Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2014) 38: 233 – 239

doi:10.3906/mat-1211-42

Generalized derivations on Jordan ideals in prime rings

Mahmmoud EL-SOUFI, Ahmed ABOUBAKR*
Department of Mathematics, Faculty of Science, Fayoum University, Fayoum, Egypt

Received: 27.11.2012 • **Accepted:** 27.01.2013 • **Published Online:** 27.01.2014 • **Printed:** 24.02.2014

Abstract: Let R be a 2-torsion free prime ring with center $Z(R)$, J be a nonzero Jordan ideal also a subring of R, and F be a generalized derivation with associated derivation d. In the present paper, we shall show that $J \subseteq Z(R)$ if any one of the following properties holds: (i) $[F(u), u] \in Z(R)$, (ii) $F(u)u = ud(u)$, (iii) $d(u^2) = 2F(u)u$, (iv) $F(u^2) - 2uF(u) = d(u^2) - 2ud(u)$, (v) $F^2(u) + 3d^2(u) = 2Fd(u) + 2dF(u)$, (vi) $F(u^2) = 2uF(u)$ for all $u \in J$.

Key words: Prime rings, Jordan ideals, generalized derivations, derivations

1. Introduction

Let R denote an associative ring with center $Z(R)$. For any $x, y \in R$, we write the commutator $[x, y] = xy - yx$, and the Jordan product $x \circ y = xy + yx$. We recall that a ring R is called prime if for any $a, b \in R$, $aRb = (0)$ implies that either $a = 0$ or $b = 0$; it is called a semiprime if $aRa = (0)$ implies that $a = 0$. A prime ring is clearly a semiprime ring. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. An additive mapping $F : R \to R$ is called a generalized derivation if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. A ring R is said to be n-torsion free, where $n \neq 0$ is a positive integer, if whenever $na = 0$, with $a \in R$, then $a = 0$. An additive subgroup J is said to be a Jordan ideal of R if $uor \in J$, for all $u \in J$ and $r \in R$. One may observe that every ideal of R is a Jordan ideal of R but the converse need not be true. An additive subgroup U of R is said to be a Lie ideal of R if $[u, r] \in U$, for all $u \in U$ and $r \in R$. It is clear that if $charR = 2$, then the Jordan ideal and Lie ideal of R are the same. In [4] Huang proved: Let R be an associative prime ring with char$R \neq 2$, U a Lie ideal of R such that $u^2 \in U$ for all $u \in U$, and F a generalized derivation associated with $d \neq 0$. If any one of the following conditions holds: (1) $[d(x), F(y)] = 0$, (2) $d(x) \circ F(y) = 0$, (3) either $d(x) \circ F(y) = x \circ y$ or $d(x) \circ F(y) + x \circ y = 0$, (4) either $[d(x), F(y)] = [x, y]$ or $[d(x), F(y)] + [x, y] = 0$, (5) either $[d(x), F(y)] = (x \circ y)$ or $[d(x), F(y)] + (x \circ y) = 0$, (6) either $d(x) \circ F(y) = [x, y]$ or $d(x) \circ F(y) + [x, y]$, (7) either $d(x) \circ F(y) + xy \in Z(R)$ or $d(x) \circ F(y) - xy \in Z(R)$for all $x, y \in U$, then either $d = 0$ or $U \subseteq Z(R)$.

Motivated by the results of Huang, we continue this line of investigation. In this paper, we study generalized derivation F with derivation d if any one of the following conditions holds: (i) $[F(u), u] \in Z(R)$, (ii) $F(u)u = ud(u)$, (iii) $d(u^2) = 2F(u)u$, (iv) $F(u^2) - 2uF(u) = d(u^2) - 2ud(u)$, (v) $F^2(u) + 3d^2(u) = 2Fd(u) + 2dF(u)$, (vi) $F(u^2) = 2uF(u)$ for all u in a Jordan ideal that is also a subring of R.

*Correspondence: afs00@fayoum.edu.eg
This paper is a part of the MSc thesis under the supervision of Prof M.N. Daif.
2010 *AMS Mathematics Subject Classification:* 16W25, 16N60, 16U80.

