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An existence result for a quasilinear system with gradient term under the Keller–Osserman conditions

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Abstract: We use some new technical tools to obtain the existence of entire solutions for the quasilinear elliptic system of type $\Delta_p u_i + h_i(|x|)|\nabla u_i|^{p-1} = a_i(|x|) f_i(u_1, u_2)$ on $\mathbb{R}^N$ $(N \geq 3,\ i = 1,2)$ where $N-1 \geq p > 1$, Δ_p is the p-Laplacian operator, and h_i, a_i, f_i are suitable functions. The results of this paper supplement the existing results in the literature and complete those obtained by Jesse D Peterson and Aihua W Wood (Large solutions to non-monotone semilinear elliptic systems, Journal of Mathematical Analysis and Applications, Volume 384, pages 284–292, 2011).

Key words: Entire solution, large solution, elliptic system

1. Introduction

In this paper we establish a new result concerning the existence of solutions for the quasilinear elliptic system

$$\begin{cases} \Delta_p u_1(r) + h_1(r)|\nabla u_1(r)|^{p-1} = a_1(r) f_1(u_1(r), u_2(r)) , \\ \Delta_p u_2(r) + h_2(r)|\nabla u_2(r)|^{p-1} = a_2(r) f_2(u_1(r), u_2(r)) , \end{cases} \tag{1}$$

where $r := |x|$ $(x \in \mathbb{R}^N,\ N-1 \geq p > 1)$ is the Euclidean norm, and Δ_p is the so-called p-Laplace operator defined by $\Delta_p u := \mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right)$.

It will be assumed throughout this paper that:

P1) $h_j,\ a_j$ $(j = 1,2)$ are nonnegative nontrivial $C\left(\mathbb{R}^N\right)$ functions,

while

P2) $f_j : [0,\infty)^2 \to [0,\infty)$ $(j = 1,2)$ are continuous and nondecreasing functions in each variable and verify $f_j(s_1, s_2) > 0$ whenever $s_i > 0$ for some $i = 1,2$ together with the Keller–Osserman type condition

$$I(\infty) := \lim_{r\to\infty} I(r) = \infty, \tag{2}$$

where $I(r) := \int_a^r (F(s))^{-1/p} ds$ for $a > 0$, $F(s) := \int_0^s \sum_{i=1}^{2} f_i(t,t)\, dt$.

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There is by now broad literature regarding the study of solutions for (1). For a single equation of the form $\Delta u = f(u)$ where $f(u)$ is a positive, real continuous function defined for all real u and nondecreasing, the existence of entire large solutions is equivalent to a condition on f known as the Keller–Osserman condition:

$$\int_{u_0}^{\infty} \left(\int_0^t f(s)\, ds \right)^{-1/2} dt = \infty \text{ for } \; u_0 > 0 \tag{3}$$

(see [6], [10]). In particular, Keller and Osserman proved that a necessary and sufficient condition for the considered problem to have an entire solution is that f satisfies (3). Such a solution will necessarily satisfy $\lim_{|x|\to\infty} u(x) = \infty$ and hence be a large solution. Moreover, Keller applied the results to electrohydrodynamics, namely to the problem of the equilibrium of a charged gas in a conducting container; see [7].

For the systems case, basic results in the study of solutions have been obtained in the works of Bandl -Marcus [1], the present author [3], Dkhil-Zeddini [4], Goncalves-Jiazheng [5], Matero [8,9], Zhang-Liu [14], and Peterson-Wood [11] and their references. We comment below on a few further results closer to our interests in the present article.

Regarding Zhang-Liu [14] and Dkhil-Zeddini [4], we observe from their works that they studied the existence of entire large positive solutions of the system (1), in the semilinear case [14] and the quasilinear case [4]. In [4] and [13], the authors imposed on a_1, a_2, f_1, and f_2 satisfying the above conditions instead of the Keller-Osserman condition the following:

$$\int_a^{\infty} \frac{ds}{f_1^{1/(p_1-1)}(s,s) + f_2^{1/(p_2-1)}(s,s)} = \infty \text{ for } a > 0,\, 1 < p_1, p_2 < \infty, \tag{4}$$

where Δ_p in (1) is replaced by Δ_{p_i} $(i = 1, 2)$. Obviously, (4) completes (2). For more details, the interested reader can consult, for example, [12, pp. 55] or [2].

