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Hölder regularity for weak solutions of diagonal divergence quasilinear degenerate elliptic systems

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Abstract: In this paper, we establish Hölder regularity for weak solutions of a class of diagonal divergence quasilinear degenerate elliptic systems of Hörmander's vector fields when the coefficients belong to the class of VMO_X functions with respect to x and uniformly with respect to u, and the lower order terms satisfy a natural growth condition.

Key words: Quasilinear degenerate elliptic system, Hölder regularity, Hörmander's vector fields, VMO_X function, natural growth

1. Introduction

Let $X = (X_1, \cdots, X_q)$ be a family of real smooth vector fields in a neighborhood $\tilde{\Omega}$ of some bounded domain $\Omega \subset \mathbb{R}^n$ $(q \leqslant n)$ with the form

$$X_j = \sum_{k=1}^{n} b_{jk} \frac{\partial}{\partial x_k}, \quad j = 1, 2, \cdots, q,$$

where $b_{jk} \in C^\infty$, and satisfying Hörmander's condition (see Section 2) free up to the order s. The paper is devoted to the study of the partial regularity of weak solutions to the following diagonal divergence quasilinear degenerate elliptic system constructed by Hörmander's vector fields $\{X_j\}(j = 1, \cdots, q)$

$$-X_\alpha^*(a_i^{\alpha\beta}(x, u)X_\beta u^i) = g_i(x, u, Xu) - X_\alpha^* f_i^\alpha(x), \tag{1.1}$$

where $i = 1, 2, \cdots, N$; $\alpha, \beta = 1, 2, \cdots, q$; $X_j^* = -X_j + c_j$ $(c_j = -\sum_{k=1}^{n} \frac{\partial b_{jk}}{\partial x_k} \in C^\infty(\Omega))$ is the transposed vector field of X_j, and Ω is the bounded domain in $\mathbb{R}^n$.

In recent years, the regularity of weak solutions for divergence linear and nonlinear eliptic equations or systems in the classic Euclidean spaces $(q = n, X_i = \frac{\partial}{\partial x_i})$ has been widely considered. De Giorgi [3] and Nash [18] investigated Hölder regularity for weak solutions of divergence linear elliptic equations with bounded and measurable coefficients by applying the technique of De Giorgi–Nash–Moser's iteration. Then many authors

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obtained analogous results [12,14,16] for divergence nonlinear elliptic equations. In order to treat partial Hölder regularity and get a sharp Hölder exponent of weak solutions of nonlinear systems, authors in [10,11,19] always derived a higher integrability for weak solutions by using the reverse Hölder inequality. In the discussions of partial Hölder regularity of quasilinear equations and systems, one should consider what growth conditions the lower order terms satisfy. The authors of [22] exhibited a sharp Hölder exponent for weak solutions of the elliptic systems with discontinuous coefficients and the natural condition.

Recently, the regularity of divergence linear degenerate elliptic equations or systems of Hörmander's vector fields has received extensive attention. Di Fazio and Fanciullo in [4] showed gradient estimates in Morrey spaces when $p=2$ and Hölder regularity by using an isomorphic relationship for a divergence linear degenerate elliptic system in Lipschitz vector fields. The interior Hölder regularity for weak solutions of linear degenerate elliptic systems was concluded by Dong and Niu in [5] with energy estimates and reverse Hölder inequality on the homogeneous spaces.

For nonlinear degenerate elliptic systems of Hörmander's vector fields, Xu and Zuily [20] attained interior regularity for weak solutions. Partial regularity for weak solutions was examined by Gao and Niu in [6].

Our purpose in this paper is to establish Hölder regularity for weak solutions of (1.1), where the coefficients of (1.1) satisfy VMO_X conditions in x and are continuous with respect to u, and the lower order terms satisfy a natural growth condition. To do so, we first prove a Caccioppoli inequality (see Section 3) and then consider the homogeneous system with constant coefficients and the inhomogeneous system related to (1.1), respectively. Finally, partial Hölder regularity with a sharp Hölder exponent for weak solutions of (1.1) is deduced by using Morrey's lemma in our setting.

To state the main result of the paper, let us first specify the definitions of Sobolev spaces, Hölder spaces, and BMO_X and VMO_X function spaces induced by Hörmander's vector fields. Then the assumptions on $a_i^{\alpha\beta}(z,u)$, $g_i(z,u,Xu)$ and $f_i^\alpha(z)$ in (1.1) are listed.

Definition 1.1 (Sobolev spaces) *Let $1\leqslant p\leqslant+\infty$, k be a positive integer. If $u\in L^p(\Omega,\mathbb{R}^N)$ satisfies*

$$\|u\|_{S_X^{k,p}(\Omega,\mathbb{R}^N)}\equiv\|u\|_{L^p(\Omega,\mathbb{R}^N)}+\sum_{h=1}^{k}\sum_{j_h=1}^{q}\|X_{j_1}X_{j_2}\cdots X_{j_h}u\|_{L^p(\Omega,\mathbb{R}^N)}<+\infty,$$

then we say that u belongs to the Sobolev space $S_X^{k,p}(\Omega,\mathbb{R}^N)$.

The space $S_{X,0}^{k,p}(\Omega,\mathbb{R}^N)$ is the closure of $C_0^\infty(\Omega,\mathbb{R}^N)$ in $S_X^{k,p}(\Omega,\mathbb{R}^N)$ with respect to the above norm. If $p=2$ and $k=1$, then we simply denote $S_X^{1,2}(\Omega,\mathbb{R}^N)$ by $S_X^1(\Omega,\mathbb{R}^N)$.

Definition 1.2 *Let $\alpha\in(0,1)$. The space $C_X^{0,\alpha}(\bar\Omega,\mathbb{R}^N)$ is the set of functions satisfying*

$$\|u\|_{C_X^{0,\alpha}(\bar\Omega,\mathbb{R}^N)}=\sup_{\Omega}|u|+\sup_{\bar\Omega}\frac{|u(x)-u(y)|}{[d(x,y)]^\alpha}<+\infty.$$

Clearly, it is a Banach space under the norm above.

Definition 1.3 (BMO_X and VMO_X spaces) *Let $u\in L^1_{loc}(\Omega,\mathbb{R}^N)$. If*

$$\|u\|_{BMO_X(\Omega,\mathbb{R}^N)}=\sup_{x_0\in\Omega,0<R<d_0}\frac{1}{|\Omega\cap B(x_0,R)|}\int_{\Omega\cap B(x_0,R)}|u-u_R|dx<+\infty,$$

then we say that $u \in BMO_X(\Omega, \mathbb{R}^N)$ *(Bounded Mean Oscillation). Moreover, if* $u \in BMO_X(\Omega, \mathbb{R}^N)$ *and*

$$\eta_R(u) = \sup_{x_0 \in \Omega, 0<\rho<R} \frac{1}{|\Omega \cap B(x_0,\rho)|} \int_{\Omega \cap B(x_0,\rho)} |u - u_\rho| dx \to 0, \quad R \to 0,$$

then we say that $u \in VMO_X(\Omega, \mathbb{R}^N)$ *(Vanishing Mean Oscillation).*

The following are the relevant assumptions.

