another class of monomial ideals of S, so-called Veronese bi-type ideals. They are an extension of the ideals of Veronese type [11] in a polynomial ring in 2 sets of variables. The ideals of Veronese bi-type are monomial ideals of S generated in the same degree: $L_{q,s} = \sum_{k+r=q} I_{k,s} J_{r,s}$, with $k, r \geq 1$, where

*Correspondence: monicalb@unime.it
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$I_{k,s}$ is the Veronese-type ideal generated on degree k by the set $\{x_1^{a_{i_1}} \cdots x_n^{a_{i_n}} \mid \sum_{j=1}^n a_{i_j} = k,\ \ 0 \le a_{i_j} \le s,\ \ s \in \{1,\dots,k\}\}$ and $J_{r,s}$ is the Veronese-type ideal generated on degree r by $\{y_1^{b_{i_1}} \cdots y_m^{b_{i_m}} \mid \sum_{j=1}^m b_{i_j} = r,\ \ 0 \le b_{i_j} \le s,\ \ s \in \{1,\dots,r\}\}$ [5], [7].

In [6] and [8], the symmetric algebra of these classes of monomial ideals was studied. More precisely, the authors investigated in which cases such ideals are generated by s-sequences. The notion of s-sequence has been employed to compute the standard invariants of the symmetric algebra. In this paper we are interested in studying the Rees algebra of these monomial ideals and we investigate in which cases they are of linear type, generalizing the results stated in [9].

The paper is organized in the following way. The first section contains notations and terminology. In the second section we study classes of monomial ideals generated by s-sequences of linear type. We investigate the ideals of mixed products and the ideals of Veronese bi-type, and as results we state a classification of these monomial ideals that are of linear type.

2. Preliminary notions

Let R be a Noetherian ring and let $I = (f_1, \dots, f_s)$ be an ideal of R.

The Rees algebra $\Re(I)$ of I is defined to be the R-graded algebra $\bigoplus_{i\ge 0} I^i$. It can be identified with the R-subalgebra of $R[t]$ generated by It, where t is an indeterminate on R. Let us consider the epimorphism of graded R-algebras:

$$\varphi : R[T_1, \dots, T_s] \to \Re(I) = R[f_1 t, \dots, f_s t],$$

defined by $\varphi(T_i) = f_i t$, $i = 1, \dots, s$.

The ideal $N = ker\varphi$ of $R[T_1, \dots, T_s]$ is R-homogeneous and we denote N_i the R-homogeneous component of degree i of N. The elements of N_1 are called linear relations. If $A = (a_{ij})$, $i = 1, \dots, r$, $j = 1, \dots, s$ is the relation matrix of I, then $g_i = \sum_{j=1}^s a_{ij} T_j$, $i = 1, \dots, r$, belongs to N and $R[T_1, \dots, T_s]/J$, with $J = (g_1, \dots, g_r)$, is isomorphic to the symmetric algebra $Sym_R(I)$ of I. The generators g_i of J are all linear in the variables T_j.

The natural map $\psi : Sym_R(I) \to \Re(I)$ is a surjective homomorphism of R-algebras. I is called of the *linear type* if ψ is an isomorphism, that is, $N = J$.

Several classes of ideals of R of linear type are known. For instance, ideals generated by d-sequences are of the linear type [4], [12].

Now, let K be a field, $R = K[x_1, \dots, x_n]$ be the polynomial ring, and $I \subset R$ be an equigenerated graded ideal that is a graded ideal whose generators $f_1, \dots, f_s$ are all of the same degree. Then the Rees algebra

$$\Re(I) = \oplus_{j\ge 0} I^i t^i = R[f_1 t, \dots, f_s t] \subset R[t]$$

is naturally bigraded with $deg(x_i) = (1, 0)$ for $i = 1, \dots, n$ and $deg(f_i t) = (0, 1)$ for $i = 1, \dots, s$.

Let $R[T_1, \dots, T_s]$ be the polynomial ring over R in the variables $T_1, \dots, T_s$. Then we define a bigrading by setting $deg(x_i) = (1, 0)$ for $i = 1, \dots, n$ and $deg(T_j) = (0, 1)$ for $j = 1, \dots, s$.

If $I = (f_1, \dots, f_s) \subset R$ is a monomial ideal, for all $1 \le i < j \le s$ we set

$$f_{ij} = \frac{f_i}{GCD(f_i, f_j)}$$

and

$$g_{ij} = f_{ij}T_j - f_{ji}T_i,$$

and then J is generated by $\{g_{ij}\}_{1\leq i<j\leq s}$ in $R[t_1,\ldots,T_s]$.

In this paper our aim is to investigate classes of monomial ideals for which the linear relations g_{ij} form a system of generators for $ker\varphi$ (this means that $J = ker\varphi$ and the ideals are of linear type).

