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Counting pseudo-Anosov mapping classes on the 3-punctured projective plane

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Abstract: We prove that in the pure mapping class group of the 3-punctured projective plane equipped with the word metric induced by certain generating set, the ratio of the number of pseudo-Anosov elements to the number of all elements in a ball centered at the identity tends to one, as the radius of the ball tends to infinity. We also compute growth functions of the sets of reducible and pseudo-Anosov elements.

Key words: Mapping class group, nonorientable surface, growth functions

1. Introduction

Let G be a group with a finite generating set A. For $x \in G$ the *length* of x with respect to A is defined to be the minimum number of factors needed to express x as a product of elements of A and their inverses. We denote it by $||x||_A$. The *word metric* on G with respect to A is defined as $d_A(x,y) = ||xy^{-1}||_A$ for $x, y \in G$. For a subset $X \subset G$, the *growth function* of X with respect to A is the function $f(z)$ defined by the power series $\sum_{n=0}^{\infty} C_n z^n$, where the coefficient C_n is equal to the number of elements of length n in X. The *density* $d(X)$ of X with respect to A is defined as

$$d(X) = \lim_{n\to\infty} \frac{\#(\mathcal{B}(n) \cap X)}{\#\mathcal{B}(n)},$$

where $\mathcal{B}(n)$ is the set of elements of G of length at most n (it is the ball of radius n, centered at the identity, with respect to the word metric induced by A), and $\#$ denotes the cardinality.

Let S be a compact surface with a finite set P of distinguished points in the interior of S called *punctures*. We denote as $\mathrm{Homeo}(S,P)$ the topological group of all, orientation preserving if S is orientable, homeomorphisms of S that preserve P and fix the boundary of S pointwise. The *mapping class group* of (S,P) is $\mathcal{M}(S,P) = \pi_0\mathrm{Homeo}(S,P)$. Elements of $\mathcal{M}(S,P)$ are isotopy classes of homeomorphisms in $\mathrm{Homeo}(S,P)$. By the *pure mapping class group* of (S,P) we understand in this paper the subgroup $\mathcal{PM}(S,P)$ of $\mathcal{M}(S,P)$ consisting of the isotopy classes of homeomorphisms fixing every puncture and also preserving local orientation at every puncture. Since the groups $\mathcal{M}(S,P)$ and $\mathcal{PM}(S,P)$ are finitely generated, it makes sense to study growth functions and densities of their subsets, with respect to various finite generating sets.

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Suppose that $\partial S = \emptyset$ and the Euler characteristic of $S\backslash P$ is negative. Let $\mathcal{C}(S,P)$ denote the set of isotopy classes of simple closed curves on $S\backslash P$ not bounding a disc with less than 2 punctures. The group $\mathcal{M}(S,P)$ acts on $\mathcal{C}(S,P)$. An element of $\mathcal{M}(S,P)$ is called *reducible* if it fixes a nonempty finite collection of pairwise disjoint elements of $\mathcal{C}(S,P)$. An element of $\mathcal{M}(S,P)$ that has infinite order and is not reducible is called *pseudo-Anosov*. By the Nielsen–Thurston classification of surface homeomorphisms (see [4, Chapter 13]), a pseudo-Anosov mapping class can be represented by a pseudo-Anosov homeomorphism h, such that there is a pair F^s, F^u of transverse measured foliations on S, such that $h(F^s) = \lambda^{-1}F^s$ and $h(F^u) = \lambda F^u$ for some $\lambda > 1$.

In this paper we consider the case when (S,P) is the projective plane with 3 punctures. The pure mapping class group $\mathcal{PM}(S,P)$ is free of rank 3. We fix free generators of $\mathcal{PM}(S,P)$ and consider the induced word metric. We prove the following results.

Theorem 1.1 *The growth functions of the sets of reducible and pseudo-Anosov elements in $\mathcal{PM}(S,P)$ are rational.*

We compute these growth functions explicitly.

Theorem 1.2 *Let $\mathcal{P}$ be the set of pseudo-Anosov elements in $\mathcal{PM}(S,P)$. Then $d(\mathcal{P}) = 1$.*

Analogous results were proved in [10] in the case when S is the torus, and in [1] for the 4-holed sphere. Our results, as well as those in [1, 10], give a partial answer to Question 3.14 and confirm Conjecture 3.15 in [3] in a special case. Similar results on "genericity" of pseudo-Anosovs, in the sense of random walks and not the word metric, were proved in the papers [6, 7, 8]. This paper seems to be the first in which problems of this type are considered for a nonorientable surface.

