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Some notes on nil-semicommutative rings

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Abstract: A ring R is defined to be nil-semicommutative if $ab \in N(R)$ implies $arb \in N(R)$ for $a, b, r \in R$, where $N(R)$ stands for the set of nilpotents of R. Nil-semicommutative rings are generalization of NI rings. It is proved that (1) R is strongly regular if and only if R is von Neumann regular and nil-semicommutative; (2) Exchange nil-semicommutative rings are clean and have stable range 1; (3) If R is a nil-semicommutative right $MC2$ ring whose simple singular right modules are $YJ-$injective, then R is a reduced weakly regular ring; (4) Let R be a nil-semicommutative $\pi-$regular ring. Then R is an $(S,2)$-ring if and only if $\mathbb{Z}/2\mathbb{Z}$ is not a homomorphic image of R.

Key words: Nil-semicommutative rings, clean rings, von Neumann regular rings, $(S,2)-$rings

1. Introduction

All rings considered in this article are associative with identity, and all modules are unital. The symbols $J(R)$, $P(R)$, $N(R)$, $U(R)$, $E(R)$, $Max_r(R)$, $S_l(R)$, and $S_r(R)$ will stand respectively for the Jacobson radical, the prime radical, the set of all nilpotent elements, the set of all invertible elements, the set of all idempotent elements, the set of all maximal right ideals of R, the left socle of R, and the right socle of R. For any nonempty subset X of a ring R, $r(X) = r_R(X)$ and $l(X) = l_R(X)$ denote the right annihilators of X and the left annihilators of X, respectively.

Recall that a ring R is nil-semicommutative [3] if for any $a, b \in R$, $ab \in N(R)$ implies that $arb \in N(R)$ holds for each $r \in R$. A ring R is $2-primal$ if $N(R) = P(R)$, and R is said to be an $NI-$ring if $N(R)$ forms an ideal of R. A ring R is semicommutative if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. It is known that the condition semicommutativity implies $2-primal$, while $2-primal$ implies NI, and no reversal holds by [11]. By [3], NI rings are nil-semicommutative, but whether the converse holds is an open problem posed in [3]. Proposition 2.4 points out that a ring R is NI if and only if R is nil-semicommutative and $(N(R),+)$ is a subgroup of $(R,+)$. [3, Example 2.2] implies that nil-semicommutativity is a proper generalization of semicommutativity. In this paper, many properties of nil-semicommutative rings are introduced and many known results on semicommutative rings are extended.

An element a of R is called exchange if there exists $e \in E(R)$ such that $e \in aR$ and $1 - e \in (1 - a)R$, and a is said to be clean if a is a sum of a unit and an idempotent of R. It is known from [13, Proposition

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1.8] that clean elements are always exchange, and the converse holds when R is an Abelian ring. R is called an exchange ring if every element of R is an exchange element, and R is said to be clean if every element of R is clean. Clearly, every clean ring is exchange and the converse is not true unless R satisfies one of the following conditions: (1) R is an Abelian ring ([13]); (2) R is a left quasi-duo ring ([22]); (3) R is a quasi-normal ring ([18]); (4) R is a weakly normal ring ([20]). In this paper, we shall show that nil-semicommutative exchange rings are clean.

According to [15], a ring R is called to have stable range 1 if for any $a, b \in R$ satisfying $aR + bR = R$, there exists $y \in R$ such that $a + by \in U(R)$. [23, Theorem 6] showed that exchange rings with all idempotents central have stable range 1. In particular, it is proved that left quasi-duo exchange rings have stable range 1. [18, Theorem 4.8] showed that quasi-normal exchange rings have stable range 1. In this paper, we shall show that nil-semicommutative exchange rings have stable range 1.

2. Characterizations and properties

Obviously, a ring R is nil-semicommutative if and only if for any $n \geq 2$ and $a_1, a_2, \cdots, a_n \in R$, $a_1 a_2 \cdots a_n \in N(R)$ implies $a_1 r_1 a_2 r_2 \cdots a_{n-1} r_{n-1} a_n \in N(R)$ for any $r_1, r_2, \cdots, r_{n-1} \in R$. In particular, if $a \in N(R)$ then $aR, Ra \subseteq N(R)$. Hence, if R is a nil-semicommutative ring, then $N(R) \subseteq J(R)$. In fact, we have the following proposition.

Proposition 2.1 *The following conditions are equivalent for a ring R:*

(1) *R is a nil-semicommutative ring;*

(2) *$aR \subseteq N(R)$ for any $a \in N(R)$;*

(3) *$Ra \subseteq N(R)$ for any $a \in N(R)$.*

In each case, $N(R) \subseteq J(R)$.

Proof (1) $\Longrightarrow$ (2) and (1) $\Longrightarrow$ (3) are trivial.

(2) $\Longrightarrow$ (1) Assume that $ab \in N(R)$. Clearly, $ba \in N(R)$; so, by (2), $baR \subseteq N(R)$. Hence for any $r \in R$, $bar \in N(R)$; this leads to $arb \in N(R)$. Therefore, R is a nil-semicommutative ring.

Similarly, we can show (3) $\Longrightarrow$ (1). □

A ring R is called directly finite if for any $a, b \in R$, $ab = 1$ implies $ba = 1$. [3, Proposition 2.8] showed that nil-semicommutative rings are directly finite. By Proposition 2.1, we give another proof as follows.

