The new method of determining Koebe domains for the class of typically real functions under Montel's normalization

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Abstract: We consider the class $T(r)$ of typically real functions with the normalization $f(0) = 0$ and $f(r) = r$ for a fixed $r \in (0, 1)$. In the limiting case, when r tends to 0, the class $T(r)$ coincides with the class T of typically real functions normalized by $f(0) = f'(0) - 1 = 0$. In 1980, Lewandowski and Miazga determined the Koebe domain for $T(r)$, i.e. the set of the form $\bigcap_{f \in T(r)} f(\Delta)$. They used the method applied earlier by Goodman. In this paper we present a new, complete method of determining this set. As a corollary, we obtain the Koebe set for T.

Key words: Typically real functions, Koebe domain, Montel's normalization

1. Introduction

In this paper we focus on typically real functions. Recall that an analytic function in the unit disk $\Delta = \{z \in \mathbf{C} : |z| < 1\}$ is called typically real if $\operatorname{Im} z \operatorname{Im} f(z) \geq 0$ for $z \in \Delta$. One usually considers the class T consisting of typically real functions, which satisfy the normalization condition

$$f(0) = 0 \quad \text{and} \quad f'(0) = 1 . \tag{1.1}$$

A different type of normalization was suggested by Montel in [7], namely

$$f(0) = 0 \quad \text{and} \quad f(r) = 1 \tag{1.2}$$

or

$$f(0) = 0 \quad \text{and} \quad f(r) = r . \tag{1.3}$$

In these conditions it is assumed that $r \in (0, 1)$.

The class of typically real functions with the normalization (1.3) is denoted by $T(r)$. This class was discussed by Piłat in [8]. She gave the representation formula for a function in $T(r)$ and proved various distortion properties in this class.

Our main goal is to determine the Koebe set for $T(r)$, i.e. the set of the form $\bigcap_{f \in T(r)} f(\Delta)$. The sets of this type for various classes of analytic functions have been discussed since the 1960s. Mainly, functions normalized by (1.1) were considered, but one can also find papers concerning functions under Montel's normalization; see, for example [11, 10, 3, 6, 5, 9]. It occurred that in majority of cases, these sets are domains;

however, sometimes a Koebe set is the union of disjoint domains (compare, for example, [3, 10]). We shall prove that the Koebe set for $T(r)$ is simply connected and starlike; hence, it is a domain. The idea of the proof of the main result is similar to that in [2], which was applied to determine the Koebe set for T in a different way than Goodman had done it in [1]. In the paper by Goodman, so-called universal typically real functions appeared. However, these functions are not univalent. The author applied them as majorants in subordination on the Riemann surfaces. Lewandowski and Miazga repeated in their work [4] the ideas of Goodman. In our new method we find extremal functions directly from the geometric properties of functions in $T(r)$. Due to this method, we can exchange subordination on the Riemann surfaces by typical subordination to a univalent majorant.

2. Main results

Let $r \in (0, 1)$ be arbitrarily fixed.

Theorem 2.1 *A function $f \in T(r)$ omits 2 values, $\varrho e^{i\theta}$ and $\varrho e^{-i\theta}$, where $\varrho > 0$, $\theta \in (0, \pi)$ if and only if there exists a function h analytic in Δ such that*

1. *h is typically real in Δ ,*

2. *$h(0) = 2\theta$, $h(r) = -2 \operatorname{arccot} \frac{r - \varrho \cos \theta}{\varrho \sin \theta}$,*

3. *$0 < h(x) < 2\pi$ for $-1 < x < 1$,*

4. *$f(z) = \varrho \frac{e^{i\theta} - e^{-i\theta} e^{ih(z)}}{1 - e^{ih(z)}}$.*

Proof $(\Rightarrow)$

Let $f \in T(r)$ omit 2 values, $\varrho e^{i\theta}$ and $\varrho e^{-i\theta}$, where $\varrho > 0$, $\theta \in (0, \pi)$. Let us discuss the function

$$h(z) = \frac{1}{i} \log \frac{f(z) - \varrho e^{i\theta}}{f(z) - \varrho e^{-i\theta}} . \tag{2.1}$$