Finally, we note that the study of large solutions for (1) when the integral in (2) is finite has been the subject of articles by Goncalves-Jiazheng [5] and Keller [6] for the scalar case and recently by Peterson and Wood [8] in the systems case, where the authors obtained the existence of solutions for the case when (3) fails to hold.

Motivated by [3], [4], [5], [11], [13], and [14], we are interested in another type of nonlinearity, f_i $(i = 1, 2)$, in order to obtain the existence of entire large/bounded positive solutions of (1).

The main result of this article is:

Theorem 1.1 *Under the above hypotheses, P1 and P2, there are infinitely positive entire radial solutions of system (1). If in addition*

$$s^{\frac{p(N-1)}{p-1}} \sum_{j=1}^{2} a_j(s) \text{ is nondecreasing for large } s,$$

then the solutions:

i) are bounded if there exists a positive number ε such that

$$\int_0^{\infty} t^{1+\varepsilon} \left(\sum_{j=1}^{2} a_j(t) \right)^{2/p} dt < \infty, \tag{5}$$

ii) are large if

$$\int_0^\infty \left(\frac{e^{-\int_0^t h_j(s)ds}}{t^{N-1}} \int_0^t s^{N-1} e^{\int_0^s h_j(t)dt} a_j\left(s\right) ds \right)^{1/(p-1)} dt = \infty \text{ for all } j = 1, 2 \tag{6}$$

hold.

We mention that we can prove similar results for f_1 and f_2 being nonmonotonic, as in former papers [6] or [8] or more recently in [5], respectively [11]. Since in this case the proof is as for the monotone case, we omit it.

2. Proof of the Theorem 1.1

We start by showing that the simplest equation,

$$(p-1)\, z'\left(r\right)^{p-2} z''\left(r\right) + \frac{N-1}{r} z'\left(r\right)^{p-1} = \sum_{j=1}^{2} a_j\left(r\right) \sum_{i=1}^{2} f_i\left(z\left(r\right), z\left(r\right)\right), \tag{7}$$

has positive radial solutions. Next we construct a solution for (7) with $z'\left(r\right) \geq 0$. Therefore, we see that radial solutions of (7) are any positive solutions z of the integral equation

$$z\left(r\right) = b + \int_0^r \left(\frac{1}{t^{N-1}} \int_0^t s^{N-1} \sum_{j=1}^{2} a_j\left(s\right) \sum_{i=1}^{2} f_i\left(z\left(s\right), z\left(s\right)\right) ds \right)^{1/(p-1)} dt, \tag{8}$$

where $b \geq a > 0$. An idea of how to solve (8) is to regard this as an operator equation:

$$T\left(z(r)\right) = z(r),\ T : C\left[0, \infty\right) \to C\left[0, \infty\right),$$

defined by

$$T\left(z(r)\right) = b + \int_0^r \left(\frac{1}{t^{N-1}} \int_0^t s^{N-1} \sum_{j=1}^{2} a_j\left(s\right) \sum_{i=1}^{2} f_i\left(z\left(s\right), z\left(s\right)\right) ds \right)^{1/(p-1)} dt, \tag{9}$$

where $z(0) = b$ are the central values for the problem. The integration in this operator implies that a fixed point $z \in C\left[0, \infty\right)$ is in fact in the space $C^1\left[0, \infty\right)$. We immediately see that a solution of (7) is a fixed point of the operator (9). To establish a solution to this operator, we use successive approximation. We define, recursively, sequences $\left\{z^k\right\}^{k \geq 1}$ on $\left[0, \infty\right)$ by

$$z^0 = b \text{ for all } r \geq 0$$

and

$$\begin{aligned} z^k\left(r\right) &= T\left(z^{k-1}\left(r\right)\right) \\ &= b + \int_0^r \left(\frac{1}{t^{N-1}} \int_0^t s^{N-1} \sum_{j=1}^{2} a_j\left(s\right) \sum_{i=1}^{2} f_i\left(z^{k-1}\left(s\right), z^{k-1}\left(s\right)\right) ds \right)^{1/(p-1)} dt. \end{aligned}$$

We remark that, for all $r \geq 0$ and $k \in N$,

$$z^k\left(r\right) \geq b,$$

and from the monotonicity of f_i it follows that $\left\{z^k\right\}^{k\geq 1}$ is a nondecreasing sequence of nonnegative functions.

