Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2014) 38: 240 – 245
© TÜBİTAK
doi:10.3906/mat-1305-12

On NR^*-subgroups of finite groups

Xianggui ZHONG*
Department of Mathematics, Guangxi Normal University, Guilin, Guangxi, P.R. China

Received: 08.05.2013 • **Accepted:** 23.08.2013 • **Published Online:** 27.01.2014 • **Printed:** 24.02.2014

Abstract: Let G be a finite group and let H be a subgroup of G. H is said to be an NR^*-subgroup of G if there exists a normal subgroup T of G such that $G = HT$ and if whenever $K \lhd H$ and $g \in G$, then $K^g \cap H \cap T \leq K$. A number of new characterizations of a group G are given, under the assumption that all Sylow subgroups of certain subgroups of G are NR^*-subgroups.

Key words: Finite group, NR^*-subgroup, the generalized Fitting subgroup, saturated formation

1. Introduction

All groups considered in this paper are finite. Throughout the following, G always stands for a finite group and if H and K are subgroups of G and K normalizes H then we shall use the notation $[H]K$ to indicate that $H \cap K = 1$, i.e the product HK is a split extension of the normal subgroup H by the complement K. Other unexplained notations and terminology are standard, as in [6, 7]. A topic of some interest is to investigate the structure of G under certain restrictions on its certain subgroups (see [1-3, 8-10]). Following Berkovich [2], a subgroup H of G is called an NR-*subgroup* in G if whenever $K \lhd H$, then $K^G \cap H = K$. NR-subgroups play an important role in the following result of Berkovich (see [2, Proposition 11]).

Theorem 1.1 *If all Sylow subgroups of a group G are NR-subgroups, then G is supersoluble.*

The following example indicates that it is not necessary that all Sylow subgroups of a supersolvable group G are NR-subgroups of G.

Example 1.1. Let H be an elementary abelian 3-group of order 3^2 and L be a cyclic group of order 2. Denote $G = [H]L$ to be the corresponding semidirect product, where $H = \langle a, b | a^3 = b^3 = 1 = [a, b] \rangle$, $L = \langle x \rangle$, and $b^x = b, a^x = a^{-1}$. Observe that a chief series $1 \lhd \langle a \rangle \lhd \langle a, x \rangle \lhd G$ implies that G is supersoluble. However, H is not an NR-subgroup of G. As an illustration, let $K = \langle ab \rangle$ be a maximal subgroup of H, and it is easy to see that $K \neq H = K^G \cap H$.

We hope to weaken the conditions on Sylow subgroups of G to generalize Theorem 1.1. In this article we work in this direction. We first analyze the counterexample G above. Note that $\langle b, x \rangle$ is an NR-subgroup of prime index in G. This is the case that Tong-Viet studied (see [8, 9]). In fact, Tong-Viet in [9] proved that if

*Correspondence: xgzhong@mailbox.gxnu.edu.cn
The author is supported by the National Natural Science Foundation of China (11261007) and the Natural Science Foundation of Guangxi Autonomous Region (2013GXNSFBA019003).
2010 *AMS Mathematics Subject Classification:* 20D20.

G has an NR-subgroup H of prime index in G and H is supersolvable, then G is supersolvable. On the other hand, notice that $\{\langle a\rangle, \langle b\rangle, \langle ab\rangle, \langle a^{-1}b\rangle\}$ is the set of nontrivial normal subgroups of H. Moreover, if $L = \langle ab\rangle$ or $\langle a^{-1}b\rangle$, then there exists a normal subgroup $T = \langle b, x\rangle$ of G such that $G = LT$ and $L^g \cap H \cap T = 1$ for all $g \in G$, and if $L = \langle a\rangle$ or $\langle b\rangle$, then there exists a normal subgroup $T = G$ of G such that $G = LT$ and $L^g \cap H \cap T = L$ for all $g \in G$.

We start with the following new concept.

Definition 1.1. Let G be a group and let $K \lhd H \leq G$. A triple (G, H, K) is said to be *quasispecial* in G if there exists a normal subgroup T of G such that $G = KT$ and $K^g \cap H \cap T \leq K$ for all $g \in G$. A subgroup H is said to be an NR^**-subgroup* of G if, whenever K is normal in H, the triple (G, H, K) is quasispecial in G.