Corollary 2.2 *Nil-semicommutative rings are directly finite. In particular, both NI rings and $2-primal$ rings are directly finite.*

Proof Let $a, b \in R$ and $ab = 1$. Set $e = ba$; then $ae = a$. Write $h = a - ea$. Then $he = h$, $eh = 0$, and $h^2 = 0$. Since R is a nil-semicommutative ring, $hb \in N(R)$ by Proposition 2.1, that is $1 - e = (a - ea)b = hb \in N(R)$. Thus $ba = e = 1$ and so R is directly finite. □

Recall that a ring R is NCI [6] if either $N(R) = 0$ or $N(R)$ contains a nonzero ideal of R. Clearly, NI rings are NCI. According to [6], NCI rings need not be directly finite. Hence, by Corollary 2.2, NCI rings need not be nil-semicommutative.

[6, Remark 2] pointed out that the subring of NCI rings need not be NCI, but Proposition 2.1 implies that the subrings of nil-semicommutative rings are nil-semicommutative.

Recall that a ring R is right quasi-duo if every maximal right ideal of R is an ideal of R. According to [9, Theorem 3.2], a ring R is right quasi-duo if and only if for any $a, b \in R$, $aR + (ba-1)R = R$. [9, Example 5.5] gave a reduced ring that is not right quasi-duo. Hence, nil-semicommutative rings need not be right quasi-duo by Corollary 2.2.

According to [10], a ring R is called weakly semicommutative if $ab = 0$ implies $arb \in N(R)$ for all $a, b \in R$. Clearly, nil-semicommutative rings are weakly semicommutative, but the converse is not true by [3, Example 2.2]. The following corollary is an immediate result of Proposition 2.1 for a nil-semicommutative ring. In fact, the reviewer points out that it also holds for weakly semicommutative rings. Hence we have

Corollary 2.3 *If R is a weakly semicommutative ring and $e \in E(R)$, then*

(1) $eR(1-e) \subseteq J(R)$.

(2) *If* $ReR = R$, *then* $e = 1$.

(3) *If* $M \in Max_r(R)$ *and* $e \in E(R)$, *then either* $e \in M$ *or* $1-e \in M$

Proof (1) and (2) are trivial.

(3) Let $M \in Max_r(R)$ and $e \in E(R)$. By (1), $(1-e)Re \subseteq M$. If $e \notin M$, then $eR + M = R$. Thus $R(1-e) = eR(1-e) + M(1-e) \subseteq M$, which implies $1-e \in M$. □

With the help of Proposition 2.1, we can give a characterization of NI rings.

Proposition 2.4 *A ring R is NI if and only if R is nil-semicommutative and $(N(R), +)$ is a subgroup of $(R, +)$.*

[7, Proposition 2] showed that semiprimitive right quasi-duo rings are reduced. By Proposition 2.1, we have:

Proposition 2.5 *Let R be a nil-semicommutative ring. Then*

(1) *For* $b \in N(R)$ *and* $a \in R$, $(ba-1)R = R$.

(2) eRe *is nil-semicommutative for each* $e \in E(R)$.

(3) *If R is a semiprimitive ring, then R is reduced.*

(4) *If* $x, z \in R$ *satisfy* $x + z \in N(R)xz$, *then* $Rx = Rz$.

Proof (1) Let $b \in N(R)$ and $a \in R$. If $(ba-1)R \neq R$, then there exists $M \in Max_r(R)$ containing $(ba-1)R$. Since R is a nil-semicommutative ring and $b \in N(R)$, by Proposition 2.1, $b \in J(R)$, this leads to $ba \in J(R) \subseteq M$. Since $ba - 1 \in M$, $1 \in M$, which is a contradiction. Thus $(ba-1)R = R$.

(2) and (3) are evident.

(4) Let $x + z = yxz$ for some $y \in N(R)$. Then $x = (yx-1)z$. Since R is a nil-semicommutative ring, $R = (yx-1)R$ by (1). Hence by Corollary 2.2, $yx - 1$ is invertible; this gives $Rx = R(yx-1)z = Rz$. □

Proposition 2.6 *A ring R is nil-semicommutative if and only if for any $a, b, c \in R$, $abc \in N(R)$ implies $ar_1cr_2b \in N(R)$ for any $r_1, r_2 \in R$.*

Proof Assume that R is nil-semicommutative and $abc \in N(R)$. Hence $acbac \in N(R)$. By Proposition 2.1, $acbacb \in N(R)$, that is $(acb)^2 \in N(R)$. Hence $acb \in N(R)$; this implies $ar_1cr_2b \in N(R)$ for any $r_1, r_2 \in R$.

Conversely, if $ab \in N(R)$, then $ab1 \in N(R)$, and so $arb = ar11b \in N(R)$ by hypothesis. Hence R is nil-semicommutative. □

According to [18], a ring R is called quasi-normal if for any $a \in R$ and $e \in E(R)$, $ae = 0$ implies $eaRe = 0$. Clearly, Abelian rings are quasi-normal. [18, Theorem 2.1] shows that a ring R is quasi-normal if and only if $eR(1-e)Re = 0$ for any $e \in E(R)$.

Let F be a field and $R = \begin{pmatrix} F & F & F \\ 0 & F & F \\ 0 & 0 & F \end{pmatrix}$. Consider the idempotent $e = e_{11} + e_{33}$; by computing, we can see that $eR(1-e)Re = \begin{pmatrix} 0 & 0 & F \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 0$, and so R is not quasi-normal by [18, Theorem 2.1]. Since $N(R) = \begin{pmatrix} 0 & F & F \\ 0 & 0 & F \\ 0 & 0 & 0 \end{pmatrix}$ is an ideal of R, R is NI; this implies R is nil-semicommutative. Hence there exists a nil-semicommutative ring that is not quasi-normal. Therefore nil-semicommutative rings need not be Abelian.