We note that $\left\{z^k\right\}^{k\geq 1}$ satisfies

$$(p-1)\left[\left(z^k\right)'\right]^{p-2}\left(z^k\right)''+\frac{N-1}{r}\left[\left(z^k\right)'\right]^{p-1}=\sum_{j=1}^{2}a_j\left(r\right)\sum_{i=1}^{2}f_i\left(z^{k-1}\left(r\right),z^{k-1}\left(r\right)\right). \tag{10}$$

Using the monotonicity of $\left\{z^k\right\}^{k\geq 1}$ yields

$$\sum_{j=1}^{2}a_j\left(r\right)\sum_{i=1}^{2}f_i\left(z^{k-1}\left(r\right),z^{k-1}\left(r\right)\right)\leq\sum_{j=1}^{2}a_j\left(r\right)\sum_{i=1}^{2}f_i\left(z^k,z^k\right),$$

and so

$$(p-1)\left[\left(z^k\left(r\right)\right)'\right]^{p-1}\left(z^k\right)''+\frac{N-1}{r}\left[\left(z^k\left(r\right)\right)'\right]^{p}\leq\sum_{j=1}^{2}a_j\left(r\right)\sum_{i=1}^{2}f_i\left(z^k,z^k\right)\left(z^k\left(r\right)\right)'. \tag{11}$$

Define

$$a^R=\max\{\sum_{j=1}^{2}a_j\left(r\right):0\leq r\leq R\}.$$

We prove that $z^k\left(R\right)$ and $\left(z^k\left(R\right)\right)'$, both of which are nonnegative, are bounded above independently of k. Using this and the fact that $\left(z^k\right)'\geq 0$, we note that (11) yields

$$(p-1)\left[\left(z^k\left(r\right)\right)'\right]^{p-1}\left(z^k\right)''\leq a^R\sum_{i=1}^{2}f_i\left(z^k,z^k\right)\left(z^k\left(r\right)\right)'$$

or, equivalently,

$$\frac{p-1}{p}\left\{\left[\left(z^k\left(r\right)\right)'\right]^{p}\right\}'\leq a^R\sum_{i=1}^{2}f_i\left(z^k,z^k\right)\left(z^k\left(r\right)\right)'.$$

Integrate this inequality from 0 to r to rewrite

$$\left[\left(z^k\left(r\right)\right)'\right]^{p}\leq\frac{p}{p-1}a^R\int_b^{z^k(r)}\sum_{i=1}^{2}f_i\left(s,s\right)ds\leq\frac{p}{p-1}a^R\int_0^{z^k(r)}\sum_{i=1}^{2}f_i\left(s,s\right)ds \tag{12}$$

into the form

$$\left(z^k\left(r\right)\right)'\leq\sqrt[p]{\frac{p}{p-1}a^R}\left(\int_0^{z^k(r)}\sum_{i=1}^{2}f_i\left(s,s\right)ds\right)^{1/p},\ 0\leq r\leq R. \tag{13}$$

Integrating the above inequality between 0 and R, we get the estimate

$$\int_b^{z^k(R)}\left[\int_0^t\sum_{i=1}^{2}f_i\left(s,s\right)ds\right]^{-1/p}dt=I\left(z^k\left(R\right)\right)-I\left(b\right)\leq\sqrt[p]{\frac{p}{p-1}a^R}R.$$

Since I is a bijection with I^{-1} increasing, this means that

$$z^k\left(R\right)\leq I^{-1}\left(\sqrt[p]{\frac{p}{p-1}a^R}R+I\left(b\right)\right)\text{ for all }r\geq 0. \tag{14}$$