This paper is organised as follows: In Section 2 we give an algebraic characterisation of reducible elements in the pure mapping class group of the 3-punctured projective plane. In Section 3 we count for each $n \geq 1$ the numbers of reducible elements of length n and also determine growth functions of certain sets of reducible elements. The main results are proved in Section 4.

2. Pure mapping class group of the 3-punctured projective plane

For the rest of this paper let S be the projective plane obtained from the standard unit disc $D = \{z \in \mathbb{C} : |z| \leq 1\}$ by identifying antipodal points on ∂D. Let z_1, z_2, z_3 denote the images in S of the points $-\frac{3}{4}i, \frac{3}{4}e^{\frac{\pi i}{6}}, \frac{3}{4}e^{\frac{5\pi i}{6}}$ respectively. We fix $P = \{z_1, z_2, z_3\}$ and denote $\mathcal{PM}(S,P)$ simply as $\mathcal{PM}(S)$. We also fix the local orientation at each puncture z_i induced by the standard orientation of D.

A simple closed curve γ on S is called *nonseparating* if $S\backslash\gamma$ is connected, and *separating* otherwise. Every nonseparating curve on S is *one-sided*, which means that its regular neighbourhood is a Möbius strip. Let μ_0 be the image of ∂D in S and let μ_1, μ_2, μ_3 be the images in S of the line segments respectively $t, te^{\frac{2\pi i}{3}}, te^{\frac{\pi i}{3}}$ for $t \in [-1, 1]$. Note that these are one-sided curves. For $i = 0, 1, 2, 3$ let D_i be the disc obtained by cutting S along μ_i ($D_0 = D$) and fix the orientation of D_i induced by the local orientation at z_1. For $j = 1, 2, 3$ let α_j and β_j be the separating curves in the Figure. We fix Dehn twists T_{α_j}, T_{β_j}, such that T_{α_j} are right with respect to the orientation of D_0, T_{β_2} and T_{β_3} are right with respect to the orientation of D_1, and T_{β_1} is right

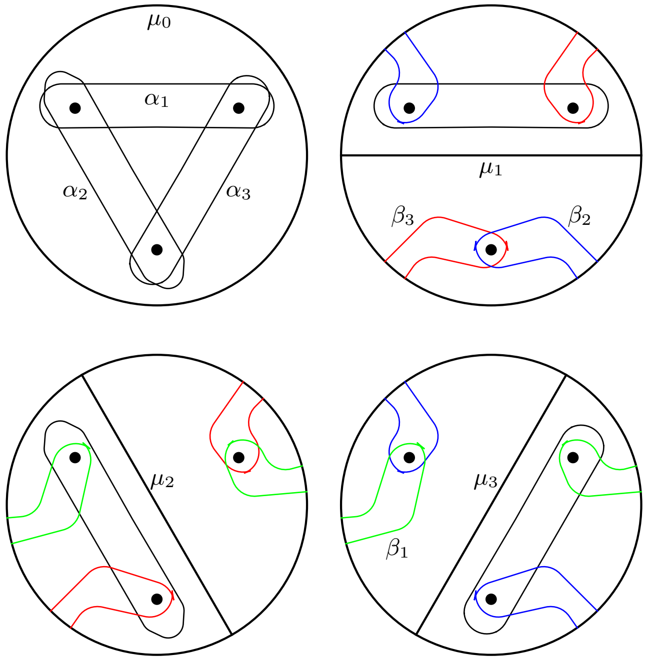

Figure. Curves on the 3-punctured projective plane S.

with respect to the orientation of D_2. Then in $\mathcal{PM}(S)$ we have the following relations:

$$\text{(L1)}\quad T_{\alpha_1}T_{\alpha_2}T_{\alpha_3}=1,\qquad \text{(L2)}\quad T_{\alpha_1}^{-1}T_{\beta_2}T_{\beta_3}=1,$$
$$\text{(L3)}\quad T_{\alpha_2}T_{\beta_1}T_{\beta_3}=1,\qquad \text{(L4)}\quad T_{\alpha_3}T_{\beta_2}T_{\beta_1}^{-1}=1.$$

They all follow from the well-known lantern relation between Dehn twists supported on a 4-holed sphere (see [4, Proposition 5.1]). In the lantern relation one has a product of 3 twists on one side of the equality and a product of 4 twists about the boundary components of the sphere on the other side. In our situation, however, the 4 twists are trivial, because they are about curves bounding once-punctured discs and a Möbius band.