This function is properly defined since $\frac{f(z) - \varrho e^{i\theta}}{f(z) - \varrho e^{-i\theta}}$ is an analytic function omitting points 0 and 1. In (2.1) we take the branch of the logarithm, which $z = 0$ associates with $2i\theta$. Hence

$$h(0) = 2\theta . \tag{2.2}$$

From Montel's normalization of f, it follows that

$$h(r) = \frac{1}{i} \log \frac{r - \varrho e^{i\theta}}{r - \varrho e^{-i\theta}} = -2 \operatorname{Arg}(r - \varrho e^{-i\theta}) ,$$

and hence

$$h(r) = -2 \operatorname{arccot} \frac{r - \varrho \cos \theta}{\varrho \sin \theta} . \tag{2.3}$$

From (2.1) we derive

$$f(z) = \varrho \frac{e^{i\theta} - e^{-i\theta} e^{ih(z)}}{1 - e^{ih(z)}} . \tag{2.4}$$

For $\theta \in (0, \pi)$, the numerator and the denominator of the right-hand side of (2.4) cannot vanish at the same time. Indeed, if there existed z_0 such that $1 - e^{ih(z_0)} = 0$, then the numerator $e^{i\theta} - e^{-i\theta}e^{ih(z_0)} = 2i\sin\theta$ would be different from 0. Hence, the analytic function f has no removable singularities. Thus, the denominator in (2.4) is different from 0. Consequently,

$$h(z) \neq 2k\pi, \ k \in \mathbf{Z} \ . \tag{2.5}$$

Taking (2.4) into account, one can obtain

$$\operatorname{Im} f(z) = \frac{\varrho}{|1 - e^{ih(z)}|^2}\left(1 - e^{-2\operatorname{Im} h(z)}\right)\sin\theta \ . \tag{2.6}$$

This means that h is a typically real function satisfying the normalization conditions of (2.2) and (2.3).

Combining it with (2.5), we get

$$0 < h(x) < 2\pi \quad \text{for} \quad -1 < x < 1 \ .$$

$(\Leftarrow)$

If a function h analytic in Δ satisfies conditions 1 through 4, then f is typically real and has Montel's normalization. Therefore, $f \in T(r)$ and f does not take the values $\varrho e^{i\theta}$ and $\varrho e^{-i\theta}$. $\qquad\square$

Lemma 2.1 *If $f \in T(r)$ omits $\varrho e^{i\theta}$ and $\varrho e^{-i\theta}$, where $\varrho > 0$, $\theta \in (0, \pi)$, then h defined by (2.1) is subordinated to H_θ given by*

$$H_\theta(z) = 2\theta + 8\theta\left(1 - \frac{\theta}{\pi}\right)\frac{z}{(1-z)^2 + 4\frac{\theta}{\pi}z} \ . \tag{2.7}$$

Proof From the assumptions of Lemma 2.1 and Theorem 2.1 we conclude that

$$h(\Delta) \subset \mathbf{C} \setminus \{x \in \mathbf{R} : x \leq 0 \vee x \geq 2\pi\} \ .$$

Moreover,

$$H_\theta(\Delta) = \mathbf{C} \setminus \{x \in \mathbf{R} : x \leq 0 \vee x \geq 2\pi\} \ ,$$

which means that h is subordinated to H_θ. $\qquad\square$

Observe that H_θ is a composition of a univalent starlike function and an affine function, and has real coefficients. Thus, H_θ is typically real.

Now we can state the main result.

Theorem 2.2 *The Koebe domain for the class $T(r)$, $r \in (0, 1)$, is a bounded domain, symmetric with respect to the real axis. Its boundary in the upper half plane is given by the polar equation $w = \varrho(\theta)e^{i\theta}$, $\theta \in [0, \pi]$, where*

$$\varrho(\theta) = \begin{cases} \frac{1}{4}(1+r)^2 & , \theta = 0 \\ r\frac{\sin(\theta + \Psi(\theta))}{\sin(\Psi(\theta))} & , \theta \in (0, \pi) \\ \frac{1}{4}(1-r)^2 & , \theta = \pi \end{cases} \tag{2.8}$$

and

$$\Psi(\theta) = 4\theta\left(1 - \frac{\theta}{\pi}\right)\frac{r}{(1-r)^2 + 4r\frac{\theta}{\pi}} \ . \tag{2.9}$$

Proof Assume that $f \in T(r)$, $r \in (0,1)$, and f omits 2 values, $\varrho e^{i\theta}$ and $\varrho e^{-i\theta}$, $\varrho > 0$, $\theta \in (0, \pi)$. Although, at the beginning, we do not make any assumption about omitted points, later on we find a correspondence between ϱ and θ, which is essentially important in the proof.