In [2], Conca and De Negri introduced the monomial M-sequences and they proved that an M-sequence is always of the linear type. An M-sequence is an s-sequence, but an ideal generated by an s-sequence need not be of linear type [2], [3]. Now we study classes of monomial ideals generated by s-sequences of the linear type. More precisely, we investigate the following classes of monomial ideals:

1) the ideals of mixed products;

2) the ideals of Veronese bi-type.

Let $S = K[x_1,\ldots,x_n;y_1,\ldots,y_m]$ be the polynomial ring over a field K in 2 sets of variables with each $\deg(x_i) = 1$, $\deg(y_j) = 1$, for all $i = 1,\ldots,n$, $j = 1,\ldots,m$.

Given the nonnegative integers k,r,s,t such that $k+r = s+t$, in [10] the authors introduced the square-free monomial ideals of S:

$$L = I_kJ_r + I_sJ_t,$$

where I_k (resp. J_r) is the monomial ideal of S generated by all square-free monomials of degree k (resp. r) in the variables $x_1,\ldots,x_n$ (resp. $y_1,\ldots,y_m$).
These ideals are called *ideals of mixed products*. Setting $I_0 = J_0 = S$, we then consider the following cases:

1) $L = I_kJ_r$, with $1 \leq k \leq n$, $1 \leq r \leq m$

2) $L = I_kJ_r + I_{k+1}J_{r-1}$, with $1 \leq k \leq n$, $2 \leq r \leq m$

3) $L = J_r + I_sJ_t$, with $r = s+t$, $1 \leq s \leq n$, $1 \leq r \leq m$, $t \geq 1$.

Example 2.1 1) $S = K[x_1,x_2,x_3;y_1,y_2]$ $L = I_2J_1 = (x_1x_2y_1, x_1x_3y_1, x_2x_3y_1, x_1x_2y_2, x_1x_3y_2, x_2x_3y_2)$.

2) $S = K[x_1,x_2;y_1,y_2,y_3]$ $L = I_1J_2 + I_2J_1 = (x_1y_1y_2, x_1y_1y_3, x_1y_2y_3, x_2y_1y_2, x_2y_1y_3, x_2y_2y_3, x_1x_2y_1, x_1x_2y_2, x_1x_2y_3)$.

In [5] the *ideals of Veronese bi-type* of degree q are defined as the monomial ideals of S:

$$L_{q,s} = \sum_{r+k=q} I_{k,s}J_{r,s}, \quad r,k \geq 1,$$

where $I_{k,s}$ is the ideal of Veronese-type of degree k in the variables $x_1,\ldots,x_n$ and $J_{r,s}$ is the ideal of Veronese-type of degree r in the variables $y_1,\ldots,y_m$.

Remark 2.1 In general $I_{k,s} \subseteq I_k$, where I_k is the *Veronese ideal* of degree k generated by all the monomials in the variables $x_1,\ldots,x_n$ of degree k ([12]).

One has $I_{k,s} = I_k$ for any $k \leq s$. If $s = 1$, $I_{k,1}$ is the square-free Veronese ideal of degree k generated by all the square-free monomials in the variables $x_1,\ldots,x_n$ of degree k. Similar considerations hold for $J_{r,s} \subset K[y_1,\ldots,y_m]$.

Example 2.2 Let $S = K[x_1, x_2; y_1, y_2]$ be a polynomial ring:

1) $L_{2,2} = I_{1,2}J_{1,2} = I_1J_1 = (x_1y_1, x_1y_2, x_2y_1, x_2y_2)$;

2) $L_{4,2} = I_{3,2}J_{1,2} + I_{1,2}J_{3,2} + I_{2,2}J_{2,2} = I_{3,2}J_1 + I_1J_{3,2} + I_2J_2 = (x_1^2x_2y_1, x_1^2x_2y_2, x_1x_2^2y_1, x_1x_2^2y_2, x_1y_1^2y_2, x_2y_1^2y_2, x_1y_1y_2^2, x_2y_1y_2^2, x_1^2y_1^2, x_1^2y_1y_2, x_1^2y_2^2, x_2^2y_1^2, x_2^2y_2^2, x_2^2y_1y_2, x_1x_2y_1^2, x_1x_2y_2^2, x_1x_2y_1y_2)$.

3. Monomial ideals of linear type

In this section our aim is to investigate in which cases the ideals of mixed products and the ideals of Veronese bi-type are of linear type.

At first we consider the ideal I_k in $K[x_1, \dots, x_n]$ (resp. J_r in $K[y_1, \dots, y_m]$), that is, the square-free Veronese ideal of degree k (resp. r).