Let F be a field, $F < X, Y >$ the free algebra on X, Y over F and $S = F < X, Y > /(X^2)$, where (X^2) is the ideal of $F < X, Y >$ generated by (X^2). By [1, Example 4.8], S is an Armendariz ring and so S is an Abelian ring, but S is not nil-semicommutative by [3, Example 2.2]. Hence there exists a quasi-normal ring that is not nil-semicommutative.

Let R be a ring. Write $ME_r(R) = \{e \in E(R) \mid eR$ is a minimal right ideal of $R\}$. Similarly, we can define $ME_l(R)$. A ring R is called right min-abelian if every element of $ME_r(R)$ is right semicentral in R, a ring R is said to be strongly right min-abelian if every element of $ME_r(R)$ is left semicentral, and a ring R is said to be right $MC2$ if $ME_r(R) \subseteq ME_l(R)$. Abelian rings are strongly right min-abelian. [16, Theorem 1.8] showed that a ring R is strongly right min-abelian if and only if R is right min-abelian and right $MC2$. [16, Theorem 1.2] showed that a ring R is right quasi-duo if and only if R is right min-abelian and $MERT$. Now, we can show the following proposition.

Proposition 2.7 *Nil-semicommutative rings are right min-abelian.*

Proof Let $e \in ME_r(R)$ and $a \in R$. Write $h = ea - eae$. Then $eh = h$, $he = 0$, and $h \in N(R)$. If $h \neq 0$, Then $hR = eR$ because eR is minimal right ideal of R. By Proposition 2.1, $eR = hR \subseteq N(R)$, which is a contradiction. Hence $ea = eae$ for all $a \in R$, which implies that R is right min-abelian. □

Clearly, for any ring R, the polynomial ring $R[x]$ is always right min-abelian. However, [3, Theorem 2.6] gave a ring R such that the polynomial ring $R[x]$ is not nil-semicommutative. Hence the converse of Proposition 2.7 is not true, in general.

Let F be a field and $S = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$. Then S is a right quasi-duo ring and so S is right min-abelian. Consider the idempotent $e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$; by computing, we can see that $e \in ME_r(S)$ and e is not left semicentral. Hence S is not strongly right min-abel. Since $N(S) = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ is an ideal of S, S is a nil-semicommutative ring. Hence there exists a nil-semicommutative ring that is not strongly right min-abelian and so there exists a nil-semicommutative ring that is not right $MC2$ by Proposition 2.7.

Let R be an algebra over a commutative ring S. Recall that the *Dorroh* extension of R by S is the ring $R\times S$ with operations $(r_1;s_1)+(r_2;s_2)=(r_1+r_2;s_1+s_2)$ and $(r_1;s_1)(r_2;s_2)=(r_1r_2+s_1r_2+r_1s_2;s_1s_2)$; where $r_i\in R$ and $s_i\in S$.

Theorem 2.8 *Let R be an algebra over a commutative reduced ring S, and D be the Dorroh extension of R by S. If R is nil-semicommutative, then D is also nil-semicommutative.*

Proof Let $x=(a;s)\in N(D)$. Assume that $n\geq 1$ such that $x^n=0$; then $s^n=0$. Since S is reduced, $s=0$. Hence $a\in N(R)$ because $x^n=(a^n;0)$. Since R is nil-semicommutative, $aR\subseteq N(R)$. For any $y=(b;t)\in D$, $xy=(ab+at;0)$. Since $at+ab=a(1t+b)\in aR\subseteq N(R)$, $xy\in N(D)$. Hence $xD\subseteq N(D)$; by Proposition 2.1, D is nil-semicommutative. □

Proposition 2.9 *Let R be a quasi-normal ring and $e\in E(R)$. If eRe and $(1-e)R(1-e)$ are nil-semicommutative rings and $(N(R),+)$ is a subgroup of $(R,+)$, then R is a NI ring.*

Proof Let $ab\in N(R)$. Then there exists $n\geq 1$ such that $(ab)^n=0$. Since R is quasi-normal, $eabe=eaebe$ and $(eabe)^n=e(ab)^ne=0$ by [18, Corollary 2.2]. Hence $(eae)(ebe)\in N(eRe)$. Since eRe is nil-semicommutative, $(eae)(ere)(ebe)\in N(eRe)$ for each $r\in R$. Thus, by [18, Corollary 2.2], $earbe\in N(eRe)\subseteq N(R)$ for each $r\in R$. Similarly, $(1-e)arb(1-e)\in N(R)$ because $(1-e)R(1-e)$ is nil-semicommutative. Since $earb(1-e),(1-e)arbe\in N(R)$ and $(N(R),+)$ is a subgroup of $(R,+)$, $earbe+earb(1-e)+(1-e)arbe+(1-e)arb(1-e)\in N(R)$, that is, $arb\in N(R)$. Hence R is nil-semicommutative; by Proposition 2.4, R is NI. □

Proposition 2.10 *If R is a subdirect product of a finite family of nil-semicommutative rings $\{R_i|1\leq i\leq n\}$, then R is nil-semicommutative.*