The coefficients of f are real. For this reason, the Koebe set for $T(r)$ is symmetric with respect to the real axis and it is enough to find the upper half of this set.

Firstly, we shall find a typically real function that omits only 2 conjugated values having arguments θ and $-\theta$, respectively. We associate with f the function h of the form (2.1). From Theorem 2.1 the function h satisfies conditions 1–4 of this theorem. By Lemma 2.1, $h \prec H_\theta$, where H_θ is a typically real function defined by (2.7). If, additionally, we assume the same Montel normalization of h and H_θ, i.e.

$$h(r) = H_\theta(r) \ , \tag{2.10}$$

then, combining (2.7) and (2.3), we obtain the following relation connecting ϱ and θ:

$$\varrho = r \frac{\sin\left(\theta + 4\theta \left(1 - \frac{\theta}{\pi}\right) \frac{r}{(1-r)^2 + 4r\frac{\theta}{\pi}}\right)}{\sin\left(4\theta \left(1 - \frac{\theta}{\pi}\right) \frac{r}{(1-r)^2 + 4r\frac{\theta}{\pi}}\right)} \ . \tag{2.11}$$

Applying Theorem 2.1 once again, we can see that there exists a function $F_\theta \in T(r)$ such that F_θ omits $\varrho e^{i\theta}$ and $\varrho e^{-i\theta}$, with $\varrho = \varrho(\theta)$ given by (2.8). The formula connecting F_θ and H_θ is of the form

$$F_\theta(z) = \varrho(\theta) \frac{e^{i\theta} - e^{-i\theta} e^{iH_\theta(z)}}{1 - e^{iH_\theta(z)}} \ , \tag{2.12}$$

or equivalently,

$$F_\theta(z) = \varrho(\theta) \left(\cos\theta - \sin\theta \cot\left(\frac{1}{2} H_\theta(z)\right)\right) \ .$$

Since $H_\theta(\Delta)$ is the complement of 2 horizontal rays with endpoints in 0 and 2π, the function $\cot\left(\frac{1}{2} H_\theta(z)\right)$ maps Δ onto the whole plane without 2 points i and $-i$. Hence

$$F_\theta(\Delta) = \mathbf{C} \setminus \left\{\varrho(\theta) e^{i\theta}, \varrho(\theta) e^{-i\theta}\right\} \ . \tag{2.13}$$

Moreover, F_θ is locally univalent in Δ and univalent on $(-1, 1)$.

Let us denote by K a domain containing the origin, which is symmetric with respect to the real axis, and its boundary in the upper half plane is given in polar coordinates by (2.8). We claim that K is the Koebe domain for $T(r)$.

Let f be an arbitrary function in $T(r)$ such that f omits $\varrho e^{i\theta}$ and $\varrho e^{-i\theta}$. From (2.13) we get

$$\frac{\varrho(\theta)}{\varrho} f \prec F_\theta \ . \tag{2.14}$$

This means that there exists an analytic function ω such that $|\omega(z)| < 1$ and

$$\frac{\varrho(\theta)}{\varrho} f(z) = F_\theta(\omega(z)) \ .$$

Since f and F_θ are typically real, ω is also typically real. It is known that any typically real function is strictly increasing on $(-1, 1)$. Hence

$$\frac{\varrho(\theta)}{\varrho} r = \frac{\varrho(\theta)}{\varrho} f(r) = F_\theta(\omega(r)) \le F_\theta(r) = r \ , \tag{2.15}$$

which means that for $\theta \in (0, \pi)$ there is

$$\varrho \ge \varrho(\theta) \ . \tag{2.16}$$

Taking into account suitable limits, one can observe that functions F_θ for $\theta = 0$ and $\theta = \pi$ are of the form $F_0(z) = \frac{z(1+r)^2}{(1+z)^2}$ and $F_\pi(z) = \frac{z(1-r)^2}{(1-z)^2}$, respectively. These functions belong to $T(r)$ and satisfy $F_0(1) = \frac{1}{4}(1+r)^2$ and $F_\pi(-1) = -\frac{1}{4}(1-r)^2$. Moreover, $F_0(\Delta) = \mathbf{C} \setminus \left[\frac{1}{4}(1+r)^2, \infty\right)$ and $F_\pi(\Delta) = \mathbf{C} \setminus \left(-\infty, -\frac{1}{4}(1-r)^2\right]$.