We now return to the Keller–Osserman condition (2) to conclude that $z^k(R)$ is uniformly bounded above independently of k and then the sequences $z^k(r)$ are uniformly bounded above independently of k (since $r \le R$ and $z^k(r)$ is nondecreasing sequence). It remains to be proven that $z^k(r)$ is also equicontinuous on $[0,R]$. Indeed, since $z^k(r)$ is uniformly bounded above independent of k, there exists M such that $z^k(r) \le M$ for all k and all $r \in [0,R]$. It is sufficient to observe that

$$\begin{aligned}\left(z^k(r)\right)' &\le \sqrt[p]{\frac{p}{p-1}a^R}\left(\int_0^{z^k(r)} \sum_{i=1}^{2} f_i(s,s)\,ds\right)^{1/p}, \; 0 \le r \le R.\\ &\le \sqrt[p]{\frac{p}{p-1}a^R}\left(\max_{0\le t\le M}\sum_{i=1}^{2} f_i(t,t)\int_0^M ds\right)^{1/p}\\ &\le \sqrt[p]{\frac{p}{p-1}a^R}\left(M\max_{0\le t\le M}\sum_{i=1}^{2} f_i(t,t)\right)^{1/p} < \infty.\end{aligned}$$

We also clearly have $z^k(r) > 0$ for all $r \ge 0$, and so our sequence is equicontinuous on $[0,R]$ for arbitrary $R > 0$. Putting the fact that $z^k(r)$ is a monotonic, uniformly bounded, equicontinuous sequence of functions on $[0,R]$ together, we get that there exists a function $z \in C([0,R])$ such that $z^k(r) \to z(r)$ uniformly. Hence, z_i is a fixed point of (9) in $C([0,R])$. Next, we extend this result to show that T has a fixed point in $C^1([0,\infty))$. Let $\left\{z^k(r)\right\}^{k\ge 1}$ be a sequence of fixed points defined by

$$z^k(r) = T\left(z^k(r)\right) \text{ on } [0,k],\; z^k(r) \in C([0,k]), \tag{15}$$

for $k = 1,2,3,\ldots$. As earlier, we may show that both $z^k(r)$ are bounded and equicontinuous on $[0,1]$. Thus, by applying the Arzela–Ascoli Theorem to each sequence separately, we can derive that $\left\{z^k(r)\right\}^{k\ge 1}$ contains a convergent subsequence, $z^{k^1}(r)$, that converges uniformly on $[0,1]$. Let

$$z^{k^1}(r) \to z^1 \text{ uniformly on } [0,1] \text{ as } k^1 \to \infty.$$

Likewise, the subsequences $z^{k^1}(r)$ are bounded and equicontinous on $[0,2]$, so there exists a subsequence $z^{k^2}(r)$ of $z^{k^1}(r)$ such that

$$z^{k^2}(r) \to z^2 \text{ uniformly on } [0,2] \text{ as } k^2 \to \infty.$$

Notice that

$$\left\{z^{k^2}(r)\right\} \subseteq \left\{z^{k^1}(r)\right\} \subseteq \left\{z^k(r)\right\}_{k\ge 2}^{\infty},$$

so $z^2 = z^1$ on $[0,1]$. Continuing this line of reasoning, we obtain a sequence, denoted by $\left\{z^k(r)\right\}$, such that

$$\begin{aligned} z^k(r) &\in C([0,k]),\; k = 1,2,..\\ z^k(r) &= z^1(r) \text{ for } r \in [0,1]\\ z^k(r) &= z^2(r) \text{ for } r \in [0,2]\\ &\ldots\\ z^k(r) &= z^{k-1}(r) \text{ for } r \in [0,k-1],\end{aligned}$$

and these functions are radially symmetric. Therefore, $z^k(r)$ converges pointwise to some $z(r)$, which satisfies

$$z(r) = z^k(r) \text{ if } 0 \le r \le k.$$

Hence, $z(r)$ is radially symmetric. Further, since $z^k(r)$ is in the form of (15), we have that $z^k(r)$ is also equicontinuous. Pointwise convergence and equicontinuity imply uniform convergence and thus the convergence is uniform on bounded sets. Thus,

$$z(r) \in C^1([0,\infty))$$

is a fixed point of (9) and a solution to (7) with central value b. Since $b \ge a > 0$ was chosen arbitrarily, it means that

$$\Delta_p z(r) = \sum_{j=1}^{2} a_j(r) \sum_{i=1}^{2} f_i(z(r), z(r)) \;\; r := |x|, \, x \in \mathbb{R}^N \tag{16}$$

has infinitely many positive entire solutions. Moreover, for each $R > 0$, there exist $c_R > 0$ such that $z(R) \le c_R$. Due to the fact that any z is radial, we have