Theorem 2.1 ([9, Theorem 7.5]) *The group $\mathcal{PM}(S)$ is freely generated by T_{α_1}, T_{α_2}, T_{β_1}.*

Since a free group is torsion free, every element of $\mathcal{PM}(S)$ is either reducible or pseudo-Anosov.

Lemma 2.2 *Let M be the Möbius strip with one puncture $p \in M$. Then $\mathcal{PM}(M,\{p\})$ is generated by a Dehn twist about the boundary of M.*

Proof Let F be the projective plane obtained from M by gluing a disc with a puncture q along ∂M. Since every $h \in \mathrm{Homeo}(M,\{p\})$ may be extended by the identity on the disc to $h' \in \mathrm{Homeo}(F,\{p,q\})$, we have a homomorphism $\mathcal{PM}(M,\{p\}) \to \mathcal{PM}(F,\{p,q\})$, which fits in the following short exact sequence (see [9, Section 7])

$$1 \to \mathbb{Z} \to \mathcal{PM}(M,\{p\}) \to \mathcal{PM}(F,\{p,q\}) \to 1,$$

where $\mathbb{Z}$ is generated by a Dehn twist $T_{\partial M}$. By [5, Corollary 4.6], $\mathcal{M}(F,\{p,q\})$ is isomorphic to the dihedral group of order 8, and since $\mathcal{PM}(F,\{p,q\})$ is a subgroup of index 8, thus it is trivial (note that in [5] a slightly different definition of the pure mapping class group of a nonorientable surface is used; its elements are allowed to reverse local orientation at the punctures). □

Proposition 2.3 *An element of $\mathcal{PM}(S)$ is reducible if and only if it fixes an isotopy class of one-sided curves.*

Proof Let h be a reducible homeomorphism of S. By definition, there is a set C of disjoint nonisotopic simple closed curves such that $h(C) = C$. If C contains a one-sided curve, then since any 2 one-sided curves on S intersect, C contains only one such curve, and this curve is fixed by h. If C does not contain a one-sided curve, then it consists of a single separating curve γ. Let E and M be the connected components of the surface obtained by cutting S along γ, where E is a punctured disc and M is a Möbius strip with at most one puncture. Clearly h preserves M and E, and since it preserves local orientation at the punctures, it also preserves orientation of E. It follows that h preserves orientation of γ and changing h by an isotopy we may assume that it is equal to the identity on γ. Let $h' = h|_M$. If there is no puncture in M then h' is isotopic to the identity on M by an isotopy fixing ∂M (see [2, Theorem 3.4]), while if there is a puncture in M, then h' is isotopic to some power of a Dehn twist about ∂M, by Lemma 2.2. In particular h is isotopic to a homeomorphism fixing a one-sided curve on M. □

We say that 2 simple closed curves γ_1 and γ_2 are $\mathcal{PM}(S)$*-equivalent* if $\gamma_1 = h(\gamma_2)$ for some $h \in \mathrm{Homeo}(S, P)$ fixing every puncture and preserving local orientation at every puncture.

Lemma 2.4 *Every one-sided simple closed curve on S is $\mathcal{PM}(S)$-equivalent to μ_i for some $i \in \{0, 1, 2, 3\}$.*

Proof Let γ be a one-sided simple closed curve and let E be the disc obtained by cutting S along γ. Fix the orientation of E induced by the local orientation at z_1. Let us compare the local orientations at z_2 and z_3 to the orientation of E. There are 4 cases.

Case 1. The local orientations at z_2 and z_3 agree with the orientation of E. Then there is an orientation preserving homeomorphism $f\colon D_0 \to E$, preserving the punctures, which commutes with the gluings giving back S. Thus f induces $h \in \mathrm{Homeo}(S, P)$ such that $h(\mu_0) = \gamma$.

Case 2. The local orientations at z_2 and z_3 are opposite to the orientation of E. Then there is an orientation preserving homeomorphism $f\colon D_1 \to E$ inducing $h \in \mathrm{Homeo}(S, P)$ such that $h(\mu_1) = \gamma$.