(H1) (ellipticity condition) there exists a positive constant δ*, such that for a.e.* $x \in \Omega$ *and for any* $\xi \in \mathbb{R}^{qN}$*,*

$$a_i^{\alpha\beta}(x,u)\xi_\alpha^i \xi_\beta^i \geqslant \delta |\xi|^2;$$

(H2) $(VMO_X \cap L^\infty)$ *assume that* $a_i^{\alpha\beta}(x,u_0) \in VMO_X \cap L^\infty(\Omega)$ *uniformly on* x *for some fixed* $u_0 \in \mathbb{R}^N$*, that is,*

$$\lim_{R\to 0} \eta_R \left(a_i^{\alpha\beta}(\cdot, u_0) \right) = 0,$$

and there exists a constant $L > 0$*, such that for any* $x \in \Omega$ *and* $u \in \mathbb{R}^N$*, it holds* $\left| a_i^{\alpha\beta}(x,u) \right| \leqslant L$*;*

(H3) (continuity) there exist a positive constant c *and a continuous concave function* $\omega : \mathbb{R}^+ \to \mathbb{R}^+$*,* $\omega(0) = 0$*,* $0 \leqslant \omega \leqslant 1$*, such that for any* $x \in \Omega$*,* $u, v \in \mathbb{R}^N$*,*

$$\left| a_i^{\alpha\beta}(x,u) - a_i^{\alpha\beta}(x,v) \right| \leqslant c\omega\left(|u-v|^2 \right);$$

(H4) (natural growth condition) let $u \in S_X^1(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N)$*,* $f_i^\alpha \in L^m(\Omega)$ *with* $m > \frac{pQ}{2}$ *(*$p \in [2, 2 + \frac{2Q}{2+Q}\varepsilon)$*,* Q *is the homogeneous dimension relative to* Ω*, see Section 2),* $g_i(x,u,Xu)$ *satisfies*

$$|g_i(x,u,Xu)| \leqslant \mu(M) \left(|Xu|^2 + g^i(x) \right),$$

where $g^i(x) \in L^m(\Omega)(i = 1, 2, \ldots, N)$*,* $M = \sup_{x\in\Omega} |u(x)|$*.*

Definition 1.4 *If* $u \in S_X^1(\Omega, \mathbb{R}^N)$ *and for any* $\varphi \in C_0^\infty(\Omega, \mathbb{R}^N)$*,*

$$-\int_\Omega a_i^{\alpha\beta}(x,u) X_\beta u^i X_\alpha \varphi^i dx = \int_\Omega (g_i(x,u,Xu)\varphi^i - f_i^\alpha X_\alpha \varphi^i) dx,$$

then we say that u *is a weak solution of (1.1).*

Our main result in the paper is as follows.

Theorem 1.5 *Let* $u \in S_X^1(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N)$ *be a weak solution of (1.1). Suppose that assumptions (H1–H4) are satisfied and* $2\mu(M)M < \delta$ *for any* $|u| \leqslant M$*. Then there exists an open subset* $\Omega_0 \subset \Omega$ *with* $\dim_H(\Omega \backslash \Omega_0) \leqslant Q - 2$ *(here* $\dim_H(\Omega\backslash\Omega_0)$ *is the Hausdorff dimension of the set* $\Omega\backslash\Omega_0$*) such that*

$$u \in C^{0,k}(\Omega_0, \mathbb{R}^N),$$

where $k = 1 - \frac{pQ}{2m}$*.*

This paper is organized as follows. In section 2, we introduce Hörmander's vector fields and properties. Some definitions and known conclusions are described. Section 3 is devoted to establishing a Caccioppoli inequality and 2 useful results, Lemma 3.1 and 3.2, for the homogeneous system with constant coefficients and the inhomogeneous system related to (1.1), respectively. The proof of Theorem 1.1 is given in Section 4.

2. Preliminaries and some known results

For every multi-index $\beta = (\beta_1, \beta_2, \cdots, \beta_d)(1 \leqslant \beta_i \leqslant q,\ i = 1, \cdots, d,\ |\beta| = d)$, write the commutator of length d by $\mathrm{X}_\beta = \left[X_{\beta_d}, \left[X_{\beta_{d-1}}, \cdots [X_{\beta_2}, X_{\beta_1}], \cdots\right]\right]$.

Definition 2.1 *If for every* $x \in \Omega \subset \mathbb{R}^n$, $\{X_\beta(x)\}_{|\beta| \leqslant s}$ *spans* $\mathbb{R}^n$, *then we say that the system* $X = (X_1, \cdots, X_q)$ *satisfies the Hörmander's condition of step* s.

Following [20], we can assume that Hörmander's vector fields $X_1, \cdots, X_q$ *are free up to the order* s. *For every multi-index* $I = (i_1, i_2, \ldots, i_k)$, *the length of* I *is denoted by* $|I| = k$. *If* $i_k \leqslant q$, *then we set*

$$X_I = X_{i_i} X_{i_2} \ldots X_{i_k}.$$

Definition 2.2 (Carnot–Carathéodory distance) *Let* Ω *be a bounded set in* $\mathbb{R}^n$. *An absolutely continuous curve* $\gamma : [0, T] \to \Omega$ *is called a sub-unit curve with respect to the system* $X = (X_1, \cdots, X_q)$, *if* $\gamma'(t)$ *exists for a.e.* t *and satisfies*

$$< \gamma'(t), \xi >^2 \leqslant \sum_{j=1}^{q} < X_j(\gamma(t)), \xi >^2, \quad a.e.\ t \in [0, T].$$

We denote the length of this curve by $l_S(\gamma) = T$. *Given any* $x, y \in \Omega$, *let* $\Phi(x, y)$ *be the collection of all sub-unit curves connecting* x *and* y. *The Carnot–Carathéodory distance induced by the system X is defined by*

$$d_{\mathrm{X}}(x, y) = \inf\{l_S(\gamma) : \gamma \in \Phi(x, y)\}.$$

Let us denote by

$$B(x_0, R) = \{x \in \Omega : d(x_0, x) < R\}$$

the metric ball centered at x_0 *of radius* R. *When one does not consider the center of a ball, we simply write* B_R *instead of* $B(x, R)$.

It is well known that the doubling property (see [17]) for metric balls holds true, i.e., there exist positive constants C_D *and* R_D, *such that for any* $x_0 \in \Omega$, $0 < 2R < R_D$, $B_{2R} \subset \Omega$, *one has*

$$|B(x_0, 2R)| \leqslant C_D |B(x_0, R)|.$$

Furthermore, it follows that for any $R \leqslant R_D$ *and* $t \in (0, 1)$,

$$|B_{tR}| \geqslant C_D^{-1} t^Q |B_R|.$$

The number $Q = \log_2 C_D$ *is called a locally homogeneous dimension relative to* Ω. *As in [20], we assume that there exist 2 positive constants* C_1 *and* C_2, *such that*

$$C_1 R^Q \leqslant |B_R| \leqslant C_2 R^Q. \tag{2.1}$$

Theorem 2.3 (Sobolev–Poincaré inequality, see [7], [15]) *For the open set Ω, there exist positive constants R_0 and c, such that for any $0 < R \leqslant R_0$, $B_R \subset \Omega$ and $u \in C^\infty(\overline{B_R})$, it holds*

$$\left(\frac{1}{|B_R|}\int_{B_R}|u-u_R|^{p'}dx\right)^{\frac{1}{p'}} \leqslant cR\left(\frac{1}{|B_R|}\int_{B_R}|Xu|^p dx\right)^{\frac{1}{p}},$$

where $1 < p < Q$, $1 \leqslant p' \leqslant \frac{pQ}{Q-p}$, $u_R = \frac{1}{|B_R|}\int_{B_R}u(x)dx$, R_0 and c depend on Ω.