2. Preliminaries

Throughout the present paper, we shall make use of the following 2 basic identities without any specific mention:

$$[xy, z] = x[y, z] + [x, z]y, \text{ for all } x, y, z \in R. \tag{2.1}$$

$$[x, yz] = y[x, z] + [x, y]z, \text{ for all } x, y, z \in R. \tag{2.2}$$

We begin with the following known results, which will be used to prove our theorems.

Lemma 2.1 [[5], Lemma 2.7]**.** *Let R be a prime ring with char$R \neq 2$ and J a nonzero Jordan ideal of R. If J is a commutative Jordan ideal, then $J \subseteq Z(R)$.*

Lemma 2.2 [[5], Lemma 2.5]**.** *Let R be a prime ring and J a nonzero Jordan ideal of R. If $a \in R$ and $aJ = (0)$ (or $Ja = (0)$), then $a = 0$.*

Lemma 2.3 [[5], Lemma 2.6]**.** *Let R be a prime ring with char$R \neq 2$ and J a nonzero Jordan ideal of R. If $a, b \in R$ and $aJb = (0)$, then either $a = 0$ or $b = 0$.*

Lemma 2.4 [[1], Lemma 2.5]**.** *Let R be a prime ring with char$R \neq 2$ and J a nonzero Jordan ideal of R. Suppose that θ, ϕ are automorphisms of R. If R admits a (θ, ϕ)-derivation d such that $d(J) = (0)$, then either $d = 0$ or $J \subseteq Z(R)$.*

Lemma 2.5 [[5], Theorem 3.1]**.** *Let R be a prime ring with char$R \neq 2$ and J be both a Jordan ideal and a subring of R. If θ is an automorphism of R and $G : R \to R$ is an additive mapping satisfying $G(u^2) = 2\theta(u)G(u)$ for all $u \in J$, then either $J \subseteq Z(R)$ or $G(J) = 0$.*

Lemma 2.6 [[4], Lemma 2.6]**.** *A group cannot be the union of 2 of its proper subgroups.*

Now, we will prove the following 2 lemmas, which will be used to prove our theorems.

Lemma 2.7 *Let R be a ring. If R admits a generalized derivation F associated with derivation $d \neq 0$, then the mapping $F - d$ is a left centralizer on R.*

Proof Let $G = F - d$. It is clear that G is an additive mapping and for all $x, y \in R$, we have

$$\begin{aligned} G(xy) &= (F - d)(xy) = F(xy) - d(xy) \\ &= F(x)y + xd(y) - d(x)y - xd(y) \\ &= (F(x) - d(x))y = G(x)y. \end{aligned} \tag{2.3}$$

Therefore, G is a left centralizer on R. □

Lemma 2.8 *Let R be a prime ring and J a nonzero Jordan ideal of R. If G is a left centralizer of R such that $G(u) = 0$ for all $u \in J$, then $G(r) = 0$ for all $r \in R$.*

Proof Since J is a Jordan ideal of R, $ur+ru \in J$ for all $u \in J$ and $r \in R$. By hypotheses,

$$F(u) = 0 \, \text{for all} \, u \in J. \tag{2.4}$$

Replacing u by $ur+ru$, $r \in R$, in (2.4) and using (2.4), we get $G(r)u = 0 \quad \forall u \in J, r \in R$, and hence $G(r)J = (0)$ for all $r \in R$. Thus, by Lemma 2.2, we get $G(r) = (0)$ for all $r \in R$. □

Remark 2.9 The assumption that J is both a Jordan ideal and a subring of R seems close to assuming that J is an ideal of the ring. However, we can see that there exists a Jordan ideal and a subring of R, which is not an ideal of R.

Example 2.10 [2]. *Let R be a ring of all 2×2 matrices with entries form $GF(2)$. Consider $J = \{\begin{pmatrix} a & b \\ b & a \end{pmatrix} | a, b \in GF(2)\}$ we can verify that J is both a Jordan ideal and a subring of R, but it is not an ideal of R.*

3. Main results

We start by the following theorem, which is the proposition 3.1 in [3] neglecting the condition subring on a subset.