Case 3. The local orientation at z_3 agrees with the orientation of E, whereas that at z_2 is opposite. Then there is an orientation preserving homeomorphism $f\colon D_2 \to E$ inducing $h \in \mathrm{Homeo}(S, P)$ such that $h(\mu_2) = \gamma$.

Case 4. The local orientation at z_2 agrees with the orientation of E, whereas that at z_3 is opposite. Then there is $h \in \mathrm{Homeo}(S, P)$ such that $h(\mu_3) = \gamma$. □

The following corollary follows immediately from Proposition 2.3 and Lemma 2.4.

Corollary 2.5 *An element of $\mathcal{PM}(S)$ is reducible if and only if it is conjugate to an element fixing the isotopy class of μ_i for some $i \in \{0, 1, 2, 3\}$.*

For a group G and elements $x_1, \ldots, x_k \in G$ we denote by $\langle x_1, \ldots, x_k \rangle$ the subgroup of G generated by $x_1, \ldots, x_k$.

Proposition 2.6 *For $i = 0, 1, 2, 3$ let $\mathcal{S}_i$ denote the stabiliser in $\mathcal{PM}(S)$ of the isotopy class of μ_i. Then* $\mathcal{S}_0 = \langle T_{\alpha_1}, T_{\alpha_2} \rangle$, $\mathcal{S}_1 = \langle T_{\alpha_1}, T_{\alpha_2} T_{\beta_1} \rangle$, $\mathcal{S}_2 = \langle T_{\alpha_2}, T_{\beta_1} \rangle$, $\mathcal{S}_3 = \langle T_{\alpha_1} T_{\alpha_2}, T_{\beta_1} \rangle$.

Proof Fix $i \in \{0,1,2,3\}$ and consider the group $\mathcal{PM}(D_i, P)$. Since every homeomorphism of D_i equal to the identity on ∂D_i induces a homeomorphism of S, we have a homomorphism $\varphi_i\colon \mathcal{PM}(D_i,P) \to \mathcal{PM}(S,P)$. The image of φ_i is equal to $\mathcal{S}_i$, because every homeomorphism of S that fixes μ_i and preserves local orientation at the punctures must also preserve orientation of μ_i, and thus it is isotopic to a homeomorphism equal to the identity on μ_i. The group $\mathcal{PM}(D_i, P)$ is well known to be isomorphic to the pure braid group on 3 strands, and it is generated by Dehn twists about 3 curves, each curve surrounding 2 punctures, and each 2 curves intersecting each other twice (see [4, Chapter 9]). It follows that $\mathcal{S}_0 = \langle T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\rangle$, $\mathcal{S}_1 = \langle T_{\alpha_1}, T_{\beta_2}, T_{\beta_3}\rangle$, $\mathcal{S}_2 = \langle T_{\beta_1}, T_{\alpha_2}, T_{\beta_3}\rangle$, $\mathcal{S}_3 = \langle T_{\beta_1}, T_{\beta_2}, T_{\alpha_3}\rangle$. By the lantern relations (L1–L4) only 2 twists are needed to generate $\mathcal{S}_i$, and since $T_{\alpha_1}T_{\alpha_2} = T_{\alpha_3}^{-1}$ by (L1) and $T_{\alpha_2}T_{\beta_1} = T_{\beta_3}^{-1}$ by (L2), the proposition follows. □

3. Counting some words in the free group of rank 3

Let $\mathcal{F} = \mathcal{F}(a,b,c)$ be the free group on generators a,b,c. The elements of $\mathcal{F}$ are reduced words in the letters $a, a^{-1}, b, b^{-1}, c, c^{-1}$. By a word in $\mathcal{F}$ we always mean a reduced word. A word is *cyclically reduced* if its first letter is different from the inverse of its last letter. The number of letters in a word $w \in \mathcal{F}$ is *the length of* w denoted as $|w|$.

The following well-known theorem is the solution to the conjugacy problem in a free group.