It is clear that every NR-subgroup of G is an NR^*-subgroup. The converse is not true in general. For instance, let $G = [H]L$, where $H = \langle a, b | a^3 = b^3 = 1 = [a, b]\rangle$, $L = \langle x\rangle$, and $b^x = b, a^x = a^{-1}$. Then H is an NR^*-subgroup in G. However, we know that H is not an NR-subgroup.

We extend the former result by replacing conditions on "NR-subgroups" by conditions referring to only some "NR^*-subgroups". Furthermore, we work within the framework of formation theory and use NR^*-subgroup conditions on the Sylow subgroups of $F^*(G)$ to characterize the structure of a group G. Our main results are Theorems 3.7 and 3.8 (see Section 3). These results generalize some classical and recent results as particular cases.

2. Preliminaries

The following 2 lemmas will be used frequently and without comment.

Lemma 2.1. *Let K be a subgroup and let H be an NR^*-subgroup of G. Then the following holds:*

(1) *If $H \leq K$, then H is an NR^*-subgroup of K.*

(2) *If $N \lhd G$ and $N \leq H$, then H/N is an NR^*-subgroup of G/N.*

Proof (1) Let L be any normal subgroup of H. Since H is an NR^*-subgroup of G, the triple (G, H, K) is quasispecial in G, and hence there exists a normal subgroup T of G such that $G = LT$ and $L^g \cap H \cap T \leq L$ for all $g \in G$. Note that $H \leq K$; we have $K = K \cap LT = L(K \cap T)$ by Dedekind's law and hence $K \cap T \lhd K$. This implies that H is an NR^*-subgroup of K.

(2) Let L be any normal subgroup of H. Since H is an NR^*-subgroup of G, there exists a normal subgroup T of G such that $G = LT$ and $L^g \cap H \cap T \leq L$ for all $g \in G$. It follows that $G/N = (L/N)(TN/N)$ and whenever $L/N \lhd H/N$ and $g \in G$, then $(L/N)^{gN} \cap H/N \cap TN/N = (L^g \cap H \cap TN)/N = (L^g \cap H \cap T)N/N \leq L/N$. Thus, H/N is an NR^*-subgroup of G/N. □

Lemma 2.2. *Let H be a p-subgroup of G and let N be a normal p'-subgroup. Then H is an NR^*-subgroup of G if and only if HN/N is an NR^*-subgroup of G/N.*

Proof Let L be any normal subgroup of H. Assume that H is an NR^*-subgroup of G and the triple (G, H, K) is quasispecial in G; hence, there exists a normal subgroup T of G such that $G = LT$ and $L^g \cap H \cap T \leq L$ for all $g \in G$. By the assumption that H is a p-group and N is a p'-group, we have $N \leq T$, since $(|N|, |G/T|) = 1$, and hence $G/N = (LN/N)(T/N)$. Let M/N be a normal subgroup of HN/N. Then there exists a normal subgroup L of H such that $M = LN$. Note that $M^g \cap HN \cap T = L^g N \cap (H \cap T)N = (L^g N \cap H \cap T)N$. Let P be a Sylow p-subgroup of $L^g N$. Since $L^g N \cap H \cap T$ is a p-group of $L^g N$ and $L^g N$ contains a Sylow p-subgroup L^g, we have $(L^g N \cap H \cap T)^{n_1} \leq P \leq L^{gn_2}$ for suitable $n_1, n_2 \in N$. Then it follows that $L^g N \cap H \cap T \leq L^{gn_2 n_1^{-1}} \cap H \cap T$, and hence $(L^g N \cap H \cap T)N \leq (L^{gn_2 n_1^{-1}} \cap H \cap T)N \leq LN = M$. Thus, $(M/N)^{gN} \cap (HN/N) \cap T/N = (L^g \cap H \cap T)N/N \leq M/N$. Therefore, HN/N is an NR^*-subgroup of G/N.