Theorem 3.1 *Let R be a 2-torsion free semiprime ring, J a nonzero Jordan ideal, and F an additive mapping on R. If F is centralizing on J, then F is commuting on J.*

Proof A linearization of $[F(u), u] \in Z(R)$ gives $[F(u), v] + [F(v), u] \in Z(R)$ for all $u, v \in J$. In particular, replacing v by $2u^2$, we get $2[F(u), u^2] + 2[F(u^2), u] \in Z(R)$.
Since $[F(u), u] \in Z(R)$, we have $[F(u), u^2] = 2[F(u), u]u$. Thus

$$4[F(u), u]u + 2[F(u^2), u] \in Z(R) \, \text{for all} \, u \in J. \tag{3.1}$$

By assumption, $4[F(u^2), u^2] \in Z(R)$ for all $u \in J$. That is,

$$4[F(u^2), u]u + 4u[F(u^2), u] \in Z(R) \, \text{for all} \, u \in J. \tag{3.2}$$

Now fix $u \in J$ and let $z = [F(u), u] \in Z(R)$, $s = [F(u^2), u]$. By (3.1) we have $0 = [F(u), 4zu + 2s] = 2(2z^2 + [F(u), s])$. Thus

$$[F(u), s] = -2z^2 \tag{3.3}$$

According to (3.2) we have $0 = [F(u), 4su + 4us] = 4([F(u), s]u + s[F(u), u] + [F(u), u]s + u[F(u), s])$, and applying (3.3), we get $-4z^2 u + 2zs = 0$. Multiplying (3.3) by z from the left and using the last relation we obtain $-2z^3 = z[F(u), s] = [F(u), zs] = [F(u), 2z^2 u] = 2z^3$. Hence $z^3 = 0$. Since the center of a semiprime ring contains no nonzero nilpotent elements, we conclude that $z = 0$. This proves the theorem. □

Theorem 3.2 *Let R be a prime ring with $\operatorname{char} R \neq 2$, and J a nonzero Jordan ideal and a subring of R. If R admits a generalized derivation F with associated derivation $d \neq 0$ such that F is centralizing on J, then $J \subseteq Z(R)$.*

Proof By Theorem 3.1 we have

$$[F(u), u] = 0 \text{ for all } u \in J. \tag{3.4}$$

Linearizing (3.4) and using (3.4), we obtain

$$[F(u), v] + [F(v), u] = 0 \text{ for all } u, v \in J. \tag{3.5}$$

Replacing v by vu in (3.5) and using (3.5) we obtain

$$\begin{aligned} &[F(u), v]u + [F(v), u]u + v[d(u), u] + [v, u]d(u) \\ &= v[d(u), u] + [v, u]d(u) = 0 \text{ for all } u, v \in J. \end{aligned} \tag{3.6}$$

Again replacing v by wv in (3.6) and using (3.6), we get $[w, u]vd(u) = 0$ for all $u, v, w \in J$, and hence $[w, u]Jd(u) = (0)$. Thus, by Lemma 2.3, we find that for each $u \in J$ either $[w, u] = 0$ or $d(u) = 0$. Now let $J_1 = \{u \in J \mid d(u) = 0\}$ and $J_2 = \{u \in J \mid [w, u] = 0, \text{ for all } w \in J\}$. Thus, J_1 and J_2 are additive subgroups of J and $J = J_1 \bigcup J_2$. However, a group cannot be the union of 2 of its proper subgroups; hence $J_1 = J$ or $J_2 = J$. If $J_1 = J$, then $d(u) = 0$ for all $u \in J$. Thus, by Lemma 2.4, we get $J \subseteq Z(R)$. On the other hand, if $[w, u] = 0$ for all $w, u \in J$, then, by Lemma 2.1, we get $J \subseteq Z(R)$. □

Theorem 3.3 *Let R be a prime ring with char$R \neq 2$ and J a nonzero Jordan ideal and a subring of R. If R admits a generalized derivation F with associated derivation $d \neq 0$ such that $F(u)u = ud(u)$ for all $u \in J$, then $J \subseteq Z(R)$.*

Proof By hypothesis we have

$$F(u)u = ud(u) \text{ for all } u \in J. \tag{3.7}$$

Linearizing the above equation gives

$$F(u)v + F(v)u = ud(v) + vd(u) \text{ for all } u, v \in J. \tag{3.8}$$

Replace v by vu and use (3.8) to get

$$2vd(u)u = (u \circ v)d(u) \text{ for all } u, v \in J. \tag{3.9}$$

Replacing v by wv in (3.9) and using (3.9), we have $[u, w]vd(u) = (0)$ for all $u, v, w \in J$, so $[u, w]Jd(u) = (0)$. Thus by Lemma 2.3, we find that for each $u \in J$ either $[u, w] = 0$ or $d(u) = 0$ for all $w \in J$. Now using similar arguments as used in the proof of Theorem 3.2, we get $J \subseteq Z(R)$. □