In particular, if $p = p'$, then

$$\int_{B_R}|u-u_R|^p\,dx \leqslant cR^p\int_{B_R}|Xu|^p\,dx \leqslant c\int_{B_R}|Xu|^p\,dx. \tag{2.2}$$

When $u \in C_0^\infty(\overline{B_R})$, one has

$$\int_{B_R}|u|^p\,dx \leqslant cR^p\int_{B_R}|Xu|^p\,dx \leqslant c\int_{B_R}|Xu|^p\,dx. \tag{2.3}$$

Theorem 2.4 (reverse Hölder inequality, see [8], [21]) *Let $\tilde{g}, \tilde{f} \geqslant 0$ satisfy*

$$\tilde{g} \in L^{\tilde{q}}(\Omega)(\tilde{q} > 1), \tilde{f} \in L^{q'}(\Omega)(q' > \tilde{q}).$$

Fix a ball $B_{R_0} = B(x_0, R_0)$ and assume that for any $x \in B_{R_0}$ and $R < \frac{1}{2}dist(x, \partial B_{R_0})$, there exist $b > 0$ and $\theta \in [0, 1)$, such that

$$\frac{1}{|B_R|}\int_{B_R}\tilde{g}^{\tilde{q}}dx \leqslant b\left[\left(\frac{1}{|B_{4R/3}|}\int_{B_{4R/3}}\tilde{g}dx\right)^{\tilde{q}} + \frac{1}{|B_{4R/3}|}\int_{B_{4R/3}}\tilde{f}^{\tilde{q}}dx\right] + \theta\frac{1}{|B_{4R/3}|}\int_{B_{4R/3}}\tilde{g}^{\tilde{q}}dx.$$

Then there exist $\varepsilon > 0$ and $c > 0$ such that for any $r \in [\tilde{q}, \tilde{q}+\varepsilon)$, it yields $\tilde{g} \in L^r_{loc}(B_{R_0})$. Moreover, for any $B_{2R} \subset\subset \Omega$, we have

$$\left(\frac{1}{|B_R|}\int_{B_R}\tilde{g}^r dx\right)^{\frac{1}{r}} \leqslant c\left[\left(\frac{1}{|B_{2R}|}\int_{B_{2R}}\tilde{g}^{\tilde{q}}dx\right)^{\frac{1}{\tilde{q}}} + \left(\frac{1}{|B_{2R}|}\int_{B_{2R}}\tilde{f}^r dx\right)^{\frac{1}{r}}\right],$$

where c and ε are positive constants depending only on $b, \theta, \tilde{q}$, and q'.

Lemma 2.5 (see [1, 9, 13, 16]) *Let $H(\rho)$ be a nonnegative almost increasing function in $[0, R_0]$, $R_0 = dist(x_0, \partial\Omega)$, satisfying that for any $0 < \rho < R \leqslant R_0$,*

$$H(\rho) \leqslant A\left(\left(\frac{\rho}{R}\right)^\alpha + \varepsilon\right)H(R) + BR^\beta,$$

where A, B, α, β are positive constants and $\alpha > \beta$. Then there exist positive constants $\varepsilon_0 = \varepsilon_0(A, \alpha, \beta)$ and $c = c(A, \alpha, \beta)$, such that if $\varepsilon < \varepsilon_0$, then

$$H(\rho) \leqslant c\left(\left(\frac{\rho}{R}\right)^\beta H(R) + B\rho^\beta\right), \text{ for any } 0 < \rho < R < R_0.$$

The following Morrey-type lemma is from [23].

Lemma 2.6 *Suppose that* $u \in S_X^1(B_R, \mathbb{R}^N)$ *satisfies the following inequality*

$$\int_{B_R} |Xu|^2 dx \leqslant M_0^2 R^{Q-2+2k},$$

for any $B_R \subset \Omega$*, where* $M_0 > 0$ *and* $k \in (0,1)$*. Then we have* $u \in C_{loc}^k(\Omega, \mathbb{R}^N)$*, and for any* $\Omega' \subset\subset \Omega$*, there holds*

$$\sup_{\Omega'} |u| + \sup_{x_1, x_2 \in \Omega', x_1 \neq x_2} \frac{|u(x_1) - u(x_2)|}{d_X(x_1, x_2)^k} \leqslant c\left(M_0 + \|u\|_{L^2(\Omega)}\right),$$

where $c = c(Q, k, \Omega', \Omega) > 0$*.*

Definition 2.7 *Let* $E \subseteq \mathbb{R}^n$*,* $0 \leqslant \alpha < +\infty$*,* $0 < \delta \leqslant +\infty$*,* $\{F_j\}$ *be a system of open sets of* $\mathbb{R}^n$*. We say that*

$$H_\alpha(E) = \lim_{\delta \to 0} H_\alpha^\delta(E) = \sup_{\delta > 0} H_\alpha^\delta(E)$$

is the α*-Hausdorff dimension of* E*, where* $H_\alpha^\delta(E) = \omega_\alpha 2^{-\alpha} \inf\{\sum_j (diam\ F_j)^\alpha | \bigcup_j F_j \supseteq E, diam\ F_j < \delta\}, \omega_\alpha = \Gamma^\alpha\left(\frac{1}{2}\right)/\Gamma\left(\frac{\alpha}{2}+1\right)$.

Lemma 2.8 *Let* $u \in L_{loc}^1(\Omega, \mathbb{R}^N)$*. If for* $0 \leqslant \alpha < Q$*, it has*

$$E_\alpha = \left\{ x \in \Omega |\limsup_{\rho \to 0} \rho^{-\alpha} \int_{B_\rho(x)} |u(x)| dx > 0 \right\},$$

then $H_\alpha(E_\alpha) = 0$*.*

We omit its proof, since it is similar to [2]. The following result is classic.

Lemma 2.9 (Jensen inequality) *Suppose that* $f : U \to \mathbb{R}$ *is an integrable function for any bounded open subset* $U \subset \mathbb{R}^n$*. If* $\omega : \mathbb{R} \to \mathbb{R}$ *is a concave function, then*

$$\frac{1}{|U|} \int_U \omega(f) dx \leqslant \omega\left(\frac{1}{|U|} \int_U f dx\right).$$

Remark 2.10 *The constant* c *takes different values in different places. To simplify the notations, we denote* $\sqrt{\sum_{i=1}^q |X_i u|^2}$ *and* $B(x,R)$ *by* $|Xu|$ *and* B_R*, respectively. We also write* $u_R = \frac{1}{|B_R|}\int_{B_R} u(x)dx$*,* $\left(a_i^{\alpha\beta}(\cdot, u_R)\right)_R = \frac{1}{|B_R|}\int_{B_R} a_i^{\alpha\beta}(x, u_R) dx$*.*

3. Auxiliary theorems

Theorem 3.1 (Caccioppoli inequality) *Let $u \in S^1_X(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N)$ be a weak solution of (1.1). Then for any ρ, R: $0 < \rho < R$, $B_R \subset\subset \Omega$, we have*

$$\int_{B_\rho} |Xu|^2 \, dx \leqslant \frac{c}{(R-\rho)^2} \int_{B_R} |u - u_R|^2 \, dx + c \int_{B_R} \left(\sum_i \left| g^i \right|^2 + |f|^2 \right) dx,$$

where the positive constant c depends on $\delta, Q, L, M, \mu(M)$.

