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Highly nonconcurrent longest paths and cycles in lattices

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Abstract: We investigate here the connected graphs with the property that any pair of vertices are missed by some longest paths (or cycles), embeddable in n-dimensional lattices $\mathcal{L}^n$ where $\mathcal{L}$ denotes the set of integers.

Key words: Longest cycles, longest paths, n-dimensional lattices, Gallai's property

1. Introduction

T. Gallai [2] raised the following question in 1966. Do there exist connected graphs such that for any vertex there exists some longest path avoiding it? The problem of existence of such graphs was solved by H. Walther [6] in 1969.

In 1972 this question was refined by T. Zamfirescu [7] as follows. Do there exist k-connected graphs such that each set of j vertices is missed by some longest path (or cycle)? Soon after that, some examples [9] were constructed that provide partial answers to the above question.

A *grid graph* is a subgraph of an infinite planar lattice $\mathcal{L}^2$, constructed by taking a (finite) cycle and all vertices and edges lying on or inside that cycle (see also [3]). B. Menke [4] proved that no grid graph has Gallai's property.

For $j = 1$ this problem was investigated in [1]. We are dealing here with the case $j = 2$. A. Shabbir and T. Zamfirescu [5] considered the family of all (connected) graphs embeddable in $\mathcal{L}^2$, and found an affirmative answer to the refined version of Gallai's question (case $j = 2$). However, embeddings of these graphs yield very large orders. Our goal here is to provide examples of graphs enjoying the same property, but with smaller order, in the n-dimensional lattice $\mathcal{L}^n$ where $\mathcal{L}$ denotes the set of integers.

2. Embeddings of graphs in which every pair of vertices is missed by some longest path

A graph in which every pair of vertices is missed by some longest path was discovered by Zamfirescu [8]. It has 270 vertices and was constructed by using the graph P, the Petersen graph minus one vertex (shown in Figure 1).

Let H be the graph with 16 vertices shown in Figure 2. We call the vertices a, b, c of H end-points.

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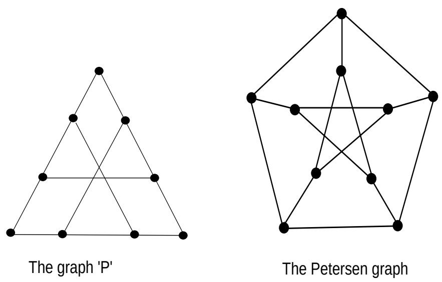

Figure 1.

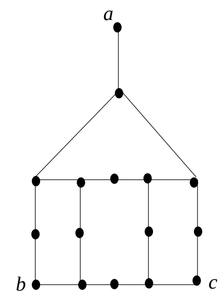

Figure 2.

Lemma 1 *Every longest path in the graph H joining 2 of its endpoints has order* 13*.*

Proof See Figure 3 in which all longest paths joining 2 vertices in $\{a, b, c\}$, are presented. All of them have order 13. □

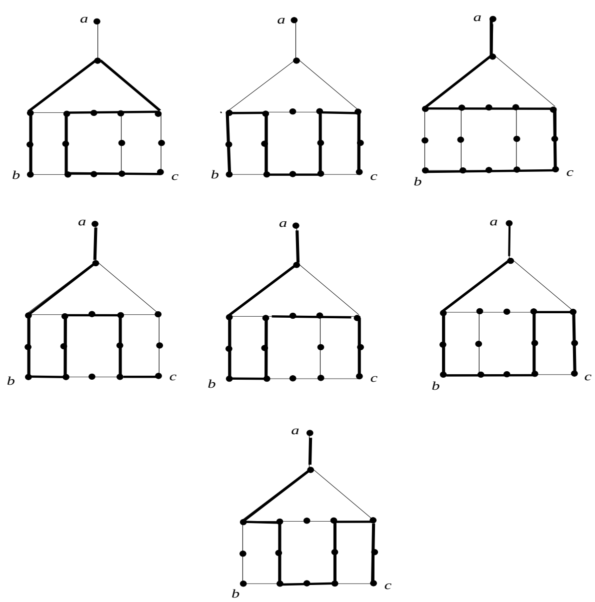

Figure 3.