Theorem 3.4 *Let R be a prime ring with char$R \neq 2$ and J a nonzero Jordan ideal and a subring of R. If R admits a generalized derivation F with associated derivation $d \neq 0$ such that $F(u^2) - 2uF(u) = d(u^2) - 2ud(u)$ for all $u \in J$, then either $J \subseteq Z(R)$ or $F = d$.*

Proof By hypothesis we have

$$F(u^2) - 2uF(u) = d(u^2) - 2ud(u) \text{ for all } u \in J. \tag{3.10}$$

Since F and d are additive mappings, (3.10) could be rewritten as

$$(F-d)(u^2) = 2u(F-d)(u) \text{ for all } u \in J. \tag{3.11}$$

Let $G = F - d$ we get $G(u^2) = 2uG(u)$ for all $u \in J$. By Lemma 2.5 (taking $\theta = I$), either $J \subseteq Z(R)$ or $G(J) = 0$. If $G(J) = 0$, by Lemma 2.7 G is a left centralizer. Using Lemma 2.8 we get $G(r) = F(r) - d(r) = 0$ for all $r \in R$; thus $F(r) = d(r)$ for all $r \in R$. □

Theorem 3.5 *Let R be a prime ring with char$R \neq 2$ and J a nonzero Jordan ideal and a subring of R. If R admits a generalized derivation F with associated derivation $d \neq 0$ such that $F^2(u)+3d^2(u) = 2Fd(u)+2dF(u)$ for all $u \in J$, then either $J \subseteq Z(R)$ or $F = d$.*

Proof By hypothesis we have

$$F^2(u) + 3d^2(u) = 2Fd(u) + 2dF(u) \text{ for all } u \in J. \tag{3.12}$$

Replacing u by uv in (3.12) we get

$$F(F(u)v + ud(v)) + 3d(d(u)v + ud(v)) = 2F(d(u)v + ud(v)) + 2d(F(u)v + ud(v)) \\ \text{for all } u, v \in J. \tag{3.13}$$

The above equation gives

$$2F(u)d(v) = 2d(u)d(v) \text{ for all } u, v \in J. \tag{3.14}$$

However, char$R \neq 2$; hence

$$(F(u) - d(u))d(v) = 0 \text{ for all } u, v \in J. \tag{3.15}$$

Again replacing v by vw and using (3.15) we get

$$(F(u) - d(u))vd(w) = 0 \text{ for all } u, v, w \in J. \tag{3.16}$$

Thus, we get $(F(u)-d(u))Jd(w) = (0)$ for all $u, w \in J$. By Lemma 2.3 we have either $d(w) = 0$ for all $w \in J$ or $F(u)-d(u) = 0$ for all $u \in J$. If $d(w) = 0$ for all $w \in J$, hence $d(J) = 0$. Thus by Lemma 2.4 we get $J \subseteq Z(R)$. On the other hand, if $F(u)-d(u) = 0$. Using the same steps in Theorem 3.4 we get $F(r) = d(r)$ for all $r \in R$. □

Theorem 3.6 *Let R be a prime ring with char$R \neq 2$ and J a nonzero Jordan ideal and a subring of R. If R admits a generalized derivation F with associated derivation $d \neq 0$ such that $d(u^2) = 2F(u)u$ for all $u \in J$, then $J \subseteq Z(R)$.*

Proof By hypothesis we have

$$d(u^2) = 2F(u)u \text{ for all } u \in J. \tag{3.17}$$

This gives

$$d(u)u + ud(u) = 2F(u)u \text{ for all } u \in J. \tag{3.18}$$

Linearizing the above equation gives

$$d(u)v + d(v)u + ud(v) + vd(u) = 2F(u)v + 2F(v)u \text{ for all } u, v \in J. \tag{3.19}$$