Proof Let $\{I_i|i=1,2,\cdots,n\}$ be ideals of R such that $\cap_{i=1}^n I_i=0$ and each R/I_i is nil-semicommutative. Assume that $ab\in N(R)$ and $r\in R$. Then, for each i, in $\bar{R}=R/I_i$, $\bar{a}\bar{b}\in N(\bar{R})$. Since $\bar{R}$ is nil-semicommutative, $\bar{a}\bar{r}\bar{b}\in N(\bar{R})$, that is, there exists $n_i\geq 1$ such that $(arb)^{n_i}\in I_i$. Set $m=max\{n_1,n_2,\cdots,n_n\}$. Then $(arb)^m\in\cap_{i=1}^n I_i=0$. Hence $arb\in N(R)$; this implies R is nil-semicommutative. □

Theorem 2.11 *Let R and S be rings and ${}_RW_S$ be a $(R,S)-$bimodule. Let $E=T(R,S,W)=\begin{pmatrix} R & W\\ 0 & S\end{pmatrix}$. Then E is nil-semicommutative if and only if R and S are nil-semicommutative.*

Proof $(\Longrightarrow)$ Take $e_1=\begin{pmatrix}1&0\\0&0\end{pmatrix}$ and $e_2=\begin{pmatrix}0&0\\0&1\end{pmatrix}$. Then clearly e_1 and e_2 are idempotents of E. Since $e_1Ee_1\cong R$ and $e_2Ee_2\cong S$, by Proposition 2.5(2), R and S are nil-semicommutative.

$(\Longleftarrow)$ Let $A=\begin{pmatrix}x&m\\0&s\end{pmatrix}, B=\begin{pmatrix}y&n\\0&t\end{pmatrix}, C=\begin{pmatrix}z&w\\0&l\end{pmatrix}\in E$ and $AB\in N(E)=\begin{pmatrix}N(R)&W\\0&N(S)\end{pmatrix}$. Then $xy\in N(R)$ and $st\in N(S)$. Since R and S are nil-semicommutative, $xzy\in N(R)$ and $slt\in N(S)$. Therefore $ACB=\begin{pmatrix}xzy & xzn+xwt+mlt\\ 0 & slt\end{pmatrix}\in\begin{pmatrix}N(R)&W\\0&N(S)\end{pmatrix}=N(E)$ and so E is nil-semicommutative. □

Let R be a ring and write $L_3(R) = \{\begin{pmatrix} 0 & a_1 & a_2 \\ 0 & a_3 & a_4 \\ 0 & 0 & 0 \end{pmatrix} | a_1, a_2, a_3, a_4 \in R\}$. Then $L_3(R)$ is a ring and $N(L_3(R)) = \{\begin{pmatrix} 0 & a_1 & a_2 \\ 0 & a_3 & a_4 \\ 0 & 0 & 0 \end{pmatrix} | a_1, a_2, a_4 \in R \text{ and } a_3 \in N(R)\}$. By Theorem 2.11, we have the following corollary.

Corollary 2.12 *(1) The following conditions are equivalent for a ring R:*

(a) R is nil-semicommutative;

(b) The $n \times n$ upper triangular matrices rings $T_n(R)$ are nil-semicommutative for any $n \geq 2$;

(c) $L_3(R)$ is nil-semicommutative.

(2) Let R be a ring and $e \in E(R)$ be left semicentral in R. If eRe and $(1-e)R(1-e)$ are nil-semicommutative, then R is nil-semicommutative.

Let R be a ring and W a bimodule over R. Write $T(R, W) = \{\begin{pmatrix} c & x \\ 0 & c \end{pmatrix} | c \in R, x \in W\}$. Then $T(R, W)$ is a subring of $T(R, R, W)$. Let $R \propto W = \{(a, m) | a \in R, m \in W\}$ with the addition componentwise and multiplication defined by $(a_1, m_1)(a_2, m_2) = (a_1a_2, a_1m_2 + m_1a_2)$. Then $R \propto W$ is a ring that is called the trivial extension of R by W. Clearly, $R \propto W$ is isomorphic to the ring $T(R, W)$ and $T(R, R)$ is also isomorphic to the ring $R[x]/(x^2)$. Hence by Theorem 2.11, we have the following corollary that appeared partly in Proposition 2.5 of [3].

Corollary 2.13 *Let W be a $(R, R)-$bimodule. Then the following conditions are equivalent:*

(1) *R is nil-semicommutative;*

(2) *$R \propto W$ is nil-semicommutative;*

(3) *$T(R, W)$ is nil-semicommutative;*

(4) *$R \propto R$ is nil-semicommutative;*

(5) *$T(R, R)$ is nil-semicommutative;*

(6) *$R[x]/(x^2)$ is nil-semicommutative.*

Let R be a ring and W a bimodule over R. Let

$$R \bowtie W = \{(a, m, b, n) | a, b \in R, m, n \in W\}$$

with the addition componentwise and multiplication defined by

$$(a_1, m_1, b_1, n_1)(a_2, m_2, b_2, n_2) = (a_1a_2, a_1m_2 + m_1a_2, a_1b_2 + b_1a_2, a_1n_2 + m_1b_2 + b_1m_2 + n_1a_2)$$

Then $R \bowtie W$ is a ring that is isomorphic to the ring $(R \propto W) \propto (R \propto W)$. Let

$$BT(R, W) = \{\begin{pmatrix} a & m & b & n \\ 0 & a & 0 & b \\ 0 & 0 & a & m \\ 0 & 0 & 0 & a \end{pmatrix} | a, b \in R, m, n \in M\}$$

Then, as rings, $BT(R,W)\cong T(T(R,W),T(R,W))$. Moreover, we have the following isomorphism as rings:

$$R[x,y]/(x^2,y^2) \longrightarrow BT(R,R)$$

$$a+bx+cy+dxy \longmapsto \begin{pmatrix} a & b & c & d \\ 0 & a & 0 & c \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{pmatrix}$$