Proof Let $\eta \in C_0^\infty(B_R)$ be a cutoff function satisfying

$$0 \leqslant \eta \leqslant 1 \text{ in } B_R, \eta = 1 \text{ on } B_\rho, |X\eta| \leqslant \frac{c}{R-\rho}.$$

Multiplying both sides of (1.1) by $(u^i - u_R)\eta^2$ and integrating on B_R leads to

$$-\int_{B_R} a_i^{\alpha\beta}(x,u) X_\beta u^i X_\alpha \left((u^i - u_R)\eta^2 \right) dx = \int_{B_R} \left[g_i (u^i - u_R)\eta^2 - f_i^\alpha X_\alpha \left((u^i - u_R)\eta^2 \right) \right] dx,$$

that is

$$\begin{aligned} &-\int_{B_R} \left(2a_i^{\alpha\beta}(x,u)\eta(u^i - u_R) X_\beta u^i X_\alpha \eta + a_i^{\alpha\beta}(x,u)\eta^2 X_\beta u^i X_\alpha u^i \right) dx \\ &= \int_{B_R} \left(g_i (u^i - u_R)\eta^2 - 2\eta(u^i - u_R) f_i^\alpha X_\alpha \eta - \eta^2 f_i^\alpha X_\alpha u^i \right) dx. \end{aligned}$$

Using (H1), (H4), and Young's inequality, one has

$$\begin{aligned} &\delta \int_{B_R} |Xu|^2 \eta^2 dx \leqslant \int_{B_R} a_i^{\alpha\beta}(x,u)\eta^2 X_\beta u^i X_\alpha u^i dx \\ &\leqslant 2L \int_{B_R} \left| (u^i - u_R) X_\alpha \eta \right| \left| \eta X_\beta u^i \right| dx + \int_{B_R} |g_i| \left| (u^i - u_R) \right| \eta^2 dx \\ &\quad + 2\int_{B_R} |\eta f_i^\alpha| \left| (u^i - u_R) X_\alpha \eta \right| dx + \int_{B_R} |\eta f_i^\alpha| \left| \eta X_\alpha u^i \right| dx \\ &\leqslant c_\varepsilon \int_{B_R} |u - u_R|^2 |X\eta|^2 dx + \varepsilon \int_{B_R} |Xu|^2 \eta^2 dx \\ &\quad + \int_{B_R} \mu(M) \left(|Xu|^2 + \sum_i \left| g^i \right| \right) |u - u_R| \eta^2 dx + c_\varepsilon \int_{B_R} |f|^2 \eta^2 dx \\ &\quad + \varepsilon \int_{B_R} |u - u_R|^2 |X\eta|^2 dx + c_\varepsilon \int_{B_R} |f|^2 \eta^2 dx + \varepsilon \int_{B_R} |Xu|^2 \eta^2 dx \\ &\leqslant c_\varepsilon \int_{B_R} |u - u_R|^2 |X\eta|^2 dx + 2\varepsilon \int_{B_R} |Xu|^2 \eta^2 dx + c_\varepsilon \int_{B_R} |f|^2 \eta^2 dx \\ &\quad + \mu(M) \int_{B_R} |Xu|^2 |u - u_R| \eta^2 dx + \mu(M) \int_{B_R} \sum_i \left| g^i \right| |u - u_R| \eta^2 dx \\ &\leqslant c_\varepsilon \int_{B_R} |u - u_R|^2 |X\eta|^2 dx + 2\varepsilon \int_{B_R} |Xu|^2 \eta^2 dx + c_\varepsilon \int_{B_R} |f|^2 \eta^2 dx \\ &\quad + 2\mu(M) M \int_{B_R} |Xu|^2 \eta^2 dx + \mu(M) c_\varepsilon \int_{B_R} \sum_i \left| g^i \right|^2 \eta^2 dx + \mu(M)\varepsilon \int_{B_R} |u - u_R|^2 \eta^2 dx, \end{aligned}$$

namely,

$$\begin{aligned} &(\delta - 2\mu(M)M - 2\varepsilon) \int_{B_R} |Xu|^2 \eta^2 dx \\ &\leqslant c_\varepsilon \int_{B_R} |u - u_R|^2 |X\eta|^2 dx + \mu(M)\varepsilon \int_{B_R} |u - u_R|^2 \eta^2 dx \\ &\quad + \mu(M) c_\varepsilon \int_{B_R} \sum_i \left| g^i \right|^2 \eta^2 dx + c_\varepsilon \int_{B_R} |f|^2 \eta^2 dx. \end{aligned}$$

Because of $2\mu(M)M < \delta$, we can choose ε small enough such that $\delta - 2\mu(M)M - 2\varepsilon > 0$, and obtain

$$\int_{B_R} |Xu|^2 \eta^2 dx \leqslant c \int_{B_R} |u - u_R|^2 \left(|X\eta|^2 + \eta^2 \right) dx + c \int_{B_R} \left(\sum_i \left| g^i \right|^2 + |f|^2 \right) \eta^2 dx.$$

By properties of η, this conclusion is proved. $\square$

By Theorem 3.1 and Theorem 2.2, we easily know that the following result is true by using a similar proof as in [5, Theorem 3.2].

Theorem 3.2 *Let $u \in S^1_X(\Omega,\mathbb{R}^N) \cap L^\infty(\Omega,\mathbb{R}^N)$ be a weak solution of (1.1). Then there exist constants $\varepsilon > 0$ and $c > 0$ such that for any $p \in [2, 2+\frac{2Q}{2+Q}\varepsilon)$, it follows $u \in S^{1,p}_{X,loc}(\Omega,\mathbb{R}^N)$ and*

$$\frac{1}{|B_R|}\int_{B_R} |Xu|^p\,dx \leqslant c\left[\left(\frac{1}{|B_{2R}|}\int_{B_{2R}} |Xu|^2\,dx\right)^{\frac{p}{2}} + \frac{1}{|B_{2R}|}\int_{B_{2R}}\left(\sum_i \left|g^i\right|^p + |f|^p\right)dx\right]$$

for any $B_{2R} \subset\subset \Omega$.

In order to prove Hölder regularity for weak solutions of (1.1), we need to consider the homogeneous system and the inhomogeneous system. Let v, $w = u - v$ be the weak solution of the following systems

$$\begin{cases} -X^*_\alpha\left(\left(a^{\alpha\beta}_i(\cdot,u_R)\right)_R X_\beta v^i\right) = 0, \; in\, B_R, \\ v = u, \qquad on\, \partial_p B_R, \end{cases} \tag{3.1}$$

and

$$\begin{cases} -X^*_\alpha\left(\left(a^{\alpha\beta}_i(\cdot,u_R)\right)_R X_\beta w^i\right) = -X^*_\alpha\left(\left(\left(a^{\alpha\beta}_i(\cdot,u_R)\right)_R - a^{\alpha\beta}_i(x,u)\right) X_\beta u^i\right) + g_i - X^*_\alpha f^\alpha_i, \; in\; B_R, \\ w = 0, \qquad on\; \partial_p B_R, \end{cases} \tag{3.2}$$

respectively. The system (3.1) allows the following result; see [5] for a detailed proof.