Theorem 3.1 *Every element of a free group is conjugate to a cyclically reduced word. Two cyclically reduced words are conjugate if and only if one is a cyclic permutation of the other.*

By Theorem 2.1, there is an isomorphism $\rho\colon \mathcal{F} \to \mathcal{PM}(S)$ given by $\rho(a) = T_{\alpha_1}$, $\rho(b) = T_{\alpha_2}$, $\rho(c) = T_{\beta_1}$, which is an isometry with respect to the word metrics induced by the generating sets $\{a,b,c\}$ of $\mathcal{F}$ and $\{T_{\alpha_1}, T_{\alpha_2}, T_{\beta_1}\}$ of $\mathcal{PM}(S)$. Via this isomorphism we identify $\mathcal{F}$ with $\mathcal{PM}(S)$.

For $w_1, \dots, w_k \in \mathcal{F}$ we denote by $\mathcal{C}(w_1,\dots,w_k)$ the set of elements of $\mathcal{F}$ that are conjugate to elements of $\langle w_1,\dots,w_k\rangle$, and by $\mathcal{C}(w_1,\dots,w_k;n)$ the subset of $\mathcal{C}(w_1,\dots,w_k)$ consisting of elements of length n.

We also introduce the following notation:

$$\begin{aligned}
A_n &= \#\mathcal{C}(b;n),\\
B_n &= \#\mathcal{C}(a,b;n),\\
C_n &= \#\mathcal{C}(abc;n),\\
D_n &= \#(\mathcal{C}(a,bc;n)\setminus(\mathcal{C}(a;n)\cup\mathcal{C}(bc;n))).
\end{aligned}$$

Lemma 3.2 *Let R_n be the number of reducible elements of length n in $\mathcal{F}$. Then, for $n \geq 1$*

$$R_n = 2B_n + 2D_n - A_n - C_n.$$

Proof From Corollary 2.5 and Proposition 2.6 we have

$$R_n = \#(\mathcal{C}(a,b;n) \cup \mathcal{C}(b,c;n) \cup \mathcal{C}(a,bc;n) \cup \mathcal{C}(ab,c;n)).$$

It follows from Theorem 3.1 that

$$\mathcal{C}(a,b)\cap\mathcal{C}(b,c)=\mathcal{C}(b),\quad \mathcal{C}(a,b)\cap\mathcal{C}(a,bc)=\mathcal{C}(a)$$
$$\mathcal{C}(a,b)\cap\mathcal{C}(ab,c)=\mathcal{C}(ab),\quad \mathcal{C}(b,c)\cap\mathcal{C}(a,bc)=\mathcal{C}(bc)$$
$$\mathcal{C}(b,c)\cap\mathcal{C}(ab,c)=\mathcal{C}(c),\quad \mathcal{C}(a,bc)\cap\mathcal{C}(ab,c)=\mathcal{C}(abc).$$

We prove the last equality; the first 5 are easily verified. Let $w\in\mathcal{C}(a,bc)\cap\mathcal{C}(ab,c)$ be nontrivial. Then w is conjugate to a word

$$w_1=a^{x_1}(bc)^{x_2}\cdots a^{x_{2k-1}}(bc)^{x_{2k}},$$

where x_i are integers, and we may assume that w_1 is cyclically reduced. Analogously, w is conjugate to a cyclically reduced word of the form

$$w_2=(ab)^{y_1}c^{y_2}\cdots(ab)^{y_{2l-1}}c^{y_{2l}}.$$

By Theorem 3.1, w_1 is a cyclic permutation of w_2. It follows that w_1 is neither a power of a nor a power of bc. Therefore we can assume $x_i\neq 0$ for $1\leq i\leq 2k$ and $k\geq 1$. By replacing w by w^{-1} if necessary, we may assume $x_1>0$. Note that none of the words aa, ac^{-1}, cb, ca^{-1} can appear as a sub-word of a cyclic permutation of w_2. It follows that $x_i=1$ for $1\leq i\leq 2k$; hence $w_1=(abc)^k$ and $w\in\mathcal{C}(abc)$. We have shown that $\mathcal{C}(a,bc)\cap\mathcal{C}(ab,c)\subseteq\mathcal{C}(abc)$, and the opposite inclusion is obvious.