Theorem 1 *There exists a connected graph of order* 207*, embeddable in* $\mathcal{L}^3$*, in which every pair of vertices is missed by some longest paths.*

Proof Consider the graph P mentioned in Figure 1 and replace each of its vertices by the graph H used in Lemma 1, attaching the 3 (or 2) incident edges to the 3 end-points a, b, and c (or just 2 of them). We obtain a graph $D = \pounds(H, P)$ of order 144 shown in Figure 4. This notation is adopted from [9]. □

This graph is not embeddable in $\mathcal{L}^3$ since it is not bipartite. In order to make it bipartite we add one vertex on each edge of D not belonging to any copy of H, as shown in Figure 5. We obtain a graph of order 156. In this graph we denote the three 1-valent vertices by x, y, and z, as shown in Figure 5. This graph has 2 different types of longest paths, joining pairs of vertices in $\{x, y, z\}$, of order 111, as follows. The longest paths of the first type avoid a whole copy of H not containing any end-points (i.e. x, y, or z), while the longest paths of the second type avoid a whole copy of H containing exactly one end-point, as shown in Figures 6 and 7, respectively.

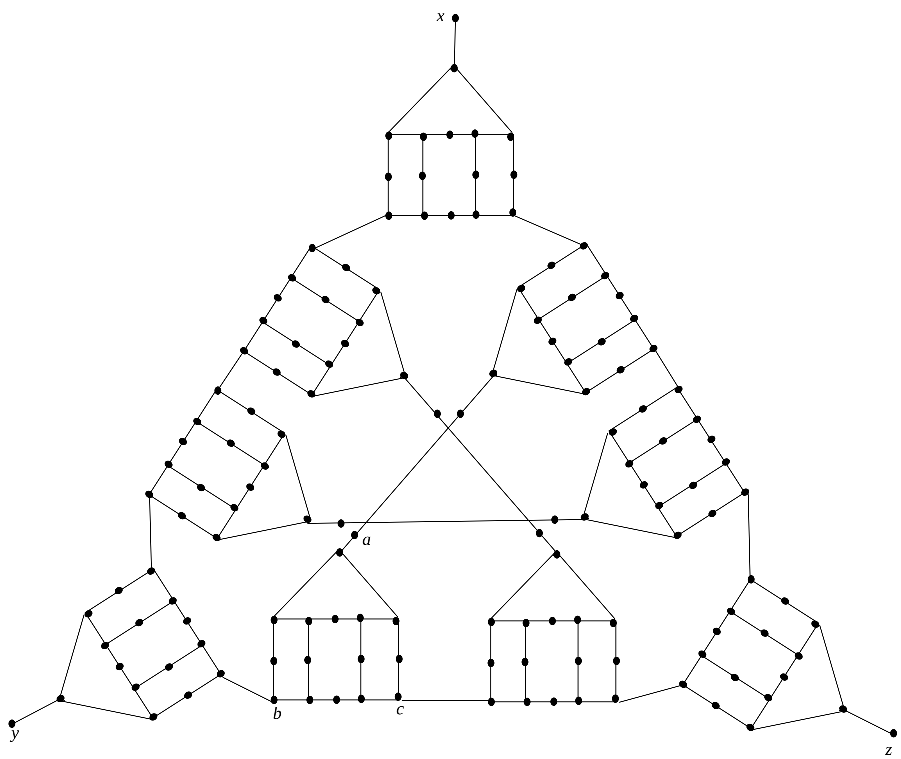

Figure 4.