Theorem 3.1 *Let $I_k \subset R = K[x_1, \dots, x_n]$, $n > 1$. I_k is of linear type if and only if $k = n - 1$.*

Proof $\Rightarrow$ Let $I_k = (x_{i_1} \cdots x_{i_k} | 1 \le i_1 < \cdots < i_k \le n)$ and $f_1, \dots, f_q$ be its generators. We assume that I_k is of linear type, i.e. $N = (g_{ij} = f_{ij}T_j - f_{ji}T_i | 1 \le i < j \le q)$. This means that all the relations among the generators of I_k are linear relations (in the variables T_i). Supposing that the condition $f_{1j} = f_{2j} = \ldots = f_{n-1,j} = x_{n-j+1}$, for all $j = 2, \dots, n$, is not verified, then it is possible to compute not-linear relations among the generators of I_k of the type $T_iT_j - T_lT_s$ for some $i, j, l, s \in \{1, \dots, q\}$. It contradicts the assumption. Hence, one has $f_{1j} = f_{2j} = \ldots = f_{n-1,j} = x_{n-j+1}$ for all $j = 2, \dots, n$. It follows that the minimal set of generators of I_k that satisfies this condition is: $f_1 = x_1x_2 \cdots x_{n-1}$, $f_2 = x_1x_2 \cdots x_{n-2}x_n$, $f_3 = x_1x_2 \cdots x_{n-3}x_{n-1}x_n$, $\dots$, $f_{n-1} = x_1x_3 \cdots x_{n-1}x_n$, $f_n = x_2x_3 \cdots x_n$. Then $k = n - 1$.

$\Leftarrow$ Let $I_{n-1} = (f_1, \dots, f_n)$, where $f_1 = x_1 \cdots x_{n-1}$, $f_2 = x_1 \cdots x_{n-2}x_n$, $f_3 = x_1 \cdots x_{n-3}x_{n-1}x_n$, $\dots$, $f_{n-1} = x_1x_3 \cdots x_n$, $f_n = x_2 \cdots x_{n-1}x_n$. We prove that the linear relations $g_{ij} = f_{ij}T_j - f_{ji}T_i$ form a Gröbner basis of N with respect to a monomial order $\prec$ on the polynomial ring $R[T_1, \dots, T_n]$. Denote by F the ideal $(f_{ij}T_j : 1 \le i < j \le n)$. To show that the linear relations g_{ij} form a Gröbner basis of N we suppose that the claim is false. Since the binomial relations are known to be a Gröbner basis of N, there exists a binomial $\underline{x}^a\underline{T}^\alpha - \underline{x}^b\underline{T}^\beta$, where $\underline{x}^a = x_1^{a_1} \cdots x_n^{a_n}$, $\underline{x}^b = x_1^{b_1} \cdots x_n^{b_n}$, $\underline{T}^\alpha = T_1^{\alpha_1} \cdots T_n^{\alpha_n}$, $\underline{T}^\beta = T_1^{\beta_1} \cdots T_n^{\beta_n}$, and the initial monomial of $\underline{x}^a\underline{T}^\alpha - \underline{x}^b\underline{T}^\beta$ is not in F. More precisely, we assume that T^α, T^β have no common factors and that both $\underline{x}^a\underline{T}^\alpha$ and $\underline{x}^b\underline{T}^\beta$ are not in F.

Let i be the smallest index such that T_i appears in $\underline{T}^\alpha$ or in $\underline{T}^\beta$. Since $\underline{x}^a\underline{T}^\alpha - \underline{x}^b\underline{T}^\beta \in N$, then f_i divides $\underline{x}^b\varphi(\underline{T}^\beta)$, where $\varphi(T_i) = f_it$. If $f_i | \underline{x}^b$, then let T_j be any of the variables of $\underline{T}^\beta$. One has $f_{ij}T_j | f_iT_j | \underline{x}^b\underline{T}^\beta$ for $i < j$. This is a contradiction by assumption (because $\underline{x}^b\underline{T}^\beta \notin F$).

Hence, $f_i \nmid \underline{x}^b$. Let $x_{i_1} \prec \ldots \prec x_{i_{n-1}}$ be a total term order on the variables of f_i, and let $f_i = x_{i_1} \cdots x_{i_{n-1}}$. Let i_k be the minimum of the indices $i_1, \dots, i_{n-1}$ such that x_{i_k} does not divide $\underline{x}^b$. Then $x_{i_1}, \dots, x_{i_{k-1}}$ divide $\underline{x}^b$. Since x_{i_k} divides $\underline{x}^b\varphi(\underline{T}^\beta)$ (because $f_i | \underline{x}^b\varphi(\underline{T}^\beta)$), then there exists j such that T_j appears in $\underline{T}^\beta$ and $x_{i_k} | f_j$. By the structure of the generators $f_1, \dots, f_n$ of I_{n-1} if $x_{i_k} | f_i$ and $x_{i_k} | f_j$ with j such that T_j is in $\underline{T}^\beta$, then $f_{ij} | x_{i_t}$ with $i_t \in \{i_1, \dots, i_{k-1}\}$ (in fact, if a variable of the monomial f_{ij} is in the monomial f_h with $h \ne i$, then such a variable belongs to any other generator f_l for all $l > h$ and $l \ne j$).