Lemma 3.3 *Let $v \in S^1_X(\Omega,\mathbb{R}^N) \cap L^\infty(\Omega,\mathbb{R}^N)$ be a weak solution of (3.1). Then for any $0 \leqslant \rho \leqslant R$, $B_R(x_0) \subset \Omega$, one has*

$$\int_{B_\rho} |Xv|^2 dx \leqslant c\left(\frac{\rho}{R}\right)^Q \int_{B_R} |Xv|^2 dx.$$

For the system (3.2), we have

Lemma 3.4 *Let $w \in S^1_{X,0}(\Omega,\mathbb{R}^N) \cap L^\infty(\Omega,\mathbb{R}^N)$ be the weak solution of (3.2). Then for any $0 \leqslant \rho \leqslant R$, $B_R(x_0) \subset \Omega$, it holds*

$$\int_{B_R} |Xw|^2 dx \leqslant c\int_{B_R} |Xu|^2 dx, \tag{3.3}$$

where the positive constant c depends on L and δ.

Proof Multiplying both sides of (3.1) by $u^i - w^i$ and integrating on B_R,

$$\int_{B_R}\left(a^{\alpha\beta}_i(\cdot,u_R)\right)_R X_\beta v^i X_\alpha\left(u^i - w^i\right) dx = 0.$$

It yields by inserting $v = u - w$ into the above equation that

$$\int_{B_R}\left(a^{\alpha\beta}_i(\cdot,u_R)\right)_R X_\beta\left(u^i - w^i\right) X_\alpha\left(u^i - w^i\right) dx = 0,$$

that is,

$$\begin{aligned}
&\textstyle\int_{B_R} \left(a_i^{\alpha\beta}(\cdot, u_R)\right)_R X_\alpha w^i X_\beta w^i dx \\
&= \textstyle\int_{B_R} \left(a_i^{\alpha\beta}(\cdot, u_R)\right)_R X_\alpha w^i X_\beta u^i dx + \int_{B_R} \left(a_i^{\alpha\beta}(\cdot, u_R)\right)_R X_\alpha u^i X_\beta w^i dx \\
&\quad - \textstyle\int_{B_R} \left(a_i^{\alpha\beta}(\cdot, u_R)\right)_R X_\alpha u^i X_\beta u^i dx.
\end{aligned}$$

By (H2) and Young's inequality,

$$\delta \int_{B_R} |Xw|^2 dx \leqslant c_\varepsilon \int_{B_R} |Xu|^2 dx + \varepsilon \int_{B_R} |Xw|^2 dx.$$

Taking ε small enough, the proof is ended. □

4. Proof of the main result

Proof of Theorem 1.1 Multiplying both sides of the system in (3.2) by w^i and integrating on B_R, it implies

$$\begin{aligned}
&\int_{B_R} \left(a_i^{\alpha\beta}(\cdot, u_R)\right)_R X_\beta w^i X_\alpha w^i dx \\
&= \int_{B_R} \left(\left(a_i^{\alpha\beta}(\cdot, u_R)\right)_R - a_i^{\alpha\beta}(x, u)\right) X_\beta u^i X_\alpha w^i dx + \int_{B_R} f_i^\alpha X_\alpha w^i dx - \int_{B_R} g_i w^i dx \\
&= \int_{B_R} \left(\left(a_i^{\alpha\beta}(\cdot, u_R)\right)_R - a_i^{\alpha\beta}(x, u_R)\right) X_\beta u^i X_\alpha w^i dx \\
&\quad + \int_{B_R} \left(a_i^{\alpha\beta}(x, u_R) - a_i^{\alpha\beta}(x, u)\right) X_\beta u^i X_\alpha w^i dx + \int_{B_R} \left(f_i^\alpha X_\alpha w^i - g_i w^i\right) dx \\
&\triangleq I + II + III.
\end{aligned} \tag{4.1}$$

By (H1), Young's inequality, and Hölder's inequality, we have

$$\begin{aligned}
I &= \int_{B_R} \left(\left(a_i^{\alpha\beta}(\cdot, u_R)\right)_R - a_i^{\alpha\beta}(x, u_R)\right) X_\beta u^i X_\alpha w^i dx \\
&\leqslant c_\varepsilon \int_{B_R} \left|\left(a_i^{\alpha\beta}(\cdot, u_R)\right)_R - a_i^{\alpha\beta}(x, u_R)\right|^2 |Xu|^2 dx + \varepsilon \int_{B_R} |Xw|^2 dx \\
&\leqslant c_\varepsilon \left(\int_{B_R} \left|\left(a_i^{\alpha\beta}(\cdot, u_R)\right)_R - a_i^{\alpha\beta}(x, u_R)\right|^{\frac{2p}{p-2}} dx\right)^{\frac{p-2}{p}} \left(\int_{B_R} |Xu|^p dx\right)^{\frac{2}{p}} \\
&\quad + \varepsilon \int_{B_R} |Xw|^2 dx.
\end{aligned} \tag{4.2}$$

Due to (H2),

$$\begin{aligned}
&\left(\int_{B_R}\left|\left(a_i^{\alpha\beta}(\cdot,u_R)\right)_R-a_i^{\alpha\beta}(x,u_R)\right|^{\frac{2p}{p-2}}dx\right)^{\frac{p-2}{p}}\\
&=\left(\int_{B_R}\left|\left(a_i^{\alpha\beta}(\cdot,u_R)\right)_R-a_i^{\alpha\beta}(x,u_R)\right|^{\frac{p+2}{p-2}}\left|\left(a_i^{\alpha\beta}(\cdot,u_R)\right)_R-a_i^{\alpha\beta}(x,u_R)\right|dx\right)^{\frac{p-2}{p}}\\
&\leqslant(2L)^{\frac{p+2}{p}}\left(\int_{B_R}\left|\left(a_i^{\alpha\beta}(\cdot,u_R)\right)_R-a_i^{\alpha\beta}(x,u_R)\right|dx\right)^{\frac{p-2}{p}}\\
&=(2L)^{\frac{p+2}{p}}|B_R|^{\frac{p-2}{p}}\left(\frac{1}{|B_R|}\int_{B_R}\left|\left(a_i^{\alpha\beta}(\cdot,u_R)\right)_R-a_i^{\alpha\beta}(x,u_R)\right|dx\right)^{\frac{p-2}{p}}\\
&\leqslant c\,|B_R|^{\frac{p-2}{p}}\left(\eta_R(a_i^{\alpha\beta}(x,u_R))\right)^{\frac{p-2}{p}}.
\end{aligned}$$

Putting the above into (4.2), it follows that

$$\begin{aligned}
I&\leqslant c_\varepsilon\,|B_R|^{\frac{p-2}{p}}\left(\eta_R(a_i^{\alpha\beta}(x,u_R))\right)^{\frac{p-2}{p}}\left(\int_{B_R}|Xu|^p dx\right)^{\frac{2}{p}}+\varepsilon\int_{B_R}|Xw|^2\,dx\\
&=c_\varepsilon\,|B_R|\left(\eta_R(a_i^{\alpha\beta}(x,u_R))\right)^{\frac{p-2}{p}}\left(\frac{1}{|B_R|}\int_{B_R}|Xu|^p dx\right)^{\frac{2}{p}}+\varepsilon\int_{B_R}|Xw|^2\,dx. \qquad (4.3)
\end{aligned}$$