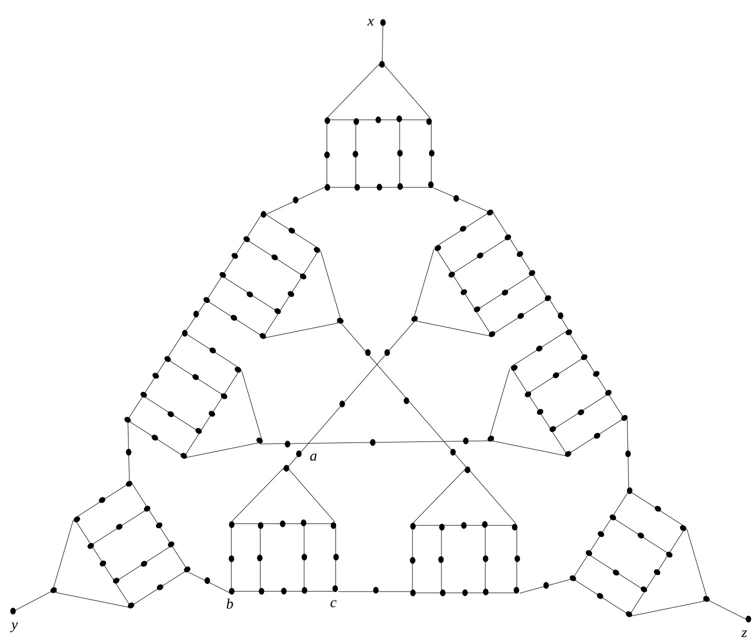

Figure 5.

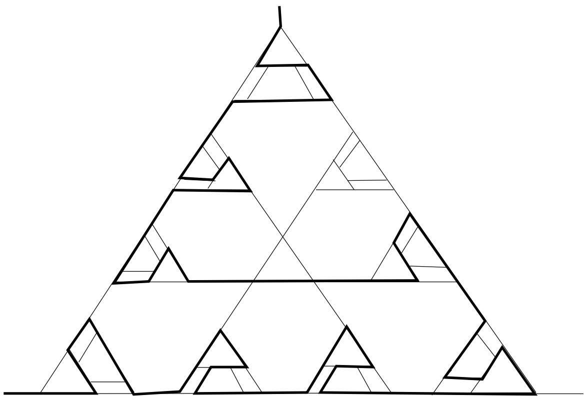

Figure 6.

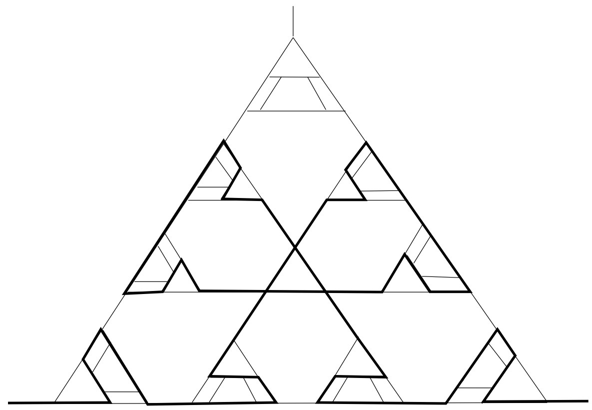

Figure 7.

D has, however, a longest path starting from x but not terminating at any of the vertices y, z, shown in Figure 8; it has order 128. Therefore, we add 17 vertices on each edge of D incident with the vertices x, y, and z. The graph D' thus obtained is homeomorphic to D and has order 207, with each longest path having 145 vertices. Hence there exists a graph of order 207 with the properties stated in Theorem 1, since every pair of vertices is missed by one of the longest paths shown in Figures 6, 7, and 8, and by using the appropriate longest paths of Lemma 1 contained in these paths.

The embedding of the graph H is shown in Figure 9. We use here embedding in Figure 9 to embed D' in $\mathcal{L}^3$ as shown in Figure 10.

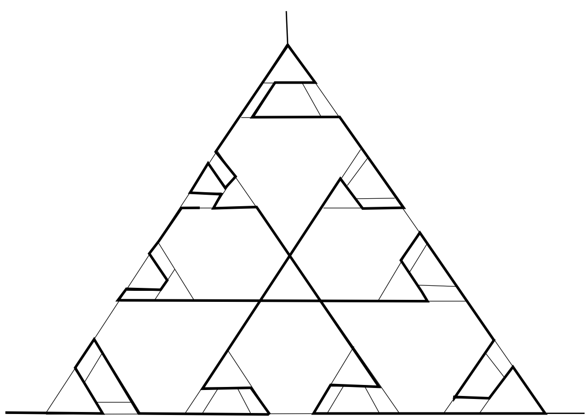

Figure 8.