Hence, $f_{ij} | \underline{x}^b$ and, as a consequence, $f_{ij}T_j | \underline{x}^b\underline{T}^\beta$, that is, a contradiction (because $\underline{x}^b\underline{T}^\beta \notin F$). It follows that $N = (g_{ij} : 1 \le i < j \le n) = J$, and hence I_{n-1} is of linear type. □

Remark 3.1 $k = n-1 \Leftrightarrow I_k$ is generated by an s-sequence [8]. Hence, I_k is generated by an s-sequence if an only if it is of linear type (by Theorem 3.1).

The following result states a classification of the ideal of mixed products $L = I_kJ_r + I_sJ_t$ of linear type. In the sequel we will suppose $L = (f_1, f_2, \ldots, f_q) \subset S = K[x_1, \ldots, x_n; y_1, \ldots, y_m]$, where $f_1 \prec f_2 \prec \cdots \prec f_q$ with respect to the monomial order $\prec_{Lex}$ on the variables $x_1, \ldots, x_n; y_1, \ldots, y_m$ and $x_1 \prec x_2 \prec \cdots \prec x_n \prec y_1 \prec y_2 \prec \cdots \prec y_m$.

Theorem 3.2 *Let $S = K[x_1, \ldots, x_n; y_1, \ldots, y_m]$, $n, m > 1$. The following conditions hold:*

1) *$L = I_kJ_r$ is of linear type if and only if $k = n-1$ and $r = m$ or $k = 1$ and $r = m$ (resp. $k = n$ and $r = m-1$ or $r = 1$).*

2) *$L = I_kJ_r + I_{k+1}J_{r-1}$ is of linear type if and only if $k = n-1$ and $r = m$.*

3) *$L = J_r + I_sJ_t$ is of linear type if and only if $r = m$, $s = n$, $t = 1$ and $m = n+1$.*

Proof $\Rightarrow$ Let L be an ideal of mixed products. Let $G(L)$ be the set of generators of L; then $|G(L)| > 1$. Let $f_1, \ldots, f_q$ be the generators of L. We assume L is of linear type, i.e.

$$N = (g_{ij} = f_{ij}T_j - f_{ji}T_i | 1 \leq i < j \leq q).$$

This means that all the relations among the generators of L are linear in the variables T_i.

1) Let $L = I_kJ_r \subset K[x_1, \ldots, x_n; y_1, \ldots, y_m]$; then $G(L)$ is

$$\{x_{i_1} \cdots x_{i_k} y_{j_1} \cdots y_{j_r} | 1 \leq i_1 < \ldots < i_k \leq n, 1 \leq j_1 < \ldots < j_r \leq m\}.$$

If supposing that none of these conditions,

i. $f_{ij} = x_{n-j+1}$ for all $j = 2, \ldots, n$, $i = 1, \ldots, n-1$,

ii. $f_{ij} = x_i$,

are verified, then it is possible to compute not-linear relations among the generators of L of the type $T_iT_j - T_lT_s$ for some $i, j, l, s \in \{1, \ldots, q\}$. It contradicts the assumption. Hence, one has $f_{ij} = x_{n-j+1}$ for all $j = 2, \ldots, n$, $i = 1, \ldots, n-1$ or $f_{ij} = x_i$. It follows that the minimal set of generators of L that satisfies these conditions is:

i. $f_1 = x_1x_2 \ldots x_{n-1}y, f_2 = x_1 \ldots x_{n-2}x_ny, f_3 = x_1 \ldots x_{n-3}x_{n-1}x_ny, \ \ldots, f_{n-1} = x_1x_3 \ldots x_ny, f_n = x_2x_3 \ldots x_ny$, where $y = y_1 \ldots y_r$. Then $k = n-1$ and $r = m$;
or

ii. $f_1 = x_1y, f_2 = x_2y, \ldots, f_n = x_ny$, where $y = y_1 \ldots y_r$. Then $k = 1$ and $r = m$.

In a similar way we prove the thesis if $k = n$ and $r = m-1$ or $r = 1$.