Replace v by vu and use (3.19) to get

$$uvd(u) + vud(u) = 2vd(u)u \text{ for all } u, v \in J. \tag{3.20}$$

Thus

$$(u \circ v)d(u) = 2vd(u)u, \text{ for all } u, v \in J. \tag{3.21}$$

Replacing v by wv in (3.21) and using (3.21), we have $[u,w]vd(u) = (0)$ for all $u, v, w \in J$; hence $[u,w]Jd(u) = (0)$. Thus, by Lemma 2.3, we find that for each $u \in J$ either $[u,w] = 0$ or $d(u) = 0$ for all $w \in J$. Now, using similar arguments as used in the proof of Theorem 3.2, we get $J \subseteq Z(R)$. □

Theorem 3.7 *Let R be a prime ring with char$R \neq 2$ and J a nonzero Jordan ideal and a subring of R. If R admits a generalized derivation F with associated derivation $d \neq 0$ such that $F(u^2) = 2uF(u)$ for all $u \in J$, then $J \subseteq Z(R)$.*

Proof By hypothesis we have

$$F(u^2) = 2uF(u) \text{ for all } u \in J. \tag{3.22}$$

Using Lemma 2.5 with $\theta = I$, we get either $J \subseteq Z(R)$ or $F(J) = 0$. If $F(J) = 0$, then

$$F(u) = 0 \text{ for all } u \in J. \tag{3.23}$$

Replacing u by uv in (3.23) we get

$$F(u)v + ud(v) = 0 \text{ for all } u \in J. \tag{3.24}$$

Using (3.23) we have $ud(v) = 0$ for all $u, v \in J$; thus $Jd(v) = 0$ for all $v \in J$. By Lemma 2.2 we get $d(v) = 0$ for all $v \in J$, and by Lemma 2.4 we get $J \subseteq Z(R)$. □

Corollary 3.8 *Let R be a prime ring with char$R \neq 2$ and I a nonzero ideal of R. Suppose that R admits a generalized derivation F associated with a nonzero derivation d such that any one of the following holds:*
(i) $F(u)u = ud(u)$ for all $u \in I$;
(ii) $d(u^2) = 2F(u)u$ for all $u \in I$;
(iii) $F(u^2) = 2uF(u)$ for all $u \in I$;
then R is commutative.
Moreover, if any one of the following holds:
(iv) $F(u^2) - 2uF(u) = d(u^2) - 2ud(u)$ for all $u \in I$;
(v) $F^2(u) + 3d^2(u) = 2Fd(u) + 2dF(u)$ for all $u \in I$;
then either R is commutative or $F = d$.

In Theorem 3.2, if we assume that J is only a subring of R, then J is not central. This can be shown by the following example.

Example 3.9 *Let R be the prime ring of all 2×2 matrices over a noncommutative prime ring S with char$S \neq 2$. Consider $J = \{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} | a, b \in S\}$. Hence U is a subring, but not a Jordan ideal of R.*

Let us define mappings $F:R\to R$ and $d:R\to R$ as follows:

$$F\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & -d \end{pmatrix}, \tag{3.25}$$

$$d\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix}. \tag{3.26}$$

Therefore, d is a nonzero derivation on R, and F is a generalized derivation on R satisfying the condition $[F(u),u]\in Z(R)$ for all $u\in J$. But $J\nsubseteq Z(R)$.

Acknowledgements

The authors are thankful to Prof M.N. Daif for the encouragement and fruitful discussion.

References

[1] Ali, A. and Kumar, D.: Generalized derivations as homomorphisms or as anti-homomorphisms in a prime ring, Hacettepe Journal of Mathematics and Statistics, 38(1), 17–20 (2009).

[2] Awtar, R.: Lie and Jordan structure in prime rings with derivations, Proceeding of the American Mathematical society, 41(1), 67–74 (1973).

[3] Bresar, M.: Centralizing mappings and derivations in prime rings, J. Algebra, 156, 385–394 (1993).

[4] Huang, S.: Generalized derivations of prime rings, International Journal of Mathematics and Mathematical Sciences, Article ID 85612, 6 pages, Volume 2007.

[5] Zaidi, S.M.A., Ashraf, M. and Ali, S.: On Jordan ideals and left (θ,θ)-derivations in prime rings, International Journal of Mathematics and Mathematical Sciences, 37, 1957–1964 (2004).