For $n\geq 1$ we have

$$R_n=\#\mathcal{C}(a,b;n)+\#\mathcal{C}(b,c;n)+\#(\mathcal{C}(a,bc;n)\backslash(\mathcal{C}(a;n)\cup\mathcal{C}(bc;n)))$$
$$+\#(\mathcal{C}(ab,c;n)\backslash(\mathcal{C}(c;n)\cup\mathcal{C}(ab;n)))-\#\mathcal{C}(b;n)-\#\mathcal{C}(abc;n).$$

The lemma follows because $\#(\mathcal{C}(ab,c;n)\backslash(\mathcal{C}(c;n)\cup\mathcal{C}(ab;n)))=D_n$ and $\#\mathcal{C}(b,c;n)=B_n$. □

Lemma 3.3 *For $k\geq 0$ we have $A_{2k+1}=A_{2k+2}=2\cdot 5^k$. The growth function of $\mathcal{C}(b)$ with respect to the generators a,b,c is $f_1(x)=\frac{1+2x-3x^2}{1-5x^2}$.*

Proof Every element of $\mathcal{C}(b)$ can be expressed uniquely in the form $w=ub^iu^{-1}$, where $i\in\mathbb{Z}$ and u is a word whose last letter is not $b^{\pm1}$. Let us fix $k\geq 0$. Observe there is a bijection $\mathcal{C}(b;2k+1)\to\mathcal{C}(b;2k+2)$ defined as $ub^iu^{-1}\mapsto ub^{i+1}u^{-1}$. Thus $A_{2k+1}=A_{2k+2}$. Let us count the words in $\mathcal{C}(b;2k+1)$. Every such word is of the form $w=ub^{\varepsilon(2i+1)}u^{-1}$, where u is a word whose last letter is not $b^{\pm1}$ of length $k-i$ for $0\leq i\leq k$ and $\varepsilon\in\{-1,1\}$. For a fixed i, there are 2 choices for ϵ, and if $i<k$ then there are $4\cdot 5^{k-i-1}$ choices for u. Thus

$$A_{2k+1}=2+\sum_{i=0}^{k-1}2\cdot 4\cdot 5^{k-i-1}=2+8\cdot 5^{k-1}\sum_{i=0}^{k-1}5^{-i}=2\cdot 5^k.$$

Now we can compute the growth function.

$$\begin{aligned}f_1(x)&=1+\sum_{k=0}^{\infty}(A_{2k+1}x^{2k+1}+A_{2k+2}x^{2k+2})=1+(1+x)\sum_{k=0}^{\infty}2\cdot 5^kx^{2k+1}\\&=1+(1+x)2x\sum_{k=0}^{\infty}(5x^2)^k=1+\frac{2x(1+x)}{1-5x^2}=\frac{1+2x-3x^2}{1-5x^2}.\end{aligned}$$

□

Lemma 3.4 *For $k \geq 0$ we have $B_{2k+1} = \frac{1}{3}B_{2k+2} = 6 \cdot 9^k - 2 \cdot 5^k$. The growth function of $\mathcal{C}(a,b)$ with respect to the generators a, b, c is*

$$f_2(x) = 1 + \frac{6x}{1-3x} - \frac{2x(1+3x)}{1-5x^2}.$$

Proof Every element of $\mathcal{C}(a,b)$ is either a word in $\langle a,b\rangle$ or it is of the form $uc^\varepsilon wc^{-\varepsilon}u^{-1}$, where $w \in \langle a,b\rangle$, $\varepsilon \in \{-1,1\}$, and u is a word whose last letter is not $c^{-\varepsilon}$. For $i \geq 1$ there are $4 \cdot 3^{i-1}$ words of length i in $\langle a,b\rangle$. It follows that $B_{2k+2} = 3B_{2k+1}$ for $k \geq 0$. Let us count words of the form $uc^\varepsilon wc^{-\varepsilon}u^{-1}$ of length $2k+1$. Suppose that $|w| = 2i+1$ for $0 \leq i \leq k-1$. Then $|u| = k-i-1$ and we have $4 \cdot 3^{2i}$ choices for w, 2 choices for ε, and 5^{k-i-1} choices for u. Thus

$$\begin{aligned} B_{2k+1} &= 4 \cdot 3^{2k} + \sum_{i=0}^{k-1} 8 \cdot 3^{2i} \cdot 5^{k-i-1} = 4 \cdot 3^{2k} + 8 \cdot 5^{k-1} \sum_{i=0}^{k-1} \left(\frac{9}{5}\right)^i = \\ &= 6 \cdot 9^k - 2 \cdot 5^k. \end{aligned}$$

$$\begin{aligned} f_2(x) &= 1 + \sum_{k=0}^{\infty} B_{2k+1}x^{2k+1} + 3B_{2k+1}x^{2k+2} \\ &= 1 + (1+3x)x \sum_{k=0}^{\infty} (6 \cdot 9^k - 2 \cdot 5^k)x^{2k} \\ &= 1 + (1+3x)x \left(\frac{6}{1-9x^2} - \frac{2}{1-5x^2}\right) = 1 + \frac{6x}{1-3x} - \frac{2x(1+3x)}{1-5x^2}. \end{aligned}$$