2) Let $L = I_kJ_r + I_{k+1}J_{r-1} \subset K[x_1,\ldots,x_n;y_1,\ldots,y_m]$; then $G(L)$ is

$$\{x_{i_1}\cdots x_{i_k}y_{j_1}\cdots y_{j_r}, x_{i_1}\cdots x_{i_{k+1}}y_{j_1}\cdots y_{j_{r-1}} | 1 \leq i_1 < \ldots < i_{k+1} \leq n,$$

$$1 \leq j_1 < \ldots < j_r \leq m\}.$$

If supposing that the conditions

$$f_{ij} = y_{m-j+1}, \quad i = 1,\ldots,m-1, \quad j = 2,\ldots,m$$

and

$$f_{ij} = x_{n+m-j+1}, \quad i = 1,\ldots,n+m-1, \quad j = m+1,\ldots,m+n.$$

are not verified, then it is possible to compute not-linear relations among the generators of L of the type $T_iT_j - T_lT_s$ for some $i,j,l,s \in \{1,\ldots,q\}$. It contradicts the assumption. The minimal set of generators of L that satisfies these conditions is: $f_1 = x_1\cdots x_ny_1\cdots y_{m-1}$, $f_2 = x_1\cdots x_ny_1\cdots y_{m-2}y_m$, $f_3 = x_1\cdots x_ny_1\cdots y_{m-3}y_{m-1}y_m$, $\ldots$, $f_{m-1} = x_1\cdots x_ny_1y_3\cdots y_m$, $f_m = x_1\cdots x_ny_2\cdots y_m$, $f_{m+1} = x_1\cdots x_{n-1}y_1\cdots y_m$, $f_{m+2} = x_1\cdots x_{n-2}x_ny_1\cdots y_m$, $f_{m+3} = x_1\cdots x_{n-3}x_{n-1}x_ny_1\cdots y_m$, $\ldots$, $f_{m+n-1} = x_1x_3\cdots x_ny_1\cdots y_m$, $f_{m+n} = x_2\cdots x_ny_1\cdots y_m$. It follows $L = I_nJ_{m-1} + I_{n-1}J_m$.

3) Let $L = J_r + I_sJ_t \subset K[x_1,\ldots,x_n;y_1,\ldots,y_m]$. Then

$$G(L) = \{y_{j_1}\cdots y_{j_r}, x_{i_1}\cdots x_{i_s}y_{j_1}\cdots y_{j_t} | 1 \leq i_1 < \ldots < i_s \leq n,$$

$$1 \leq j_1 < \ldots < j_t \leq m, 1 \leq j_1 < \ldots < j_r \leq m\}.$$

If supposing that the conditions

$$f_{ij} = y_i, \quad i = 1,\ldots,m-1 \quad j = i+1,\ldots,m,$$

$$f_{i,m+1} = x_1\ldots x_n, \quad i = 1,\ldots,m,$$

are not verified, then it is possible to compute not-linear relations among the generators of L of the type $T_iT_j - T_lT_s$ for some $i,j,l,s \in \{1,\ldots,q\}$. It contradicts the assumption. Hence, the minimal set of generators of L that satisfies these conditions is: $f_1 = x_1x_2\cdots x_ny_1$, $f_2 = x_1x_2\cdots x_ny_2$, $f_3 = x_1x_2\cdots x_ny_3$, $\ldots$, $f_m = x_1\cdots x_ny_m$, $f_{m+1} = y_1\cdots y_m$. Then $L = J_m + I_nJ_1$.

$\Leftarrow$ Let $L = (f_1, f_2, \ldots, f_q)$. We prove that the linear relations $g_{ij} = f_{ij}T_j - f_{ji}T_i$ form a Gröbner basis of N with respect to a monomial order $\prec$ on the polynomial ring $S[T_1,\ldots,T_n]$. Denote $F = (f_{ij}T_j : 1 \leq i < j \leq q)$. To show that g_{ij} form a Gröbner basis of N, we suppose that the claim is false. Since the binomial relations are known to be a Gröbner basis of N, there exists a binomial $a\underline{T}^\alpha - b\underline{T}^\beta$, where $a = x_1^{a_1}\cdots x_n^{a_n}y_1^{c_1}\cdots y_m^{c_m}$, $b = x_1^{b_1}\cdots x_n^{b_n}y_1^{d_1}\cdots y_m^{d_m}$, $\underline{T}^\alpha = T_1^{\alpha_1}\cdots T_q^{\alpha_q}$, $\underline{T}^\beta = T_1^{\beta_1}\cdots T_q^{\beta_q}$, and the initial monomial of $a\underline{T}^\alpha - b\underline{T}^\beta$ is not in F. More precisely, we assume that T^α, T^β have no common factors and that both $a\underline{T}^\alpha$ and $b\underline{T}^\beta$ are not in F.