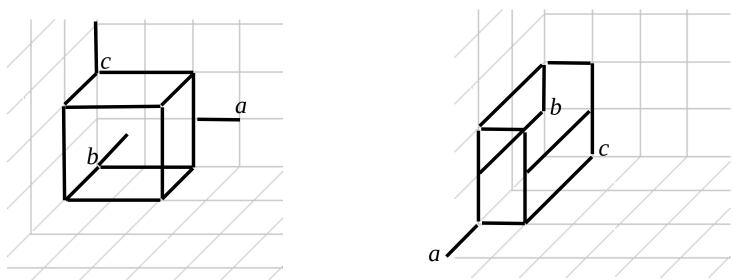

Figure 9.

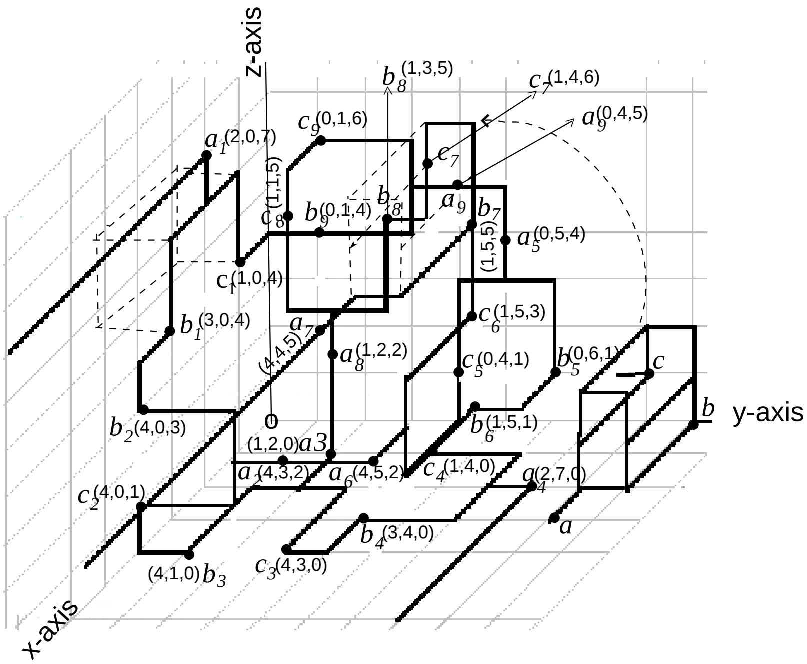

Figure 10.

3. Embeddings of graphs in which every pair of vertices is missed by some longest cycle

In this section we deal with 2-connected graphs in which every pair of vertices is avoided by some longest cycle. The smallest known planar example of such a graph was found by Zamfirescu [8]; it has order 135.

Lemma 2 *The longest paths in the graph G (shown in Figure 11) joining b and c have order 13 and those starting with the vertex a and terminating at either of the vertices b or c have order 12. Every vertex of G is missed by at least one of these paths.*

Proof This follows from Lemma 1. □

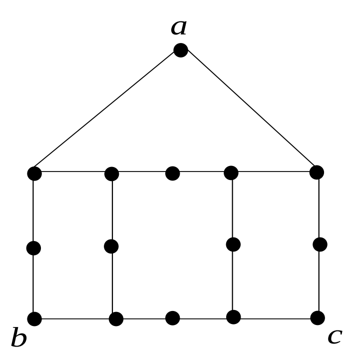

Figure 11.