Corollary 2.14 *Let R be a ring and W a bimodule over R. Then the following conditions are equivalent:*

(1) *R is a nil-semicommutative ring;*

(2) *$R\bowtie W$ is a nil-semicommutative ring;*

(3) *$BT(R,R)$ is a nil-semicommutative ring;*

(4) *$BT(R,W)$ is a nil-semicommutative ring;*

(5) *$R[x,y]/(x^2,y^2)$ is a nil-semicommutative ring;*

(6) *$R\bowtie R$ is a nil-semicommutative ring.*

Let R be a ring and write $GT_3(R)=\{\begin{pmatrix} a_1 & 0 & a_3 \\ 0 & a_2 & 0 \\ 0 & 0 & a_4 \end{pmatrix} | a_1,a_2,a_3,a_4\in R\}$. Then $GT_3(R)$ is a subring of $T_3(R)$.

Let $CT_9(R)=\{\begin{pmatrix} a_{11} & 0 & a_{13} & 0 & 0 & 0 & a_{14} & 0 & a_{15} \\ 0 & a_{21} & 0 & 0 & 0 & 0 & 0 & a_{22} & 0 \\ 0 & 0 & a_{31} & 0 & 0 & 0 & 0 & 0 & a_{32} \\ 0 & 0 & 0 & a_{41} & 0 & a_{42} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{51} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{61} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{71} & 0 & a_{72} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{81} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{91} \end{pmatrix} | a_{ij}\in R\}$. Then $CT_9(R)$ is a ring and $CT_9(R)\cong GT_3(GT_3(R))$.

Corollary 2.15 *The following conditions are equivalent for a ring R:*

(1) *R is a nil-semicommutative ring;*

(2) *$GT_3(R)$ is a nil-semicommutative ring;*

(3) *$CT_9(R)$ is a nil-semicommutative ring.*

3. Regularity of nil-semicommutative rings

Let R be a ring and $a\in R$. Then a is called $\pi-$regular, if there exists $n\geq 1$ and $b\in R$ such that $a^n=a^nba^n$; in the case of $n=1$, a is called von Neumann regular, and a is said to be strongly $\pi-$regular, if $a^n=a^{n+1}b$, and in case of $n=1$, a is called strongly regular. A ring R is called von Neumann regular, strongly regular, $\pi-$regular and strongly $\pi-$regular, if every element of R is von Neumann regular, strongly regular, $\pi-$regular, and strongly $\pi-$regular, respectively. According to [17], a ring R is called $n-$regular if every element of $N(R)$ is von Neumann regular.

Proposition 3.1 *Let R be a nil-semicommutative ring and $x \in R$. Then:*

(1) *If x is von Neumann regular, then x is strongly regular.*

(2) *If x is $\pi-$regular, then there exists $e \in E(R)$ such that ex is strongly regular and $(1-e)x \in N(R)$.*

Proof (1) Let $x = xyx$ for some $y \in R$ and put $e = xy$; so $xR = eR$. As $(1-e)x = 0$, the nil-semicommutative hypothesis implies that the element $x(1-e)ye = e - xeye$ is nilpotent, and since e minus any power of $e - xeye$ lies in $xeR = x^2R$, we obtain $xR = eR = x^2R$. Hence x is strongly regular.

(2) By hypothesis, there exists a positive integer n such that x^n is regular. By (1), x^n is strongly regular. By [12], $x^n = x^n u x^n$ and $x^n u = u x^n$ for some $u \in U(R)$. Let $e = x^n u$. Then $e \in E(R)$, $x^n = ex^n = x^n e$, and $x^n = ev$, where $v = u^{-1}$. Since $(ex)(x^{n-1}u)(ex) = ex^n uex = evuex = ex$, ex is von Neumann regular. By (1), ex is strongly regular. Since $ex = ux^n x = (ux)x^n = (ux)x^n e$ and $xe = xx^n u = x^n(xu) = ex^n(xu)$, $ex = exe = xe$. Hence $((1-e)x)^n(1-e) = (1-e)x^n(1-e) = 0$; this gives $(1-e)x \in N(R)$. □

A ring R is called right universally mininjective if every minimal right ideal of R is a direct summand, and R is said to be strongly right DS if for any minimal right ideal I of R, $N(R) \cap I = 0$. [19, Theorem 3.2] showed that a ring R is strongly right DS if and only if R is right universally mininjective and right min-abelian. By Proposition 3.1, we have the following corollary.

Corollary 3.2 *Let R be a ring. Then*

(1) *R is strongly regular if and only if R is von Neumann regular and nil-semicommutative.*

(2) *R is reduced if and only if R is $n-$regular and nil-semicommutative.*

(3) *If R is nil-semicommutative, then R is $\pi-$regular if and only if R is strongly $\pi-$regular.*

(4) *If R is nil-semicommutative, then R is strongly right DS if and only if R is right universally mininjective.*

(5) *If R is nil-semicommutative, then R is strongly right min-abelian if and only if R is right $MC2$.*

Right $R-$module M is called $Wnil-$injective [17] if for any $0 \neq a \in N(R)$ there exists $n \geq 1$ such that $a^n \neq 0$ and any right $R-$homomorphism $a^nR \longrightarrow M$ can be extended $R \longrightarrow M$. Clearly, $YJ-$injective modules are $Wnil-$injective, since semicommutative rings are nil-semicommutative and right $MC2$. Hence the following proposition generalizes Lemma 3 of [8].