Let i be the smallest index such that T_i appears in $\underline{T}^\alpha$ or in $\underline{T}^\beta$. Since $a\underline{T}^\alpha - b\underline{T}^\beta \in N$, then f_i divides $b\varphi(\underline{T}^\beta)$, where $\varphi(T_i) = f_it$. If $f_i|b$, then let T_j be any of the variables of $\underline{T}^\beta$. One has $f_{ij}T_j | f_iT_j | b\underline{T}^\beta$ for $i < j$. This is a contradiction by assumption (because $b\underline{T}^\beta \notin F$).

Hence, $f_i \nmid b$. Replace the set of variables $\{x_1, \ldots, x_n\}$ with $\{z_1, \ldots, z_n\}$ and $\{y_1, \ldots, y_m\}$ with $\{z_{n+1}, \ldots, z_{n+m}\}$ and let $z_1 \prec \ldots \prec z_{n+m}$ be a total term order on the variables of f_i. Let k be the minimum of the indices such that z_{i_k} does not divide b. Then $z_{i_1}, \ldots, z_{i_{k-1}}$ divide b. Since z_{i_k} divides $b\varphi(\underline{T}^\beta)$ (because $f_i | b\varphi(\underline{T}^\beta)$), then there exists j such that T_j appears in $\underline{T}^\beta$ and $z_{i_k} | f_j$.

One has the following cases:

1) If $L = I_{n-1}J_m$ or $L = I_1 J_m$, then, using the new variables z_i, $f_1 = z_1 z_2 \cdots z_{n-1} z_{n+1} \cdots z_{n+m}$, $f_2 = z_1 z_2 \cdots z_{n-2} z_n z_{n+1} \cdots z_{n+m}$, $f_3 = z_1 z_2 \cdots z_{n-3} z_{n-1} z_n z_{n+1} \cdots z_{n+m}$, $\ldots$, $f_{n-1} = z_1 z_3 \cdots z_n z_{n+1} \cdots z_{n+m}$, $f_n = z_2 z_3 \cdots z_n z_{n+1} \cdots z_{n+m}$ are the generators of $L = I_{n-1}J_m$ and $f_1 = z_1 z_{n+1} \cdots z_{n+m}$, $f_2 = z_2 z_{n+1} \cdots z_{n+m}$, $\ldots$, $f_n = z_n z_{n+1} \cdots z_{n+m}$ are the generators of $L = I_1 J_m$. By the structure of the generators of L if $z_{i_k} | f_i$ and $z_{i_k} | f_j$ with j such that T_j is in $\underline{T}^\beta$, then $f_{ij} | z_{i_t}$ with $i_t \in \{i_1, \ldots, i_{k-1}\}$ (in fact, if a variable of the monomial f_{ij} is in the monomial f_h with $h \neq i$, then such a variable belongs to any other generator f_l for all $l > h$ and $l \neq j$). Hence, $f_{ij} | b$ and, as a consequence, $f_{ij} T_j | b\underline{T}^\beta$, that is, a contradiction (because $b\underline{T}^\beta \notin F$). It follows that $N = (g_{ij} : 1 \leq i < j \leq n) = J$, and hence L is of the linear type.

In a similar way, the thesis follows if $k = n$ and $r = m-1$ or $r = 1$.

2) If $L = I_{n-1}J_m + I_n J_{m-1}$, the generators of L are: $f_1 = z_1 \cdots z_n z_{n+1} \cdots z_{n+m-1}$, $f_2 = z_1 \cdots z_n z_{n+1} \cdots z_{n+m-2} z_{n+m}$, $f_3 = z_1 \cdots z_n z_{n+1} z_{n+2} \cdots z_{n+m-3} z_{n+m-1} z_{n+m}$, $\ldots$, $f_{m-1} = z_1 \cdots z_n z_{n+1} z_{n+3} \cdots z_{n+m}$, $f_m = z_1 \cdots z_n z_{n+2} \cdots z_{n+m}$, $f_{m+1} = z_1 \cdots z_{n-1} z_{n+1} \cdots z_{n+m}$, $f_{m+2} = z_1 \cdots z_{n-2} z_n z_{n+1} \cdots z_{n+m}$, $f_{m+3} = z_1 \cdots z_{n-3} z_{n-1} z_n z_{n+1} \cdots z_{n+m}$, $\ldots$, $f_{m+n-1} = z_1 z_3 \cdots z_n z_{n+1} \cdots z_{n+m}$, $f_{m+n} = z_2 \cdots z_n z_{n+1} \cdots z_{n+m}$. By the structure of the generators of L if $z_{i_k} | f_i$ and $z_{i_k} | f_j$ with j such that T_j is in $\underline{T}^\beta$, then $f_{ij} | z_{i_t}$ with $i_t \in \{i_1, \ldots, i_{k-1}\}$ (in fact, if a variable of the monomial f_{ij} is in the monomial f_h with $h \neq i$, then such a variable belongs to any other generator f_l for all $l > h$ and $l \neq j$). Hence, $f_{ij} | b$ and, as a consequence, $f_{ij} T_j | b\underline{T}^\beta$, that is, a contradiction (because $b\underline{T}^\beta \notin F$). It follows that $N = (g_{ij} : 1 \leq i < j \leq n+m) = J$, and hence L is of linear type.