Theorem 2 *There exists a 2-connected graph of order 210 in $\mathcal{L}^3$, in which every pair of vertices is missed by some longest cycle.*

Proof Consider the graph G of Figure 12 (see also [8]), and add t and w vertices on several edges, as shown in Figure 12. We get a graph $G(t,w)$. □

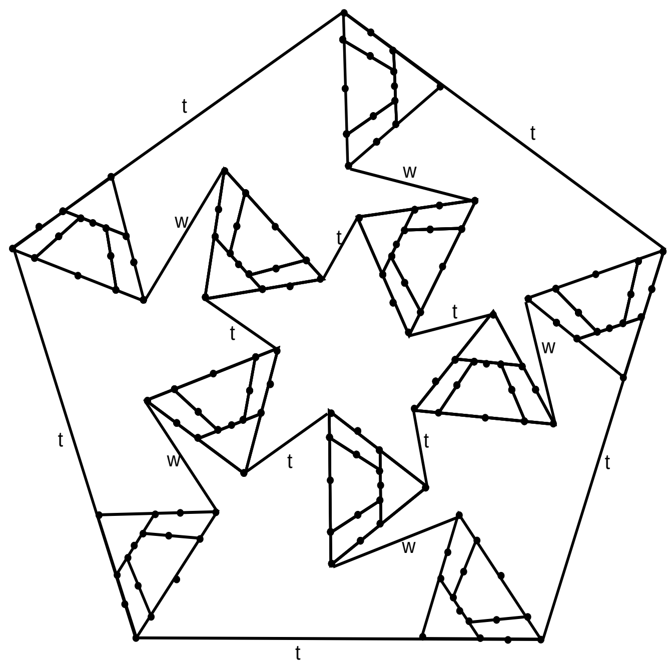

Figure 12.

In $G(t,w)$, consider the 2 different types of cycles. The longest cycles of the first type avoid and the longest cycles of the second type do not avoid the whole copy of G used in Lemma 2. One each of the first and

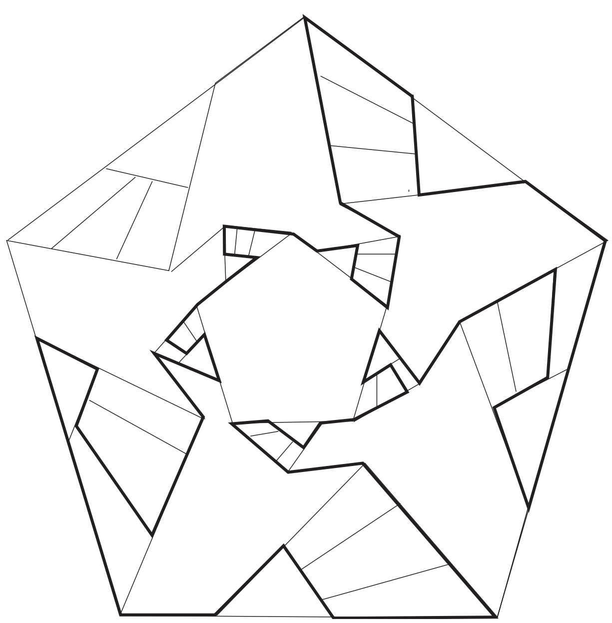

Figure 13.

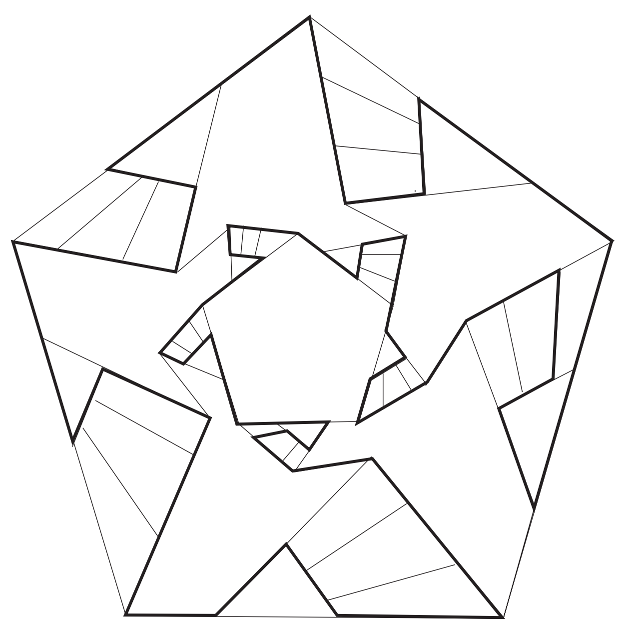

Figure 14.