$$z(r) = b + \int_0^r \frac{1}{t^{N-1}} \left(\int_0^t s^{N-1} \sum_{j=1}^{2} a_j(s) \sum_{i=1}^{2} f_i(z(s), z(s)) \, ds \right)^{\frac{1}{p-1}} dt \; ,$$

for all $r \ge 0$. After these preliminary considerations we can start with the proof of our main theorem. We choose

$$\beta_i \in (0, b] \;\; (\text{or } \beta_i \in (a, b] \text{ for } b > a), \; i = 1, 2$$

and we define the sequences $\{u_i^k\}^{k\ge1}$ on $[0,\infty)$ by

$$\begin{cases} u_1^0 = \beta_1, \;\; u_2^0 = \beta_2 \text{ and for all } r \ge 0 \\ u_1^k(r) = \beta_1 + \int_0^r \left(\frac{e^{-\int_0^t h_1(s)ds}}{t^{N-1}} \int_0^t s^{N-1} e^{\int_0^s h_1(t)dt} a_1(s) f_1\left(u_1^{k-1}, u_2^{k-1}\right) \right)^{\frac{1}{p-1}} dsdt, \\ u_2^k(r) = \beta_2 + \int_0^r \left(\frac{e^{-\int_0^t h_2(s)ds}}{t^{N-1}} \int_0^t s^{N-1} e^{\int_0^s h_2(t)dt} a_2(s) f_2\left(u_1^{k-1}, u_2^{k-1}\right) \right)^{\frac{1}{p-1}} dsdt. \end{cases}$$

A simple calculation shows that $\{u_i^k\}^{k\ge1}$ are nondecreasing sequences on $[0,\infty)$. Because $z'(r) \ge 0$, it follows that

$$0 < \beta_i \le z(0) = b \le z(r) \text{ for all } r \ge 0$$

and so

$$\begin{aligned} u_i^1(r) &= \beta_i + \int_0^r \left(\frac{e^{-\int_0^t h(s)ds}}{t^{N-1}} \int_0^t s^{N-1} e^{\int_0^s h(t)dt} a_i(s) f_i\left(u_1^0, u_2^0\right) ds \right)^{\frac{1}{p-1}} dt \\ &\le b + \int_0^r \left(\frac{1}{t^{N-1}} \int_0^t s^{N-1} \sum_{j=1}^{2} a_j(s) \sum_{i=1}^{2} f_i(z, z) \, ds \right)^{\frac{1}{p-1}} dt = z(r). \end{aligned}$$

In other words, we can see that

$$u_i^1(r) \le z(r).$$

By the previous discussion it follows that

$$u_i^k(r) \le z(r) \text{ for all } r \in [0,\infty) \text{ and } k \ge 1. \tag{17}$$

It follows that the sequences $\{u_i^k\}_{i=1,2}$ are bounded and equicontinuous on $[0,R]$ for arbitrary $R>0$. By the Arzela–Ascoli Theorem, $\{u_1^k\}$ and $\{u_2^k\}$ have subsequences converging uniformly to u_1 and u_2 on $[0,R]$. By the arbitrariness of $R>0$, we see that (u_1,u_2) are positive entire functions that satisfy for each $r\in[0,\infty)$ the system (1) with central value in (β_1,β_2).

The proof of i) To do this we prove that the solution $z(r)$ of the problem (16) is bounded and then due to (17) *any entire positive radial solution* (u_1,u_2) of system (1) is bounded. Let $R>0$ such that

$$r^{\frac{p(N-1)}{p-1}}\sum_{j=1}^{2}a_j(r) \text{ is nondecreasing for } r\geq R.$$

After some technical calculations, it is easy to see that the radial form of (16) implies

$$(p-1)\left[z'(r)\right]^{p-1}z''(r)+\frac{N-1}{r}\left[z'(r)\right]^{p}\leq\sum_{j=1}^{2}a_j(r)\sum_{i=1}^{2}f_i(z(r),z(r))(z(r))'.$$