By (H3), Young's inequality, and Hölder's inequality,

$$\begin{aligned}
II&=\int_{B_R}\left(a_i^{\alpha\beta}(x,u_R)-a_i^{\alpha\beta}(x,u)\right)X_\beta u^i X_\alpha w^i dx\\
&\leqslant c_\varepsilon\int_{B_R}\left|a_i^{\alpha\beta}(x,u_R)-a_i^{\alpha\beta}(x,u)\right|^2|Xu|^2\,dx+\varepsilon\int_{B_R}|Xw|^2\,dx\\
&\leqslant c_\varepsilon\int_{B_R}\omega^2\left(|u-u_R|^2\right)|Xu|^2\,dx+\varepsilon\int_{B_R}|Xw|^2\,dx\\
&\leqslant c_\varepsilon\left(\int_{B_R}\left(\omega\left(|u-u_R|^2\right)\right)^{\frac{2p}{p-2}}dx\right)^{\frac{p-2}{p}}\left(\int_{B_R}|Xu|^p\,dx\right)^{\frac{2}{p}}+\varepsilon\int_{B_R}|Xw|^2\,dx\\
&=c_\varepsilon\left(\int_{B_R}\left(\omega\left(|u-u_R|^2\right)\right)^{\frac{p+2}{p-2}}\omega\left(|u-u_R|^2\right)dx\right)^{\frac{p-2}{p}}\left(\int_{B_R}|Xu|^p\,dx\right)^{\frac{2}{p}}+\varepsilon\int_{B_R}|Xw|^2\,dx\\
&\leqslant c\,|B_R|\left(\frac{1}{|B_R|}\int_{B_R}\omega\left(|u-u_R|^2\right)dx\right)^{\frac{p-2}{p}}\left(\frac{1}{|B_R|}\int_{B_R}|Xu|^p\,dx\right)^{\frac{2}{p}}+\varepsilon\int_{B_R}|Xw|^2\,dx.
\end{aligned}$$

Using Lemma 2.4 leads to

$$\frac{1}{|B_R|}\int_{B_R}\omega\left(|u-u_R|^2\right)dx\leqslant\omega\left(\frac{1}{|B_R|}\int_{B_R}|u-u_R|^2\,dx\right).$$

It follows that

$$II\leqslant c\,|B_R|\,\omega^{\frac{p-2}{p}}\left(\frac{1}{|B_R|}\int_{B_R}|u-u_R|^2\,dx\right)\left(\frac{1}{|B_R|}\int_{B_R}|Xu|^p\,dx\right)^{\frac{2}{p}}+\varepsilon\int_{B_R}|Xw|^2\,dx. \qquad (4.4)$$

By (H4) and (2.3),

$$
\begin{aligned}
III &= \int_{B_R} \left(f_i^\alpha X_\alpha w^i - g_i w^i\right) dx \\
&\leqslant \int_{B_R} \left(|f_i^\alpha| \left|X_\alpha w^i\right| + |g_i| \left|w^i\right|\right) dx \\
&\leqslant \int_{B_R} |f_i^\alpha| \left|X_\alpha w^i\right| dx + \mu(M) \int_{B_R} \left(|Xu|^2 + g^i\right) |w^i| dx \\
&\leqslant c_\varepsilon \int_{B_R} |f|^2 dx + \varepsilon \int_{B_R} |Xw|^2 dx + \mu(M) \int_{B_R} |Xu|^2 |w| dx + \mu(M) \int_{B_R} \sum_i g^i |w| dx \\
&\leqslant c_\varepsilon \int_{B_R} |f|^2 dx + \varepsilon \int_{B_R} |Xw|^2 dx + \mu(M) \int_{B_R} |Xu|^2 |w| dx \\
&\quad + \mu(M) c_\varepsilon \int_{B_R} \sum_i \left|g^i\right|^2 dx + \mu(M)\varepsilon \int_{B_R} |w|^2 dx \\
&\leqslant \varepsilon \int_{B_R} |Xw|^2 dx + \mu(M) \int_{B_R} |Xu|^2 |w| dx + c \int_{B_R} \left(\sum_i \left|g^i\right|^2 + |f|^2\right) dx + \mu(M)\varepsilon R^2 \int_{B_R} |Xw|^2 dx \\
&\leqslant 2\varepsilon \int_{B_R} |Xw|^2 dx + \mu(M) \int_{B_R} |Xu|^2 |w| dx + c \int_{B_R} \left(\sum_i \left|g^i\right|^2 + |f|^2\right) dx. \qquad (4.5)
\end{aligned}
$$

In the light of (2.3) and (3.3),

$$
\begin{aligned}
&\int_{B_R} |Xu|^2 |w| dx \leqslant \left(\int_{B_R} |Xu|^p dx\right)^{\frac{2}{p}} \left(\int_{B_R} |w|^{\frac{p}{p-2}} dx\right)^{\frac{p-2}{p}} \\
&= |B_R| \left(\frac{1}{|B_R|} \int_{B_R} |Xu|^p dx\right)^{\frac{2}{p}} \left(\frac{1}{|B_R|} \int_{B_R} |w| \, |w|^{\frac{2}{p-2}} dx\right)^{\frac{p-2}{p}} \\
&\leqslant c\,|B_R| \left(\frac{1}{|B_R|} \int_{B_R} |Xu|^p dx\right)^{\frac{2}{p}} \left(\frac{1}{|B_R|} \int_{B_R} |w| dx\right)^{\frac{p-2}{p}} \\
&\leqslant c\,|B_R| \left(\frac{1}{|B_R|} \int_{B_R} |Xu|^p dx\right)^{\frac{2}{p}} \left(\frac{1}{|B_R|} \int_{B_R} |w|^2 dx\right)^{\frac{p-2}{2p}} \\
&\leqslant c\,|B_R| \left(\frac{1}{|B_R|} \int_{B_R} |Xu|^p dx\right)^{\frac{2}{p}} \left(\frac{R^2}{|B_R|} \int_{B_R} |Xw|^2 dx\right)^{\frac{p-2}{2p}} \\
&\leqslant c\,|B_R| \left(\frac{1}{|B_R|} \int_{B_R} |Xu|^p dx\right)^{\frac{2}{p}} \left(\frac{R^2}{|B_R|} \int_{B_R} |Xu|^2 dx\right)^{\frac{p-2}{2p}}.
\end{aligned}
$$

Returning to (4.5), we have

$$
\begin{aligned}
III \leqslant{}& c\,|B_R| \left(\frac{1}{|B_R|} \int_{B_R} |Xu|^p dx\right)^{\frac{2}{p}} \left(\frac{R^2}{|B_R|} \int_{B_R} |Xu|^2 dx\right)^{\frac{p-2}{2p}} \\
&+ 2\varepsilon \int_{B_R} |Xw|^2 dx + c \int_{B_R} \left(\sum_i \left|g^i\right|^2 + |f|^2\right) dx.
\end{aligned}
$$