3) If $L = J_m + I_n J_1$ with $m = n+1$, then the generators of L are $f_1 = z_1 \cdots z_n z_{n+1}$, $f_2 = z_1 \cdots z_n z_{n+2}$, $f_3 = z_1 \cdots z_n z_{n+3}$, $\ldots$, $f_m = z_1 \cdots z_n z_{n+m}$, $f_{m+1} = z_{n+1} z_{n+2} \cdots z_{n+m}$. By the structure of the generators of L if $z_{i_k} | f_i$ and $z_{i_k} | f_j$ with j such that T_j is in $\underline{T}^\beta$, then $f_{ij} | z_{i_t}$ with $i_t \in \{1, \ldots, i_{k-1}\}$ (in fact, if a variable of the monomial f_{ij} is in the monomial f_h with $h \neq i$, then such a variable belongs to any other generator f_l for all $l > h$ and $l \neq j$). Hence, $f_{ij} | b$ and, as a consequence, $f_{ij} T_j | b\underline{T}^\beta$, that is, a contradiction (because $b\underline{T}^\beta \notin F$). It follows that $N = (g_{ij} : 1 \leq i < j \leq m+1) = J$, and hence L is of linear type. □

Remark 3.2 L is generated by an s-sequence if and only if it is of linear type [8].

The following result classifies the Veronese bi-type ideals of linear type.

Theorem 3.3 *Let $S = K[x_1, \ldots, x_n; y_1, \ldots, y_m]$ be the polynomial ring over a field K. $L_{q,s}$ is of linear type if and only if $q = s(n+m) - 1$.*

Proof $\Rightarrow$ Let $L_{q,s} = (f_1, f_2, \ldots, f_t)$ where $f_1 \prec f_2 \prec \cdots \prec f_t$ with respect to the monomial order $\prec_{Lex}$ on the variables of S. We assume that $L_{q,s}$ is of linear type, i.e. $N = (g_{ij} = f_{ij} T_j - f_{ji} T_i | 1 \leq i < j \leq t)$. This means that all the relations among the generators of $L_{q,s}$ are linear relations (in the variables T_i). Supposing that the conditions $f_{1j} = f_{2j} = \ldots = f_{m-1,j} = y_{m-j+1}$ for $j = 2, \ldots, m$, $f_{mj} = f_{m+1,j} = \ldots = f_{n+m-1,j} = x_{n+m-j+1}$, for $j = m+1, \ldots, m+n$, are not verified, then it is possible to compute not-linear relations among the generators of L of the type $T_iT_j - T_lT_s$ for some $i, j, l, s \in \{1, \ldots, t\}$. It contradicts the assumption. Hence, one has $f_{1j} = f_{2j} = \ldots = f_{m-1,j} = y_{m-j+1}$ for $j = 2, \ldots, m$, $f_{mj} = f_{m+1,j} = \ldots = f_{n+m-1,j} = x_{n+m-j+1}$ for $j = m+1, \ldots, m+n$. It follows that the minimal set of generators of L that satisfies these conditions is: $f_1 = x_1^s x_2^s \cdots x_{n-2}^s x_{n-1}^s x_n^s y_1^s y_2^s \cdots y_{m-1}^s y_m^{s-1}$, $f_2 = x_1^s x_2^s \cdots x_{n-2}^s x_{n-1}^s x_n^s y_1^s y_2^s \cdots y_{m-1}^{s-1} y_m^s$, $f_3 = x_1^s x_2^s \cdots x_{n-2}^s x_{n-1}^s x_n^s y_1^s y_2^s \cdots y_{m-2}^{s-1} y_{m-1}^s y_m^s$, $\ldots$, $f_{n+m-1} = x_1^s x_2^{s-1} \cdots x_{n-2}^s x_{n-1}^s x_n^s y_1^s y_2^s \cdots y_{m-1}^s y_m^s$, $f_{n+m} = x_1^{s-1} x_2^s \cdots x_{n-2}^s x_{n-1}^s x_n^s y_1^s y_2^s \cdots y_{m-1}^s y_m^s$.

Then $q = s(n+m) - 1$.