Multiplying this equation by

$$\frac{p}{p-1}r^{\frac{p(N-1)}{p-1}},$$

we have

$$\left[r^{\frac{p(N-1)}{p-1}}(z'(r))^{p}\right]'\leq r^{\frac{p(N-1)}{p-1}}\frac{p}{p-1}\sum_{j=1}^{2}a_j(r)\sum_{i=1}^{2}f_i(z(r),z(r))(z(r))'$$

and integrating would then yield

$$\begin{aligned}&\int_R^r\left[s^{\frac{p(N-1)}{p-1}}(z'(s))^{p}\right]'ds \\ \leq\ &\int_R^r s^{\frac{p(N-1)}{p-1}}\frac{p}{p-1}\sum_{j=1}^{2}a_j(s)\sum_{i=1}^{2}f_i(z(s),z(s))z'(s)\,ds.\end{aligned}\tag{18}$$

With the use of (18) we get

$$\begin{aligned}&r^{\frac{p(N-1)}{p-1}}(z'(r))^{p}-R^{\frac{p(N-1)}{p-1}}((z'(R)))^{p} \\ \leq\ &\int_R^r s^{\frac{p(N-1)}{p-1}}\frac{p}{p-1}\sum_{j=1}^{2}a_j(s)\sum_{i=1}^{2}f_i(z(s),z(s))z'(s)\,ds,\end{aligned}$$

for $r\geq R$.

Noting that, by the monotonicity of

$$s^{\frac{p(N-1)}{p-1}}\sum_{j=1}^{2}a_j(s) \text{ for } r\geq s\geq R,$$

we get

$$r^{\frac{p(N-1)}{p-1}}(z'(r))^{p}\leq C+\frac{p}{p-1}r^{\frac{p(N-1)}{p-1}}\sum_{j=1}^{2}a_j(r)F(z(r)),$$

which yields

$$r^{\frac{N-1}{p-1}} z'(r) \leq \left[C + \frac{p}{p-1} r^{\frac{p(N-1)}{p-1}} \sum_{j=1}^{2} a_j(r) F(z(r)) \right]^{1/p} \tag{19}$$

where

$$C = R^{\frac{p(N-1)}{p-1}} \left[(z(R))' \right]^p .$$

We need to recall an important inequality,

$$(x_1 + x_2)^{1/p} \leq x_1^{1/p} + x_2^{1/p},$$

for any nonnegative constants x_i $(i = 1, 2)$ and $1/p < 1$. Therefore, by applying this inequality in (19), we get

$$z'(r) \leq \sqrt[p]{C} r^{\frac{1-N}{p-1}} + r^{\frac{1-N}{p-1}} \sqrt[p]{\frac{p}{p-1} r^{\frac{p(N-1)}{p-1}} \sum_{j=1}^{2} a_j(r)} \left[F(z(r)) \right]^{1/p} .$$

Integrating the above inequality, we get

$$\begin{aligned} & \frac{d}{dr} \int_{z(R)}^{z(r)} \left[(F(t))^{\frac{1}{p}} \right]^{-1} dt \\ \leq\ & \sqrt[p]{C} r^{\frac{1-N}{p-1}} \left[(F(z(r)))^{\frac{1}{p}} \right]^{-1} + \left(\frac{p}{p-1} \sum_{j=1}^{2} a_j(r) \right)^{1/p} . \end{aligned} \tag{20}$$

To finish the proof, it remains to be observed that inequality (20) combined with

$$\begin{aligned} \left(\sum_{j=1}^{2} a_j(s) \right)^{1/p} &= \left(s^{p(1+\varepsilon)/2} \sum_{j=1}^{2} a_j(s) s^{-p(1+\varepsilon)/2} \right)^{1/p} \\ &\leq \left(\frac{1}{2} \right)^{1/p} \left[s^{1+\varepsilon} \left(\sum_{j=1}^{2} a_j(s) \right)^{2/p} + s^{-1-\varepsilon} \right], \end{aligned}$$