Conversely, assume that HN/N is an NR^*-subgroup of G/N. Let L be a normal subgroup of H; then $LN/N \lhd HN/N$ and hence there exists a normal subgroup T/N of G/N such that $G/N = (LN/N)(T/N)$ and $(LN/N)^{gN} \cap HN/N \cap T/N \leq LN/N$. It follows that $L^g \cap H \cap T \leq (L^g N \cap H \cap T)N \leq LN$, so $G = LT$ and $L^g \cap H \cap T \leq LN \cap H = L(N \cap H) = L$, since $N \cap H = 1$. This follows easily from the assumption that H is a p-group and N is a p'-group. Thus, H is an NR^*-subgroup of G. □

Recall that a subgroup H of a group G is an $\mathcal{H}$-subgroup in G if $H^g \cap N_G(H) \leq H$ for all g in G. A subgroup H of G is called weakly $\mathcal{H}$-subgroup in G [1] if there exists a normal subgroup T of G such that $G = HT$ and $H \cap T$ is an $\mathcal{H}$-subgroup in G. For groups $H \leq T \leq G$ we say that H is strongly closed in T with respect to G if $H^g \cap T \leq H$ for all $g \in G$. Noting if H is a p-subgroup of G, then H is an $\mathcal{H}$-subgroup of G if and only if H is strongly closed in P with respect to G for some Sylow p-subgroup P of G containing H. We obtain the next result.

Lemma 2.3. *Let N be a normal subgroup of G. Assume that p is a prime dividing $|G|$ and P is a Sylow p-subgroup of N. If H is normal in P and (G, P, H) is quasispecial in G, then H is a weakly $\mathcal{H}$-subgroup of G.*

Proof By the hypotheses, for any normal subgroup H of P, there exists a normal subgroup T of G such that $G = HT$ and $H^g \cap P \cap T \leq H$ for all $g \in G$. Let S be a Sylow p-subgroup of G containing P. Observe that $N \cap S = P$ implies that $(H \cap T)^g \cap S = H^g \cap T \cap N \cap S = H^g \cap T \cap P \leq H$ for every $g \in G$. Thus, $H \cap T$ is an $\mathcal{H}$-subgroup of G. This ends the proof. □

Lemma 2.4. *Let N be a minimal normal subgroup of G and let H be a subgroup of N. If H is an NR^*-subgroup of G, then H is an NR-subgroup of G.*

Proof By our hypotheses, for any normal subgroup L of H, there exists a normal subgroup T of G such that $G = LT$ and $L^g \cap H \cap T \leq L$ for all $g \in G$. Since $N = L(N \cap T)$ by Dedekind's law and $N \cap T \lhd G$, the minimality of N implies that $N \leq T$ or $N \cap T = 1$. If $N \leq T$, then $G = T$ and so H is an NR-subgroup in G. If $N \cap T = 1$, then $N = L \leq H$ and hence $H = N$ is normal in G and, of course, an NR-subgroup in G. □

We shall need the following lemma.

Lemma 2.5 ([10, Lemma 3.1]). *Let P be a Sylow p-subgroup of a group G and let T be a normal subgroup G. If $N_G(P)$ is p-nilpotent, then $\langle T \cap P \cap (P')^g | g \in G\rangle = T \cap P \cap \langle (P')^g | g \in G\rangle$.*

3. Main results

Our main result in this section gives detailed information about the NR^*-subgroup conditions of certain subgroups of G.

Theorem 3.1 *Let p be a prime dividing $|G|$ and let P be a Sylow p-subgroup of G. Then G is p-nilpotent if and only if P is an NR^*-subgroup of G and $N_G(P)$ is p-nilpotent.*

Proof Suppose, first, that G is p-nilpotent. Then there exists a normal subgroup T of G such that $G = PT$ and $P \cap T = 1$. Let P_1 be any normal subgroup of P and let $g \in G$. Then $g = at$ with $a \in P$ and $t \in T$. Clearly, $P_1^g = P_1^t$ and P_1T is a normal subgroup of G. Observe that $P_1 \lhd N_{P_1T}(P_1)$ and $P_1 \in \mathrm{Syl}_p(N_{P_1T}(P_1))$ implies that $P_1^t \cap P \leq P_1$ and P_1 is NR^*-subgroup of G. Clearly, $N_G(P)$ is p-nilpotent.