Proposition 3.3 *A ring R is reduced if and only if R is right $MC2$, nil-semicommutative, and every simple singular right $R-$module is $Wnil-$injective.*

Proof The necessity is clear.

Now let $a^2 = 0$. If $a \neq 0$, then there exists $M \in Max_r(R)$ such that $r(a) \subseteq M$. We claim that M is essential in R_R. If not, then $M = eR$ for some $0 \neq e \in E(R)$. Clearly, $1 - e \in ME_r(R)$. Since R is a right $MC2$ ring and nil-semicommutative ring, by Corollary 3.2(5), $1-e$ is central. Hence $a \in r(a) \subseteq eR$ implies $a(1-e) = (1-e)a = 0$, and so $1 - e \in r(a) \subseteq eR$, which is a contradiction. Therefore M is essential in R_R; by hypothesis, R/M is $Wnil-$injective. Then the well-defined right $R-$homomorphism

$aR \longrightarrow R/M$

$ar \longmapsto r + M$

can be extended $R \longrightarrow M$, that is, there exists $c \in R$ such that $1 - ca \in M$. Since $a \in N(R)$, by Proposition 2.1, $ca \in N(R)$. Hence $1 - ca \in U(R)$. This is impossible. Thus $a = 0$ and so R is reduced. □

Recall that a ring R is right idempotent reflexive if $eRa = 0$ implies $aRe = 0$ for $e \in E(R)$ and $a \in R$. Clearly, semiprime rings are right idempotent reflexive and right idempotent reflexive rings are right $MC2$. By Proposition 3.3, we have the following corollary.

Corollary 3.4 *The following conditions are equivalent for a ring R:*

(1) *R is reduced;*

(2) *R is semiprime, nil-semicommutative and every simple singular right $R-$module is $Wnil-$injective;*

(3) *R is right idempotent reflexive, nil-semicommutative, and every simple singular right $R-$module is $Wnil-$injective.*

Clearly, reduced $\Longrightarrow$strongly right DS $\Longrightarrow$ right universally mininjective. By [14], right universally mininjective $\Longrightarrow$ right mininjective $\Longrightarrow$ $S_r(R) \subseteq S_l(R)$. Hence, by Proposition 3.3, we have the following corollary.

Corollary 3.5 *The following conditions are equivalent for a ring R:*

(1) *R is reduced;*

(2) *R is strongly right DS, nil-semicommutative and every simple singular right $R-$module is $Wnil-$injective;*

(3) *R is right universally mininjective, nil-semicommutative, and every simple singular right $R-$module is $Wnil-$injective;*

(4) *R is right mininjective, nil-semicommutative, and every simple singular right $R-$module is $Wnil-$injective;*

(5) *R is nil-semicommutative, $S_r(R) \subseteq S_l(R)$, and every simple singular right $R-$module is $Wnil-$injective.*

Proof It is only to show $(5) \Longrightarrow (1)$. Let $e \in ME_r(R)$. For any $0 \neq a \in R$, if $ae \neq 0$, then we claim that $(aeR)^2 \neq 0$. If not, then $RaeR \subseteq r(ae)$. Let I be a complement left ideal of $RaeR$ in R. Then $RaeR \oplus I$ is an essential left ideal of R and so $S_r(R) \subseteq S_l(R) \subseteq RaeR \oplus I$. Clearly, $aeI \subseteq I \cap RaeR = 0$ and so $I \subseteq r(ae)$; this gives $S_r(R) \subseteq r(ae)$. Since $r(e) = r(ae)$, $e \in S_r(R) \subseteq r(e)$, which is a contradiction. Hence $(aeR)^2 \neq 0$; this leads to $aeR = gR$ for some $g \in ME_r(R)$. Let $g = aec$ for some $c \in R$. Then $ae = gae = aecae$. Let $h = cae$. Then $h^2 = h$ and $Rae = Rh$. Thus $Rae = lr(h) = lr(ae) = lr(e) = Re$; this shows that Re is a minimal left ideal of R, $e \in ME_l(R)$. Thus R is a right $MC2$ ring. By Proposition 3.3, R is reduced. □

The following corollary generalizes [8, Theorem 4].

Corollary 3.6 *Let R be a right $MC2$ ring and nil-semicommutative ring whose simple singular right $R-$modules are $YJ-$ injective. Then R is a reduced weakly regular ring.*

Proof By Proposition 3.3, R is reduced and so R is semicommutative. Hence, by Theorem 4 of [8], we are done. □

It is well known that a ring R is a strongly regular ring if and only if R is a right quasi-duo ring and a weakly regular ring. Hence, Corollary 3.6 and [16, Theorem 1.2] imply the following corollary.

Corollary 3.7 *A ring R is a strongly regular ring if and only if R is a $MERT$ ring, a right $MC2$ ring, and a nil-semicommutative ring whose simple singular right $R-$modules are $YJ-$injective.*

4. Exchange rings and clean rings

Proposition 4.1 *Let R be a weakly semicommutative ring and $x \in R$. If x is exchange, then x is clean.*

Proof Since x is an exchange element of R, there exists $e \in E(R)$ such that $e = xy$ and $1-e = (1-x)z$ for some $y = ye, z = z(1-e) \in R$. Then $(x-(1-e))(y-z) = 1-ez-(1-e)y$. Since $(ez)^2 = ezez = ez(1-e)ez = 0$, $1-ez-(1-e)y = (1-(1-e)y(1+ez))(1-ez)$. Since R is a weakly semicommutative ring and $y(1-e) = 0$, $y(1+ez)(1-e) \in N(R)$. Hence $(1-e)y(1+ez) \in N(R)$, and so $1-(1-e)y(1+ez) \in U(R)$; this implies $(x-(1-e))(y-z) \in U(R)$. By Corollary 2.2, $x-(1-e) \in U(R)$; hence x is clean. □

Recall that a ring R is $VNL-$ring if for each $a \in R$, either a or $1-a$ is von Neumann regular. It is well known that $VNL-$rings are exchange. In terms of Proposition 4.1, we can obtain the following corollary.