for each $\varepsilon > 0$, yields

$$\begin{aligned} & \int_{z(R)}^{z(r)} \left[(F(t))^{\frac{1}{p}} \right]^{-1} dt \\ \leq\ & \int_R^r \sqrt[p]{C} t^{\frac{1-N}{p-1}} \left[(F(z(t)))^{\frac{1}{p}} \right]^{-1} dt \\ & + \left(\frac{1}{2} \right)^{1/p} \sqrt[p]{\frac{p}{p-1}} \left[\int_R^r t^{1+\varepsilon} \left(\sum_{j=1}^{2} a_j(t) \right)^{2/p} dt + \int_R^r t^{-1-\varepsilon} dt \right] \\ \leq\ & \sqrt[p]{C} \left[(F(z(R)))^{\frac{1}{p}} \right]^{-1} \frac{p-1}{N-p} R^{\frac{p-N}{p-1}} \\ & + \left(\frac{1}{2} \right)^{1/p} \sqrt[p]{\frac{p}{p-1}} \left[\int_R^r t^{1+\varepsilon} \left(\sum_{j=1}^{2} a_j(t) \right)^{\frac{2}{p}} dt + \frac{1}{\varepsilon R^{\varepsilon}} \right] . \end{aligned} \tag{21}$$

Since the right side of this inequality is bounded (note that $u_i(t) \geq \beta_i$), so is the left side and, hence, in light of the Keller–Osserman condition, the sequence $z(r)$ is bounded, and finally the conclusion $(u_1(r), u_2(r))$ is a bounded function. Thus, for every $x \in \mathbb{R}^N$ the function $(u_1(|x|), u_2(|x|))$ is a positive bounded solution of (1) with central value in (β_1, β_2).

The proof of ii) Since we are interested in large solutions of (1), we suppose that a_i $(i = 1, 2)$ satisfies (6). Now, let (u_1, u_2) be any positive entire radial solution of (1) determined in the first step of the proof. Since u_i $(i = 1, 2)$ is positive for all $R > 0$, we have $u_i(R) > 0$. On the other hand, since $u_i' \geq 0$ we get $u_i(r) \geq u_i(R)$ for $r \geq R$, and thus from

$$u_i(r) = \beta_i + \int_0^r \frac{e^{-\int_0^t h_i(s)ds}}{t^{N-1}} \left(\int_0^t s^{N-1} e^{\int_0^s h_i(s)ds} a_i(s) f_i(u_1(s), u_2(s))\, ds \right)^{1/(p-1)} dt,$$

we obtain

$$\begin{cases} u_i(r) = \beta_i + \int_0^r \left(\frac{e^{-\int_0^t h_i(s)ds}}{t^{N-1}} \int_0^t s^{N-1} e^{\int_0^t h_i(s)ds} a_i(s) f_i(u_1(s), u_2(s))\, ds \right)^{1/(p-1)} dt \ \geq \beta_i \\ \qquad + f_i^{1/(p-1)}(u_1(R), u_2(R)) \int_R^r \left(\frac{e^{-\int_0^t h_i(s)ds}}{t^{N-1}} \int_R^t s^{N-1} e^{\int_0^s h_i(s)ds} a_i(s)\, ds \right)^{1/(p-1)} dt \to \infty \text{ as } r \to \infty, \\ \text{for all } i = 1, 2, \end{cases}$$

and this finishes the proof.

Comments. Using the method of successive approximations, we have obtained infinite solutions for the system (1). The difference from other similar works is that the study of the existence result in Theorem 1.1 has been reduced to the study of a single equation (7). On the other hand, we see that by applying this type of reasoning, the central values of the solutions can take any parameter more or less than a defined in (2), which allows us to have a better picture about the structure of solutions of (1). The inconvenience in applying this type of reasoning is that the results cannot be generalized to the systems of equations with (p_1, p_2)-Laplacian as considered in [4].

We note that by combining the above arguments and the reasoning used by the authors of [4], we can obtain the following result:

Remark 2.1 *Assume that a_j, h_j $(j = 1, 2)$ are nonnegative nontrivial $C(\mathbb{R}^N)$ functions and $f_j : [0, \infty)^2 \to [0, \infty)$ $(j = 1, 2)$ are continuous, nondecreasing functions in each variable, and verify $f_j(s_1, s_2) > 0$ whenever $s_1, s_2 > 0$ together with the Keller–Osserman type condition*

$$I(\infty) := \lim_{r \to \infty} I(r) = \infty$$

where $I(r) := \int_a^r \left[(F_1(s))^{1/p_1} + (F_2(s))^{1/p_2} \right]^{-1} ds$ for $a > 0$, $F_i(s) := \int_0^s f_i(t, t)\, dt$, $i = 1, 2$. If in addition

$$s^{\frac{p_j(N-1)}{p_j - 1}} e^{\frac{p_j}{p_j - 1} \int_0^s h_j(t)dt} a_j(s) \ \textit{is nondecreasing for large } s, \ \ j = 1, 2,$$