Conversely, suppose that $N_G(P)$ is p-nilpotent and P is an NR^*-subgroup of G. Let $P_1, P_2, \ldots, P_s$ be maximal subgroups of P such that $\cap_{i=1}^s P_i = \Phi(P)$. Then there exists a normal subgroup T_i of G such that $G = P_iT_i$ and $P_i^g \cap T_i \cap P \leq P_i$ for all $i \in \{1, 2, \ldots, s\}$ and all $g \in G$, respectively, since $P \cap (P')^g \leq (P')^g \leq (\Phi(P))^g \leq P_i^g$. Let $N = \cap_{i=1}^s T_i$, a normal subgroup of G. Then we have $N \cap P \cap (P')^g \leq \cap_{i=1}^s (T_i \cap P_i^g \cap P) \leq \cap_{i=1}^s P_i = \Phi(P)$ for all $g \in G$. By Grun's Theorem [7, IV, Theorem 3.7], we obtain $N \cap P \cap G' = N \cap \langle P \cap (P')^g, P \cap (N_G(P))' | g \in G\rangle$. Since $N_G(P)$ is p-nilpotent, Lemma 2.5 implies that $N \cap P \cap G' = N \cap \langle P \cap (P')^g | g \in G\rangle = N \cap P \cap \langle (P')^g | g \in G\rangle = \langle N \cap P \cap (P')^g | g \in G\rangle \leq \Phi(P)$. By applying Tate's Theorem [7, IV, Theorem 4.7], we get that $N \cap G'$ is p-nilpotent. Let B be a normal p-complement of $(N \cap G')_p$ in $N \cap G'$. If $B > 1$, then B is normal in G. Consider the quotient group G/B. Since $N_{G/B}(PB/B) = N_G(P)B/B$, by Lemma 2.3(2), G/B is p-nilpotent and so is G. If $B = 1$, then $N \cap G' \leq P$ and, therefore, $N \cap G' = N \cap P \cap G' \leq \Phi(P)$. By [7, III, Theorem 3.3(a)], $N \cap G' \leq \Phi(G)$. Observe that G/N being p-nilpotent implies that $G/(N \cap G')$ is p-nilpotent. We get that G is p-nilpotent. □

Corollary 3.2. *Let p be a prime dividing $|G|$ and let P be a Sylow p-subgroup of G. Then G is p-nilpotent if and only if $N_G(P)$ is p-nilpotent and P is an NR-subgroup of G.*

Remark 3.3. In Theorem 3.1, the assumption that $N_G(P)$ is p-nilpotent is essential. In order to illustrate the situation, we consider $G = \langle a, b, c | a^9 = b^2 = c^2 = 1 = [b, c], ac = ba, ca = abc\rangle$. Then the unique subgroup of order 3 is normal in G, but G is not 3-nilpotent. However, if p is the smallest prime dividing the order of a group, then the result holds. In fact, we obtain the following result.

Theorem 3.4. *Let p be the smallest prime dividing $|G|$ and let P be a Sylow subgroup of G. Then G is p-nilpotent if and only if P is an NR^*-subgroup of G.*

Proof Since P is an NR^*-subgroup of G, by Lemma 2.1(1), P is an NR^*-subgroup of $N_G(P)$. If $N_G(P) < G$, then, by induction, $N_G(P)$ is p-nilpotent. By Theorem 3.1, G is p-nilpotent. Thus, P is normal in G. Let H be a maximal subgroup of P; then, by Lemma 2.3, H is a weakly $\mathcal{H}$-subgroup of G. By [1, Theorem 3.1], G is p-nilpotent. □

Recall that a class $\mathcal{F}$ of groups is called a *formation* provided that (i) $G \in \mathcal{F}$ and N is normal in G imply $G/N \in \mathcal{F}$, and (ii) if both G/N and G/M are in $\mathcal{F}$, then $G/(N \cap M) \in \mathcal{F}$. If, in addition, $G/\Phi(G) \in \mathcal{F}$ implies $G \in \mathcal{F}$, then we say that $\mathcal{F}$ is *saturated.* Note that, for a class $\mathcal{F}$ of groups, a chief factor H/K of G is called $\mathcal{F}$*-central* if $[H/K](G/C_G(H/K))) \in \mathcal{F}$. The symbol $Z_{\mathcal{F}}(G)$ denotes the $\mathcal{F}$-hypercenter of G, that is, the product of all such normal subgroups H of G whose G-chief factors are $\mathcal{F}$-central. We refer to [4, 5] for notation and terminology about the theory of formations. The following lemma plays a key role in the proof of Theorem 3.7.