Corollary 4.2 (1) *Let R be a weakly semicommutative ring. Then R is exchange if and only if R is clean.*

(2) *Let R be a nil-semicommutative ring. Then R is exchange if and only if R is clean.*

(3) *Let R be a NI ring. Then R is exchange if and only if R is clean.*

(4) *Nil-semicommutative $VNL-$rings are clean.*

(5) *Weakly semicommutative $VNL-$rings are clean.*

Proposition 4.3 *Let R be a nil-semicommutative ring and idempotent can be lifted modulo $J(R)$. Then*

(1) If $a \in R$ is clean, then ae is clean for any $e \in E(R)$.

(2) If both a and $-a$ are clean, then $a+e$ is clean for any $e \in E(R)$.

Proof (1) Let $a = u+f$, where $u \in U(R)$ and $f \in E(R)$. Since $eR(1-e), (1-e)Re \subseteq N(R)$, $eR(1-e), (1-e)Re \subseteq J(R)$ by Corollary 2.3, $\bar{e}$ is contained in central of $\bar{R} = R/J(R)$. Hence $\bar{a}\bar{e} = \bar{u}\bar{e} + \bar{f}\bar{e}$, where $\bar{u}\bar{e} \in U(\bar{e}\bar{R}\bar{e})$ and $\bar{f}\bar{e} \in E(\bar{e}\bar{R}\bar{e})$. Since idempotent can be lifted modulo $J(R)$, there exists $g \in E(R)$ such that $g - fe \in J(R)$; this gives $\bar{a}\bar{e} = \bar{u}\bar{e} + \bar{g}$. Let $ae = ue + g + y$ for some $y \in J(R)$. Since $(\bar{u}\bar{e} - (\bar{1}-\bar{e}))(\bar{e}\bar{u}^{-1} - (\bar{1}-\bar{e})) = \bar{u}\bar{e}\bar{u}^{-1} + \bar{1} - \bar{e} = \bar{e} + \bar{1} - \bar{e} = \bar{1}$, $(ue-(1-e))(eu^{-1}-(1-e)) = 1+z$ for some $z \in J(R)$, this gives $(ue-(1-e))(eu^{-1}-(1-e))(1+z)^{-1} = 1$. By Corollary 2.2, $v = ue-(1-e) \in U(R)$. Clearly, $ae = v+g+(1-e)+y$. Since $(\bar{g}+\bar{1}-\bar{e})^2 = \bar{g}+\bar{1}-\bar{e}$, there exists $h \in E(R)$ such that $\bar{h} = \bar{g}+\bar{1}-\bar{e}$. Let $h = g+1-e+t$ for some $t \in J(R)$. Then $ae = v+h+(y-t)$ and $y-t \in J(R)$, so $ae = v(1+v^{-1}(y-t))+h$, where $v(1+v^{-1}(y-t)) \in U(R)$ and $h \in E(R)$. Hence ae is clean.

(2) Since $-a$ is clean, $1+a$ is clean. Let $a = u+f$ and $1+a = v+g$, where $u, v \in U(R)$ and $f, g \in E(R)$. Then $a+e = (1+a)e + a(1-e) = (ve+u(1-e)) + (ge+f(1-e))$. In $\bar{R} = R/J(R)$, $\bar{g}\bar{e} + \bar{f}(\bar{1}-\bar{e}) \in E(\bar{R})$, and so there exists $h \in E(R)$ such that $ge+f(1-e) = h+y$ for some $y \in J(R)$. Since $(\bar{v}\bar{e} + \bar{u}(\bar{1}-\bar{e}) + \bar{y})(\bar{e}\bar{v}^{-1} + (\bar{1}-\bar{e})\bar{u}^{-1}) = \bar{1}$, $(ve+u(1-e)+y)(ev^{-1}+(1-e)u^{-1}) = 1+x$ for some $x \in J(R)$. This implies $ve+u(1-e)+y \in U(R)$. Clearly, $a+e = (ve+u(1-e)+y)+h$ and so $a+e$ is clean. □

Proposition 4.4 *(1) Let R be a weakly semicommutative ring and $x \in R$. If x^n is clean for some $n \geq 2$, then x is clean.*

(2) Let R be a nil-semicommutative ring and idempotent can be lifted modulo $J(R)$. If a^2 is clean, then $a+e$ is clean for any $e \in E(R)$.

Proof (1) Let $x^n = u + f$ for some $u \in U(R)$ and $f \in E(R)$. Write $e = u(1-f)u^{-1}$. Then $e \in E(R)$ and $(x^n - e)u = (u+f)u - u(1-f) = x^{2n} - x^n$; this leads to $e = x^n + (x^n - x^{2n})u^{-1} \in xR$ and $1 - e = 1 - x^n - (1-x^n)x^n u^{-1} \in (1-x)R$, and so x is an exchange element. By Proposition 4.1, x is clean.

(2) It is an immediate result of Proposition 4.3 and (1). □

In [4], Ehrlich showed that if R is a unit regular ring, then every element in R is a sum of 2 units. A ring R is called an $(S,2)$-ring (cf. [5]) if every element in R is a sum of 2 units of R. In [2, Theorem 6] it is shown that if R is an Abelian $\pi-$regular ring, then R is an $(S,2)$-ring if and only if $\mathbb{Z}/2\mathbb{Z}$ is not a homomorphic image of R.