Summing up these estimates and using (2.1), we get

$$\delta\int_{B_R}|Xw|^2\,dx\leqslant\int_{B_R}\left(a_i^{\alpha\beta}(\cdot,u_R)\right)_R X_\beta w^i X_\alpha w^i dx$$
$$\leqslant c_\varepsilon\,|B_R|\left(\eta_R(a_i^{\alpha\beta}(x,u_R))\right)^{\frac{p-2}{p}}\left(\frac{1}{|B_R|}\int_{B_R}|Xu|^p dx\right)^{\frac{2}{p}}+4\varepsilon\int_{B_R}|Xw|^2\,dx$$
$$+c\,|B_R|\,\omega^{\frac{p-2}{p}}\left(\frac{1}{|B_R|}\int_{B_R}|u-u_R|^2\,dx\right)\left(\frac{1}{|B_R|}\int_{B_R}|Xu|^p\,dx\right)^{\frac{2}{p}}$$
$$+\,c\,|B_R|\left(\frac{1}{|B_R|}\int_{B_R}|Xu|^p dx\right)^{\frac{2}{p}}\left(\frac{R^2}{|B_R|}\int_{B_R}|Xu|^2 dx\right)^{\frac{p-2}{2p}}+c\int_{B_R}\left(\sum_i\left|g^i\right|^2+|f|^2\right)dx$$
$$\leqslant c_\varepsilon\,|B_R|\left(\eta_R(a_i^{\alpha\beta}(x,u_R))\right)^{\frac{p-2}{p}}\left(\frac{1}{|B_R|}\int_{B_R}|Xu|^p dx\right)^{\frac{2}{p}}+4\varepsilon\int_{B_R}|Xw|^2\,dx$$
$$+c\,|B_R|\,\omega^{\frac{p-2}{p}}\left(R^{2-Q}\int_{B_R}|Xu|^2\,dx\right)\left(\frac{1}{|B_R|}\int_{B_R}|Xu|^p\,dx\right)^{\frac{2}{p}}$$
$$+\,c\,|B_R|\left(\frac{1}{|B_R|}\int_{B_R}|Xu|^p dx\right)^{\frac{2}{p}}\left(R^{2-Q}\int_{B_R}|Xu|^2 dx\right)^{\frac{p-2}{2p}}+c\int_{B_R}\left(\sum_i\left|g^i\right|^2+|f|^2\right)dx$$
$$\leqslant c\,|B_R|\,\vartheta(x,R)\left(\frac{1}{|B_R|}\int_{B_R}|Xu|^p\,dx\right)^{\frac{2}{p}}+4\varepsilon\int_{B_R}|Xw|^2\,dx$$
$$+c\int_{B_R}\left(\sum_i\left|g^i\right|^2+|f|^2\right)dx, \tag{4.6}$$

where $\vartheta(x,R)=\left(\eta_R(a_i^{\alpha\beta}(x,u_R))\right)^{\frac{p-2}{p}}+\omega^{\frac{p-2}{p}}\left(R^{2-Q}\int_{B_R}|Xu|^2\,dx\right)+\left(R^{2-Q}\int_{B_R}|Xu|^2dx\right)^{\frac{p-2}{2p}}$.

Denote $E(x_0,R)=R^{2-Q}\int_{B_R}|Xu|^2\,dx$ and set

$$\Omega_0=\left\{x_0\in\Omega|\liminf_{\rho\to0}\rho^{2-Q}\int_{B_\rho(x_0)}|Xu|^2dx=0\right\}.$$

For any $x_0\in\Omega$ and $\varepsilon_0>0$, there exists $R<R_0\leqslant dist(x_0,\partial\Omega)$, such that $E(R)<\varepsilon_0$. By properties of VMO_X functions, we know that for any $\sigma>0$, if $x_0\in\Omega$, then there exists a positive constant $R<R_0$, such that $\vartheta(x,R)<\sigma$. Taking small enough ε such that $\delta-4\varepsilon>0$, it infers by (4.6) that

$$\int_{B_R}|Xw|^2\,dx\leqslant c\,|B_R|\,\sigma\left(\frac{1}{|B_R|}\int_{B_R}|Xu|^p\,dx\right)^{\frac{2}{p}}+c\int_{B_R}\left(\sum_i\left|g^i\right|^2+|f|^2\right)dx. \tag{4.7}$$

From Lemma 3.1, we obtain that for any $\rho\leqslant\frac{1}{2}R$,

$$\int_{B_{2\rho}}|Xu|^2\,dx\leqslant c\int_{B_{2\rho}}|Xv|^2\,dx+c\int_{B_{2\rho}}|Xw|^2\,dx$$
$$\leqslant c\left(\frac{\rho}{R}\right)^Q\int_{B_R}|Xv|^2dx+c\int_{B_R}|Xw|^2\,dx\leqslant c\left(\frac{\rho}{R}\right)^Q\int_{B_R}|Xu|^2dx+c\int_{B_R}|Xw|^2\,dx.$$

With the aid of (4.7), it follows that

$$\begin{aligned}
&\int_{B_{2\rho}} |Xu|^2\, dx \\
&\leqslant c\,|B_R|^{\frac{p-2}{p}} \left(\frac{\rho}{R}\right)^Q \left(\int_{B_R} |Xu|^p dx\right)^{\frac{2}{p}} + c\,|B_R|\,\sigma \left(\frac{1}{|B_R|}\int_{B_R} |Xu|^p\, dx\right)^{\frac{2}{p}} \\
&\quad + c\int_{B_R}\left(\sum_i |g^i|^2 + |f|^2\right) dx \\
&\leqslant c\,|B_R|^{\frac{p-2}{p}} \left[\left(\left(\frac{\rho}{R}\right)^Q + \sigma\right)\left(\int_{B_R} |Xu|^p dx\right)^{\frac{2}{p}} + \left(\int_{B_R}\sum_i |g^i|^p dx\right)^{\frac{2}{p}} + \left(\int_{B_R} |f|^p dx\right)^{\frac{2}{p}}\right].
\end{aligned} \tag{4.8}$$

Noting

$$\int_{B_\rho} |Xu|^p\, dx \leqslant c\,|B_\rho| \left(\frac{1}{|B_{2\rho}|}\int_{B_{2\rho}} |Xu|^2 dx\right)^{\frac{p}{2}} + c\int_{B_{2\rho}}\left(\sum_i |g^i|^p + |f|^p\right) dx,$$

we have by Theorem 3.2 that

$$\begin{aligned}
&\textstyle\int_{B_\rho} |Xu|^p\, dx \\
&\leqslant c\,|B_\rho| \left(\frac{c|B_R|^{\frac{p-2}{p}}}{|B_{2\rho}|}\left[\left(\left(\frac{\rho}{R}\right)^Q + \sigma\right)\left(\int_{B_R}|Xu|^p dx\right)^{\frac{2}{p}} + \left(\int_{B_R}\sum_i |g^i|^p dx\right)^{\frac{2}{p}} + \left(\int_{B_R}|f|^p dx\right)^{\frac{2}{p}}\right]\right)^{\frac{p}{2}} \\
&\quad + c\int_{B_{2\rho}}\left(\sum_i |g^i|^p + |f|^p\right) dx \\
&\leqslant c\left(\frac{|B_R|}{|B_\rho|}\right)^{\frac{p-2}{2}}\left[\left(\left(\frac{\rho}{R}\right)^Q + \sigma\right)^{\frac{p}{2}}\int_{B_R}|Xu|^p dx + \int_{B_R}\left(\sum_i |g^i|^p + |f|^p\right) dx\right] \\
&\leqslant c\left(\frac{|B_R|}{|B_\rho|}\right)^{\frac{p-2}{2}}\left[\left(\left(\frac{\rho}{R}\right)^Q + \sigma\right)^{\frac{p}{2}}\int_{B_R}|Xu|^p dx + R^{Q-2+2k}\left(\sum_i |g^i|^p_{L^m} + \|f\|^p_{L^m}\right)\right],
\end{aligned}$$

where $k = 1 - \frac{pQ}{2m}$. Therefore,

$$\begin{aligned}
&\left(|B_\rho|^{\frac{p-2}{2}}\int_{B_\rho}|Xu|^p\, dx\right)^{\frac{2}{p}} \\
&\leqslant c\left(\left(\frac{\rho}{R}\right)^Q + \sigma\right)\left(|B_R|^{\frac{p-2}{2}}\int_{B_R}|Xu|^p dx\right)^{\frac{2}{p}} + c\left(|B_R|^{\frac{p-2}{2}} R^{Q-2+2k}\left(\sum_i |g^i|^p_{L^m} + \|f\|^p_{L^m}\right)\right)^{\frac{2}{p}} \\
&\leqslant c\left(\left(\frac{\rho}{R}\right)^Q + \sigma\right)\left(|B_R|^{\frac{p-2}{2}}\int_{B_R}|Xu|^p dx\right)^{\frac{2}{p}} + c\left(R^{\frac{pQ}{2}-2+2k}\left(\sum_i |g^i|^p_{L^m} + \|f\|^p_{L^m}\right)\right)^{\frac{2}{p}}.
\end{aligned}$$