Lemma 3.5. *Let P be a normal p-subgroup of G. If P is an NR^*-subgroup of G, then $P \leq Z_{\mathcal{U}}(G)$.*

Proof Let H be an arbitrary maximal subgroup of P. Observe that P being an NR^*-subgroup of G implies that H is a weakly $\mathcal{H}$-subgroup of G by Lemma 2.3. We conclude by [1, Lemma 3.3] that $P \leq Z_{\mathcal{U}}(G)$. □

We now prove the following main results.

Theorem 3.6. *If all Sylow subgroups of* G *are* NR^**-subgroups of* G*, then* G *is supersolvable.*

Proof Assume that the result is not true and let G be a counterexample of minimal order. By Theorem 3.4, we can conclude that G has a Sylow tower of supersolvable type. Let p be the largest prime dividing the order of G and P a Sylow p-subgroup of G. Then $P \lhd G$. If N is a nontrivial normal subgroup of G contained in P, then as G/N satisfies the hypothesis of the theorem by Lemmas 2.1(2) and 2.2, the minimality of G yields that G/N is supersolvable. Set $L = P \cap \Phi(G)$. Observe that $L \neq 1$ implies that G/L is supersolvable, and hence G must be supersolvable, which is a contradiction. Hence, L must be 1. Then P is the direct product of minimal normal subgroups of G, which are contained in P. Now we wish to show that P is the unique minimal normal subgroup of G. Otherwise, let N_1 and N_2 be 2 distinct minimal normal p-subgroups of G; arguing as above, we conclude that G/N_i are supersolvable and so G is supersolvable, which is a contradiction and our claim holds. Let P_1 be any maximal subgroup of P. By Lemma 2.3, P_1 is weakly $\mathcal{H}$-subgroup in G. Then there exists a normal subgroup K of G such that $G = P_1K$ and $P_1 \cap K$ is $\mathcal{H}$-subgroup in G. Clearly, $P_1 \cap K$ is subnormal in G. By [3, Theorem 6(2)], $P_1 \cap K \lhd G$. It follows that $P_1 \cap K = 1$ and so $P = P \cap K$ is a cyclic group of order p, which implies that G is supersolvable, a final contradiction, and the proof of the theorem is now complete. □

Theorem 3.7. *Let* $\mathcal{F}$ *be a saturated formation containing the class of supersolvable groups* $\mathcal{U}$*. Then* $G \in \mathcal{F}$ *if and only if* G *has a normal subgroup* H *such that* $G/H \in \mathcal{F}$ *and all Sylow subgroups of* H *are* NR^**-subgroups of* G*.*

Proof We need only to prove the "if" part. We use induction on $|G|$. By our hypothesis and Lemma 2.1(1), every Sylow subgroup of H is an NR^*-subgroup in H. Then H is supersolvable by Theorem 3.6. Let p be the largest prime dividing $|H|$ and P a Sylow p-subgroup of H. Then P is characteristic in H and so is normal in G. So, first by applying Lemma 2.1(2) and Lemma 2.2, all maximal subgroups of every Sylow subgroup of H/P are NR^*-subgroups in G/P. We get that $(G/P, H/P)$ satisfies the hypothesis of Theorem 3.7 and so $G/P \in \mathcal{F}$ by induction on $|G|$. Now, Lemma 3.5 applies, yielding that $P \leq Z_{\mathcal{U}}(G)$. Observe that $Z_{\mathcal{U}}(G) \leq Z_{\mathcal{F}}(G)$ by [4, Proposition 3.11] implies that $P \leq Z_{\mathcal{F}}(G)$ and so $G \in \mathcal{F}$. □

For a group G, the generalized Fitting subgroup $F^*(G)$ of G is the set of all elements of G that induce an inner automorphism on every chief factor of G. If $G \neq 1$, then $F^*(G) \neq 1$. If $N \lhd G$, then $F^*(N) \leq F^*(G)$. In particular we have $C_G(F^*(G)) \leq F(G)$, and the solvability of $F^*(G)$ implies that $F^*(G) = F(G)$ (see [1, Lemma 2.7], see also [4, 6]). With these results, now we can prove the next theorem.