If now $f \in T(r)$ omits positive real value ϱ, then (2.14) and (2.15) are still true with $\theta = 0$. Consequently, $\varrho \ge \frac{1}{4}(1+r)^2$. In an analogous way one can show that $\varrho \ge \frac{1}{4}(1-r)^2$ for $\theta = \pi$.

We have proven that (2.16) holds for all $\theta \in [0, \pi]$. This means that every function $f \in T(r)$ omitting $\varrho e^{i\theta}$ and $\varrho e^{-i\theta}$ takes each value from K.

Finally, observe that a function

$$g_{p,\theta}(z) = \frac{\varrho}{\varrho(\theta)} F_\theta(pz) \ , \ p \in [0, 1] \ ,$$

for a fixed $\theta \in [0, \pi]$ and $\varrho > \varrho(\theta)$, is typically real and $g_{p,\theta}(r) = \frac{\varrho}{\varrho(\theta)} F_\theta(pr)$. This expression takes all values in the interval $[0, \frac{\varrho}{\varrho(\theta)} F_\theta(r)] = [0, \frac{\varrho}{\varrho(\theta)} r]$, while p varies in $[0, 1]$, because F_θ is typically real and univalent in $(-1, 1)$. Therefore, there exists $p_0 \in (0, 1)$ such that $g_{p_0,\theta}(r) = r$, which means that $g_{p_0,\theta} \in T(r)$. Moreover, from $F_\theta(z) \ne \varrho(\theta)e^{i\theta}$ for $z \in \Delta$ we obtain $g_{p_0,\theta}(z) \ne \varrho e^{i\theta}$ for $z \in \Delta$. From the above we conclude that every value $\varrho e^{i\theta}$, $\varrho > \varrho(\theta)$ is omitted by a suitable chosen function $g_{p,\theta}$. This means K is the Koebe domain for $T(r)$. $\hfill\square$

Remark 2.1 *In his book [7], Montel suggested the normalization (1.2) rather than (1.3). In this case the formula (2.8) in Theorem 2.2 is slightly different:*

$$\varrho(\theta) = \begin{cases} \frac{1}{4r}(1+r)^2 & , \theta = 0 \\ \frac{\sin(\theta + \Psi(\theta))}{\sin(\Psi(\theta))} & , \theta \in (0, \pi) \\ \frac{1}{4r}(1-r)^2 & , \theta = \pi \end{cases} \ , \tag{2.17}$$

while the formula for $\Psi(\theta)$ is the same as in Theorem 2.2, i.e. (2.9).

Observe that

$$\lim_{r \to 0^+} \Psi(\theta) = 0 \quad \text{and} \quad \lim_{r \to 0^+} \frac{\sin \Psi(\theta)}{r} = 4\theta \left(1 - \frac{\theta}{\pi}\right) \ ,$$

and consequently

$$\lim_{r \to 0^+} \varrho(\theta) = \varrho_0(\theta) \ ,$$

where

$$\varrho_0(\theta) = \begin{cases} \frac{\sin\theta}{4\theta\left(1-\frac{\theta}{\pi}\right)} & , \theta \in (0,\pi) \\ \frac{1}{4} & , \theta = 0 \vee \theta = \pi \, . \end{cases} \tag{2.18}$$

For this reason from Theorem 2.2 we conclude the known result (see [1]) for the class T.

Corollary 2.1 *The Koebe domain for the class T is a bounded domain, symmetric with respect to the real axis. Its boundary in the upper half plane is given by the polar equation $w = \varrho_0(\theta)e^{i\theta}$, $\theta \in [0,\pi]$, where $\varrho_0(\theta)$ is given by (2.18).*

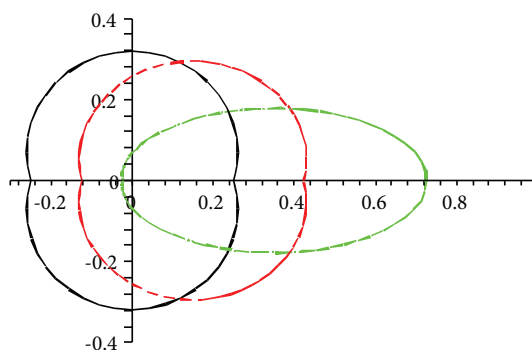This set and Koebe sets for $T(r)$, when $r = 1/4$ and $r = 2/3$, are shown in the Figure.

Figure. Koebe domains for $T(1/4)$ (green line), $T(2/3)$ (red line), and T (black line).

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