Now let us set $H(\rho) = \left(|B_\rho|^{\frac{p-2}{2}}\int_{B_\rho}|Xu|^p\, dx\right)^{\frac{2}{p}}$, $H(R) = \left(|B_R|^{\frac{p-2}{2}}\int_{B_R}|Xu|^p\, dx\right)^{\frac{2}{p}}$, $\alpha = Q$, $\beta = \left(\frac{pQ}{2} - 2 + 2k\right)\frac{2}{p}$, $\varepsilon = \sigma$, $B = c\left(\sum_i |g^i|^p_{L^m} + \|f\|^p_{L^m}\right)^{\frac{2}{p}}$. In terms of Lemma 2.1,

$$\begin{aligned}
&\left(|B_\rho|^{\frac{p-2}{2}}\int_{B_\rho}|Xu|^p\, dx\right)^{\frac{2}{p}} \\
&\leqslant c\left(\frac{\rho}{R}\right)^{\left(\frac{pQ}{2}-2+2k\right)\frac{2}{p}}\left(|B_R|^{\frac{p-2}{2}}\int_{B_R}|Xu|^p dx\right)^{\frac{2}{p}} + c\left(\rho^{\frac{pQ}{2}-2+2k}\left(\sum_i |g^i|^p_{L^m} + \|f\|^p_{L^m}\right)\right)^{\frac{2}{p}}.
\end{aligned}$$

By (2.1),

$$\begin{aligned}&\int_{B_\rho} |Xu|^p\,dx \\ &\leqslant c\left(\frac{\rho}{R}\right)^{\frac{pQ}{2}-2+2k}\left(\frac{|B_R|}{|B_\rho|}\right)^{\frac{p-2}{2}}\int_{B_R}|Xu|^p dx + c\,|B_\rho|^{\frac{2-p}{2}}\,\rho^{\frac{pQ}{2}-2+2k}\left(\sum_i \left|g^i\right|^p_{L^m}+\|f\|^p_{L^m}\right)\\ &\leqslant c\left(\frac{\rho}{R}\right)^{Q-2+2k}\int_{B_R}|Xu|^p dx + c\rho^{Q-2+2k}\left(\sum_i \left|g^i\right|^p_{L^m}+\|f\|^p_{L^m}\right)\end{aligned}$$

and thus

$$\int_{B_\rho} |Xu|^p\,dx \leqslant c\rho^{Q-2+2k}.$$

It is observed by lemma 2.2 that if $x_0 \in \Omega_0$, then

$$u(x) \in C^k(\Omega_0, \mathbb{R}^N), \quad k = 1 - \frac{pQ}{2m},$$

and from lemma 2.3,

$$H_{Q-2}(\Omega \backslash \Omega_0) = 0,$$

The proof is completed. □

References

[1] Campanato S.: Equazioni ellittiche del II ordine e spazi $\mathcal{L}^{(2,\lambda)}$. Ann. Mat. Pura Appl. 69, 321–381(1965).

[2] Chen Y.Z., Wu L.C.: Second order Elliptic Equations and Elliptic Systems. Amer. Math. Soc. 174: 27, 1991.

[3] De Giorgi E.: Sulla differenziabilitá e l'analiticitá della estremali degli integrali multipli regolari. Mem. Accad. Sci. Torino cl. Sci. Fis. Mat. Natur. 3, 25–43(1957).

[4] Di Fazio G., Fanciullo M.: Gradient estimates for elliptic systems in Carnot-Carathéodory spaces. Comment. Math. Univ. Carolinae, 43, 605–618(2002).

[5] Dong Y., Niu P.C.: Estimates in Morrey Spaces and Hölder Continuity for Weak Solutions to Degenerate Elliptic Systems. Manuscripta Math. 138, 419–437(2012).

[6] Gao D., Niu P.C., Wang J.L.: Partial regularity for degenerate subelliptic systems associated with Hörmander's vector fields. Nonlinear Anal. 73, 3209–3223(2010).

[7] Garofalo N., Nhieu D.M.: Isoperimetric and Sobolev inequalities for Carnot-Carathéodory Spaces and the existence of minimal Surfaces. Communications on Pure and Applied Mathematics, 49, 1081–1144(1996).

[8] Gianazza U.: Regularity for non-linear equations involving square Hörmander operators. NLA-TMA. 23, 49–73(1994).

[9] Giaquinta M.: Multiple integrals in calculus of variations and nonlinear elliptic systems. Ann. Of Math. Stud. 105, Princeton Univ. Press, Princeton, 1983.

[10] Giaquinta M.: Introduction to Regularity Theory for Nonlinear Elliptic Systems. Berlin: Birkhäuser Verlag, 1993.

[11] Giaquinta M., Modica G.: Regularity results for some classes of higher order nonlinear elliptic systems. J. Reine Angew. Math. 311, 145–169(1993).

[12] Gilbarg D., Trudinger N.S.: Elliptic Partial Differential Equations of Second Order. Berlin: Spinger-Verlag, 2001.

[13] Han Q., Lin F.H.: Elliptic Partial Differential Equations. Amer. Math. Soc. Providence, Rhode Island, 1997.

[14] Ladyzhenskaya O.A., Ural'tseva N.N.: Linear and Quasilinear Elliptic Equations. Moscow: Nauka, 1973.

[15] Lu G.: The sharp inequality for free vector fields: an endpoint result. Revista Matematica Iberoamericana, 10, 453–466(1994).

[16] Morrey C. B.: Multiple Integrals in the Calculus of Variations. Heidelberg: Springer-Verlag, 1966.

[17] Nagel A., Stein E.M., Wainger S.: Balls and metrics defined by vector fields I: Basic properties. Acta Mathematica, 155, 103–147(1985).

[18] Nash J.: Continuity of solutions of parabolic and elliptic equations. Amer. J. Math. 8, 931–954(1958).

[19] Palagachev D.K.: Quasilinear elliptic equations with VMO coefficients. Trans. Amer. Math. Soc. 347, 2481–2493(1995).

[20] Xu C.J., Zuily C.: Higher interior regularity for quasilinear subellliptic systems. Calc. Var. 5, 323–343(1997).

[21] Zatorska-Goldstein A.: Very weak solutions of nonlinear subelliptic equations. An. Acad Sci. Fen. Math. 30, 407–436(2005).

[22] Zheng S., Zheng X., Feng Z.: Regularity for a class of degenerate elliptic equations with discontinuous coefficients under natural growth. J. Math. Anal. Appl. 346, 359–373(2008).

[23] Zhou Z.: A generalization of Lebesgue differential theorem and its application. Acta Mathematic Scientia, 21, 109–113(2002).