Theorem 3.8. *Let* $\mathcal{F}$ *be a saturated formation containing the class of supersolvable groups* $\mathcal{U}$*. Then* $G \in \mathcal{F}$ *if and only if* G *has a normal subgroup* H *such that* $G/H \in \mathcal{F}$ *and all Sylow subgroups of* $F^*(H)$ *are* NR^**-subgroups of* G*.*

Proof We need only to prove the "if" part. By Lemma 2.2(1), we have that every Sylow subgroup of $F^*(H)$ is an NR^*-subgroup in $F^*(H)$. By Theorem 3.6 with respect to $F^*(H)$, we have that $F^*(H)$ is supersolvable and, hence, $F^*(H) = F(H)$. Since any subgroup that is characteristic in $F(G)$ is normal in G, by Lemma 3.5, $F(H) \leq Z_{\mathcal{U}}(G)$. Observe that $Z_{\mathcal{U}}(G) \leq Z_{\mathcal{F}}(G)$ implies that $F(H) \leq Z_{\mathcal{F}}(G)$. Hence, $G/C_G(F(H)) \in \mathcal{F}$ by [4, Theorem 6.10]. Since G/H and $G/C_G(F(H))$ are in $\mathcal{F}$, we have that $G/C_H(F(H)) = G/(H \cap C_G(F(H)))$ is in $\mathcal{F}$. Finally, note that $C_H(F^*(H)) \leq F(H)$ and the fact that $F^*(H) = F(H)$. Then $G/F(H)$ is an epimorphic image of $G/C_H(F(H))$, and thus $G/F(H) \in \mathcal{F}$. Now, by applying Theorem 3.7, we get $G \in \mathcal{F}$, which completes the proof. □

Theorem 3.8 immediately implies the following corollaries.

Corollary 3.9. *Let $\mathcal{F}$ be a saturated formation containing the class of supersolvable groups $\mathcal{U}$. Then $G \in \mathcal{F}$ if and only if G has a solvable normal subgroup H such that $G/H \in \mathcal{F}$ and if all Sylow subgroups of $F(H)$ are NR^*-subgroups of G.*

Corollary 3.10. *Let G be a group with a normal subgroup H such that G/H is supersolvable. If all Sylow subgroups of $F^*(H)$ are NR^*-subgroups of G, then G is supersolvable.*

Corollary 3.11. *Let G be a group. If all Sylow subgroups of $F^*(G)$ are NR^*-subgroups of G, then G is supersolvable.*

Corollary 3.12. *Let G be a group with a solvable normal subgroup H such that G/H is supersolvable. If all Sylow subgroups of $F(H)$ are NR^*-subgroups of G, then G is supersolvable.*

Corollary 3.13. *Let G be a solvable group. If all Sylow subgroups of $F(G)$ are NR^*-subgroups of G, then G is supersolvable.*

Corollary 3.13 is not true if the solvability of G is omitted, as the nonabelian simple groups show.

Acknowledgments

The author would like to thank the referee for his/her valuable suggestions and useful comments on the original version, which make the present paper readable.

References

[1] Asaad M, Heliel AA, Al-Mosa Al-Shomrani MM. On weakly $\mathcal{H}$-subgroups of finite groups. Comm Algebra 2012; 40: 3540–3550.

[2] Berkovich Y. Subgroups with the character restriction property and related topics. Houston J Math 1998; 24: 631–638.

[3] Bianchi M, Gillio Berta Mauri A, Herzog M, Verardi L. On finite solvable groups in which normality is a transitive relation. J Group Theory 2000; bf 3: 147–156.

[4] Doerk K, Hawkes T. Finite Soluble Groups. Berlin: Walter de Gruyter, 1992.

[5] Gorenstein D. Finite Groups. New York: Chelsea Publishing Company, 1968.

[6] Guo W. The Theory of Class of Groups. Beijing-New York-Dordrecht-Boston: Science Press-Kluwer Academic Publishers, 2000.

[7] Huppert B. Endliche Gruppen I. Berlin: Springer-Verlag, 1967.

[8] Tong-Viet HP. Groups with normal restriction property. Arch Math (Basel) 2009; 93: 199–203.

[9] Tong-Viet H P. Normal restriction in finite groups. Comm Algebra 2011; 39: 344–354.

[10] Wei X, Guo X. On $\mathcal{HC}$-subgroups and the structure of finite groups. Comm Algebra 2012; 40: 3245–3256.