the system

$$\left\{\begin{array}{l}\Delta_{p_1}u_1(r)+h_1(r)\left|\nabla u_1(r)\right|^{p_1-1}=a_1(r)f_1(u_1(r),u_2(r))\ ,\\ \Delta_{p_2}u_2(r)+h_2(r)\left|\nabla u_2(r)\right|^{p_2-1}=a_2(r)f_2(u_1(r),u_2(r))\ ,\end{array}\right. \quad N-1\geq p_i>1$$

has infinitely positive entire radial solutions. Moreover, the solutions:

i) are bounded if there exists a positive number ε such that

$$\int_0^\infty t^{1+\varepsilon}\left(e^{\frac{p_j}{p_j-1}\int_0^t h_j(t)dt}a_j(t)\right)^{2/p_j}dt<\infty \quad \text{for all } j=1,2,$$

ii) are large if

$$\int_0^\infty\left(\frac{e^{-\int_0^t h_j(s)ds}}{t^{N-1}}\int_0^t s^{N-1}e^{\int_0^s h_j(t)dt}a_j(s)\,ds\right)^{1/(p_j-1)}dt=\infty \text{ for all } j=1,2$$

holds.

We also have a remark that improves the arguments used recently in [13].

Remark 2.2 *The system*

$$\left\{\begin{array}{l}\Delta_p u_1(x)+\left|\nabla u_1(x)\right|^p=a_1(x)f_1(u_1(x),u_2(x))\ ,\\ \Delta_p u_2(x)+\left|\nabla u_2(x)\right|^p=a_2(x)f_2(u_1(x),u_2(x))\ ,\end{array}\right. \quad \text{in } \mathbb{R}^N, \tag{22}$$

has or no bounded/large solutions with central values more greater than or equal to one if and only if the system without gradient term

$$\left\{\begin{array}{l}\Delta_p u_1(x)=a_1(x)u_1^{p-1}(x)f_1(\ln u_1(x),\ln u_2(x)),\\ \Delta_p u_2(x)=a_2(x)u_2^{p-1}(x)f_2(\ln u_1(x),\ln u_2(x)),\end{array}\right. \quad \text{in } \mathbb{R}^N, \tag{23}$$

has or no bounded/large solutions.

Proof Indeed, via the change of variables

$$u_i(x)=e^{v_i(x)}\ (i=1,2),$$

we turn system (1) into system (7). In fact,

$$\begin{aligned}\Delta_p u_i(x) &= e^{(p-1)v_i(x)}\left[\Delta_p v_i(x)+(p-1)\left|\nabla v_i(x)\right|^p\right]\\ &= e^{(p-1)v_i(x)}\left[(1-p)\left|\nabla v_i\right|^p+a_i(x)f_i(v_1,v_2)+(p-1)\left|\nabla v_i(x)\right|^p\right]\\ &= e^{(p-1)v_i(x)}b_i(x)f_i(v_1,v_2)\\ &= (u_i(x))^{p-1}b_i(x)f_i(\ln u_1,\ln u_2)\,.\end{aligned}$$

Conversely, the change of variables

$$v_i(x)=\ln u_i(x)\ \ (i=1,2),$$

in system (7) turns back to system (1). This follows from the fact that

$$\begin{aligned}\Delta_p v_i(x) &= \frac{\Delta_p u_i(x)}{(u_i(x))^{p-1}} - (p-1)\frac{|\nabla u_i|^p}{(u_i(x))^p} \\ &= \frac{a_i(x)\, u_i^{p-1}(x)\, f_i(\ln u_1(x), \ln u_2(x))}{(u_i(x))^{p-1}} - (p-1)\frac{\left|e^{v_i(x)}\nabla v_i\right|^p}{\left(e^{v_i(x)}\right)^p} \\ &= b_i(x)\, f_i(\ln v_1(x), \ln v_2(x)) - (p-1)|\nabla v_i|^p,\end{aligned}$$

which concludes the proof. □

The existence/nonexistence of solutions for the problem (22) is then provided by the existence/nonexistence of solutions for the system (23). Then our proof of Theorem 1.1 is applicable and for the system with p-gradient terms of the type (22).

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