second types of cycles are shown in Figures 13 and 14, respectively, which are the only serious candidates to being longest cycles, and they are denoted by C_1 and C_2, respectively. The length of C_1 is $112+5t+4w$ since it contains 4 copies of the longest path of order 13 and 5 copies of the longest path of order 12 of the graph used in Lemma 2. Similarly, the length of C_2 is $122+8t+2w$. As the cycles of both types must be equally long, we get $2w-3t=10$. $t=2$, $w=8$ verify this equation. We obtain the 2-connected graph $G(2,8)$ homeomorphic to G. It has order 210 and the property mentioned in Theorem 2. The longest cycle has length 154.

An embedding of $G(2,8)$ in $\mathcal{L}^3$ is shown in Figure 15. This is done by using the embedding of the graph H shown in Figure 9.

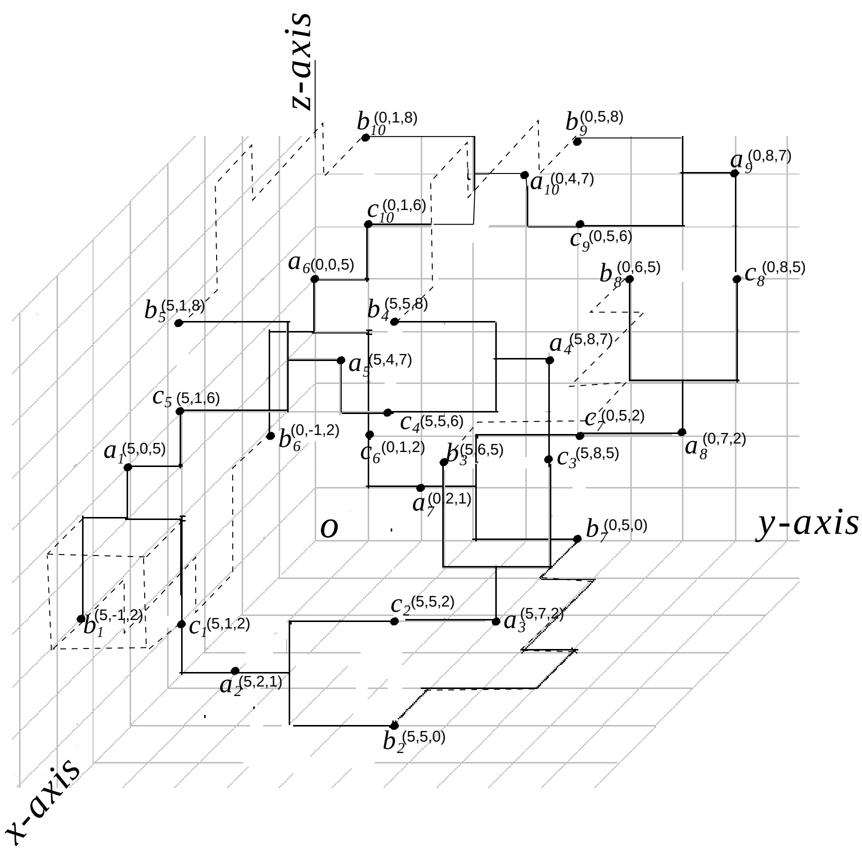

Figure 15.

Now we consider graphs in higher dimensional lattices.

Theorem 3 *There exists a* 2*-connected graph of order* 175 *in* $\mathcal{L}^4$*, in which every pair of vertices is missed by some longest cycle.*

Proof Consider the graph $G(t,w)$ of Figure 12. This time we can use the solutions $t=0$, $w=5$ of the equation $2w-3t=10$ from the proof of Theorem 2. We obtain the 2-connected graph $G(0,5)$ homeomorphic to G. It has order 175 and the property mentioned in Theorem 2, since every pair of vertices is missed by one of the cycles C_1 and C_2, which use the appropriate longest paths of Lemma 2 (see also [8]). The longest cycles have length 132.