Theorem 4.5 *Let R be a nil-semicommutative $\pi-$regular ring. Then R is an $(S,2)$-ring if and only if $\mathbb{Z}/2\mathbb{Z}$ is not a homomorphic image of R.*

Proof Since R is a nil-semicommutative $\pi-$regular ring, $R/J(R)$ is a strongly $\pi-$regular ring by Corollary 3.2(3); this implies $R/J(R)$ is reduced and so $R/J(R)$ is a strongly regular ring. By [2, Theorem 6], $R/J(R)$ is an $(S,2)$-ring if and only if $\mathbb{Z}/2\mathbb{Z}$ is not a homomorphic image of $R/J(R)$. By [18, Lemma 4.3], we are done. □

5. Stable range one

It is well known that (1) a ring R has stable range 1 if and only if $R/J(R)$ has stable range 1; (2) An exchange ring R has stable range 1 if and only if every von Neumann regular element of R is unit-regular; (3) $VNL-$rings are exchange; (4) $\pi-$regular rings and clean rings are exchange. Hence by Proposition 3.1(1), we have the following proposition.

Proposition 5.1 (1) *Nil-semicommutative exchange rings have stable range 1.*

(2) *NI exchange rings have stable range 1.*

(3) *Nil-semicommutative $VNL-$rings have stable range 1.*

(4) *Nil-semicommutative clean rings have stable range 1.*

(5) *Nil-semicommutative $\pi-$regular rings have stable range 1.*

In [21], a ring R is said to satisfy the unit one-stable condition if for any $a, b, c \in R$ with $ab+c=1$, there exists $u \in U(R)$ such that $au + c \in U(R)$. It is easy to prove that R satisfies the unit one-stable condition if and only if $R/J(R)$ satisfies the unit one-stable condition. [18, Proposition 4.10] showed that for a quasi-normal exchange ring R, R is an $(S,2)$-ring if and only if R satisfies the unit 1-stable condition.

Proposition 5.2 *Let R be a nil-semicommutative exchange ring. Then the following conditions are equivalent:*

(1) *R is an $(S,2)-$ring;*

(2) *R satisfies the unit one-stable condition;*

(3) *Every factor ring R_1 of R is an $(S,2)-$ring;*

(4) *$\mathbb{Z}_2$ is not a homomorphic image of R.*

Proof It is trivial. □

It is well known that an exchange ring R has stable range 1 if and only if for any $a, x \in R$ and $e \in E(R)$, $ax + e = 1$ implies $a + ey \in U(R)$ for some $y \in R$.

Proposition 5.3 *An exchange ring R has stable range 1 if and only if for every von Neumann regular element a of R, there exists $u \in U(R)$ such that $a - aua \in J(R)$.*

Proof The necessity is clear.

Now assume $ax + e = 1$, where $a, x \in R$ and $e \in E(R)$. Then $a = axa + ea$. If $ea = 0$, then $a = axa$. By hypothesis, there exists $u \in U(R)$ such that $a - aua \in J(R)$. Let $a = aua + z$ for some $z \in J(R)$. Then $1 - e = ax = auax + zx = au(1-e) + zx$ and $(au - e)^2 = auau - aue - eau + e = au - zu - aue + e = au(1-e) + e - zu = 1 - e - zx - zu + e = 1 - (zx + zu) \in U(R)$; this implies $au - e \in U(R)$. Let $au - e = v$ for some $v \in U(R)$. Then $a - eu^{-1} = vu^{-1} \in U(R)$. If $ea \neq 0$, then $a \neq axa$. Let $f = ax = 1 - e$ and $r = fa - a$. Then $rx = (fa - a)x = (axa - a)x = (ax - 1)ax = -e(1-e) = 0$ and $fr = f^2a - fa = 0$. Let $a' = a + r$. Then $a'x = ax + rx = ax = f$, $a'xa' = fa' = fa + fr = fa = r + a = a'$, and $a'x + e = f + e = ax + e = 1$. Since $ea' = ea + er = efa = eaxa = e(1-e)a = 0$, by a similar proof of above, there exists $w \in U(R)$ such that $a' - ew = s \in U(R)$. Since $fr = 0$, $r = (1-f)r = er$; this leads to $s = a' - ew = a + r - ew = a + e(r - w)$. Therefore R has stable range 1 □

Theorem 5.4 *Let R be an exchange ring. Then*

(1) If $eR(1-e) \subseteq J(R)$ for each $e \in E(R)$, then R has stable range 1.

(2) If R is a weakly semicommutative ring, then R has stable range 1.

Proof Let a be a von Neumann regular element of R. Then $a = aba$ for some $a \in R$. Let $e = ba$ and $g = ab$. Then $a = ae = ga$ and $e, g \in E(R)$. Since $(1-e)a = (1-e)ae \in (1-e)Re$ and $a(1-g) = ga(1-g) \in gR(1-g)$, $(1-e)a, a(1-g) \in J(R)$ by hypothesis, that is, $a - ba^2, a - a^2b \in J(R)$. Hence, in $\bar{R} = R/J(R)$, $\bar{a}$ is strongly regular and so $\bar{a}$ is unit regular. Hence there exists $u \in U(R)$ such that $a - aua \in J(R)$; by Proposition 5.3, R has stable range 1.

(2) It is an immediate result of (1) and Corollary 2.3. □

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