$\Leftarrow$ Let $q = s(n+m) - 1$. We prove that the linear relations $g_{ij} = f_{ij}T_j - f_{ji}T_i$ form a Gröbner basis of N with respect to a monomial order $\prec$ on the polynomial ring $S[T_1, \ldots, T_{n+m}]$. Denote $F = (f_{ij}T_j : 1 \leq i < j \leq n+m)$. To show that g_{ij} form a Gröbner basis of N, we suppose that the claim is false. Since the binomial relations are known to be a Gröbner basis of N, there exists a binomial $a\underline{T}^\alpha - b\underline{T}^\beta$, where $a = x_1^{a_1} \cdots x_n^{a_n} y_1^{c_1} \cdots y_m^{c_m}$, $b = x_1^{b_1} \cdots x_n^{b_n} y_1^{d_1} \cdots y_m^{d_m}$, $\underline{T}^\alpha = T_1^{\alpha_1} \cdots T_{n+m}^{\alpha_{n+m}}$, $\underline{T}^\beta = T_1^{\beta_1} \cdots T_{n+m}^{\beta_{n+m}}$, and the initial monomial of $a\underline{T}^\alpha - b\underline{T}^\beta$ is not in F. More precisely, we assume that T^α, T^β have no common factors and that both $a\underline{T}^\alpha$ and $b\underline{T}^\beta$ are not in F.

Let i be the smallest index such that T_i appears in $\underline{T}^\alpha$ or in $\underline{T}^\beta$. Since $a\underline{T}^\alpha - b\underline{T}^\beta \in N$, then f_i divides $b\varphi(\underline{T}^\beta)$, where $\varphi(T_i) = f_i t$. If $f_i | b$, then let T_j be any of the variables of $\underline{T}^\beta$. One has $f_{ij}T_j | f_iT_j | b\underline{T}^\beta$ for $i < j$. This is a contradiction by assumption (because $b\underline{T}^\beta \notin F$).

Hence, $f_i \nmid b$. Replace the set of variables $\{x_1, \ldots, x_n\}$ with $\{z_1, \ldots, z_n\}$ and $\{y_1, \ldots, y_m\}$ with $\{z_{n+1}, \ldots, z_{n+m}\}$ and let $z_1 \prec \ldots \prec z_{n+m}$ be a total term order on the variables of f_i. Let i_k be the minimum of the indices such that $z_{i_k}^{a_{i_k}}$ does not divide b, $a_{i_k} \in \{s, s-1\}$. Since $z_{i_k}^{a_{i_k}}$ divides $b\varphi(\underline{T}^\beta)$ (because $f_i | b\varphi(\underline{T}^\beta)$), then there exists j such that T_j appears in $\underline{T}^\beta$ and $z_{i_k} | f_j$.

By the structure of the generators $f_1, \ldots, f_{n+m}$ of $L_{q,s}$ if $z_{i_k} | f_i$ and $z_{i_k} | f_j$ with j such that T_j is in $\underline{T}^\beta$, then $f_{ij} | z_{i_1}^{a_{i_1}} \cdots z_{i_{k-1}}^{a_{i_{k-1}}}$, $a_{i_1}, \ldots, a_{i_{k-1}} \in \{s, s-1\}$ (in fact, if a variable of f_{ij} is in degree D in the monomial f_h, with $h \neq i, j$, then such variable in degree D belongs to any other generators f_l for all $l > h$ and $l \neq j$).

Hence, $f_{ij} | b$ and, as a consequence, $f_{ij}T_j | b\underline{T}^\beta$, that is, a contradiction (because $b\underline{T}^\beta \notin F$). It follows that $N = (g_{ij} : 1 \leq i < j \leq n+m) = J$, and hence $L_{q,s}$ is of linear type. □

Remark 3.3 $q = s(n+m) - 1 \Leftrightarrow L_{q,s}$ is generated by an s-sequence [6]. Hence, $L_{q,s}$ is generated by an s-sequence if and only if it is of linear type.

Example 3.1 $R = K[x_1, x_2; y_1, y_2]$.

$$L_{11,3} = (x_1^3x_2^3y_1^3y_2^2, x_1^3x_2^3y_1^2y_2^3, x_1^3x_2^2y_1^3y_2^3, x_1^2x_2^3y_1^3y_2^3) = (f_1, f_2, f_3, f_4)$$

$$\varphi : R[T_1, T_2, T_3, T_4] \to R[f_1t, f_2t, f_3t, f_4t]$$

$$T_i \to f_i t, \quad i = 1, \ldots, 4$$

$$Ker\varphi = N = (x_2T_3 - x_1T_4, y_1T_2 - x_1T_4, y_2T_1 - x_1T_4) = J.$$

$L_{11,3}$ is of linear type.

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