An embedding of the graph $G(0,5)$ in $\mathcal{L}^4$ is shown in Figure 16. This is done by using again the embedding of the graph H shown in Figure 9. $\square$

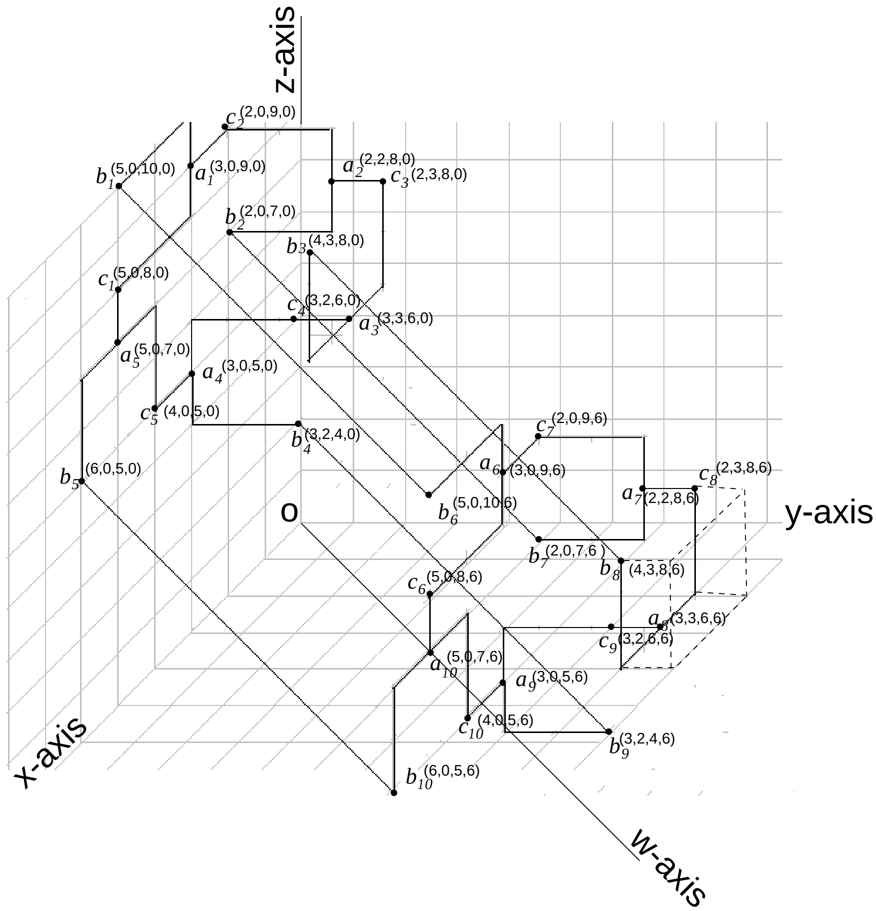

Figure 16.

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References

[1] Bashir Y, Zamfirescu T. Lattice graphs with Gallai's property. Bull Math Soc Sci Math Roumanie 2013, 104: 65–71.

[2] Gallai T. Problem 4. In: Erdös P, Katona G, editors. Theory of Graphs, Proc. Tihany 1966, New York, NY, USA: Academic Press, 1968, p. 362.

[3] Menke B, Zamfirescu Ch, Zamfirescu T. Intersection of longest cycles in grid graphs. J Graph Theory 1997; 25: 37–52.

[4] Menke B. Longest cycles in grid graphs. Studia Math Hung 2000; 36: 201–230.

[5] Shabbir A, Zamfirescu T. Highly non-concurrent longest cycles in lattice graphs. Discrete Math DOI: 10.1016/j.disc.2012.03.015.

[6] Walther H. Über die Nichtexistenz eines Knotenpunktes, durch den alle längsten Wege eines Graphen gehen. J Comb Theory 1969; 6: 1–6 (in German).

[7] Zamfirescu T. A two-connected planar graph without concurrent longest paths. J Combin Theory B 1972; 13: 116–121.

[8] Zamfirescu T. On longest paths and circuits in graphs. Math Scand 1976; 38: 211–239.

[9] Zamfirescu T. Intersecting longest paths or cycles: A short survey. Analele Univ Craiova, Seria Mat Inf 2001; 28: 1–9.