□

Lemma 3.5 *For $k \geq 0$ we have $C_{6k+3} = C_{6(k+1)} = \frac{6}{31}(5^{3k+2}+6)$, $C_{6k+5} = C_{6(k+1)+2} = 5C_{6k+3} - 6$, $C_{6(k+1)+1} = C_{6(k+1)+4} = 5C_{6k+5}$. The growth function of $\mathcal{C}(abc)$ with respect to the generators a, b, c is*

$$f_3(x) = 1 + \frac{6x^3}{31}\left(\frac{25(1+x^3)(1+5x^2+25x^4)}{1-(5x^2)^3} + \frac{6-x^2-5x^4}{1-x^3}\right).$$

Proof Every nontrivial element of $\mathcal{C}(abc)$ can be expressed uniquely in the form uv^iu^{-1}, where $i \geq 1$, $v \in \{(abc)^{\pm 1}, (bca)^{\pm 1}, (cab)^{\pm 1}\}$ and u is a word whose last letter is neither equal to the last letter of v nor to the inverse of the first letter of v.

Let us count the elements of $\mathcal{C}(abc; 6k+3)$. Every such element is of the form $uv^{2i+1}u^{-1}$, where u, v are as above, $0 \leq i \leq k$, and $|u| = 3(k-i)$. There are 6 choices for v and if $i < k$ then there are $4 \cdot 5^{3(k-i)-1}$ choices for u. Thus

$$C_{6k+3} = 6 + 24 \sum_{i=0}^{k-1} 5^{3(k-i)-1} = 6 + 24 \cdot 5^{3k-1} \sum_{i=0}^{k-1} 5^{-3i} = \frac{6}{31}(5^{3k+2}+6).$$

Every element of $\mathcal{C}(abc; 6k+5)$ is of the form $\alpha w \alpha^{-1}$ for $w \in \mathcal{C}(abc; 6k+3)$, where α is a single letter. For each w there are 4 choices for α if w is cyclically reduced, and 5 choices otherwise. There are 6 cyclically reduced words in $\mathcal{C}(abc; 6k+3)$, namely v^{2k+1} for $v \in \{(abc)^{\pm 1}, (bca)^{\pm 1}, (cab)^{\pm 1}\}$; hence $C_{6k+5} = 5C_{6k+3} - 6 = \frac{6}{31}(5^{3k+3} - 1)$.

Similarly, every element of $\mathcal{C}(abc; 6k+7)$ is of the form $\alpha w \alpha^{-1}$ for $w \in \mathcal{C}(abc; 6k+5)$, where α is a single letter. Since the words in $\mathcal{C}(abc; 6k+5)$ are not cyclically reduced, hence $C_{6k+7} = 5C_{6k+5}$.

Observe that the mapping $uv^iu^{-1} \mapsto uv^{i+1}u^{-1}$ defines bijections $\mathcal{C}(abc; 6k+3) \to \mathcal{C}(abc; 6k+6)$, $\mathcal{C}(abc; 6k+5) \to \mathcal{C}(abc; 6k+8)$ and $\mathcal{C}(abc; 6k+7) \to \mathcal{C}(abc; 6k+10)$. Thus $C_{6k+3} = C_{6k+6}$, $C_{6k+5} = C_{6k+8}$ and $C_{6k+7} = C_{6k+10}$.

Since $C_1 = C_2 = C_4 = 0$, thus

$$\begin{aligned} f_3(x) = 1 &+ \sum_{k=0}^{\infty} C_{6k+3}(x^{6k+3} + x^{6k+6}) \\ &+ \sum_{k=0}^{\infty} C_{6k+5}(x^{6k+5} + x^{6k+8} + 5x^{6k+7} + 5x^{6k+10}) \\ &= 1 + x^3(1+x^3)\sum_{k=0}^{\infty} C_{6k+3}x^{6k} + x^5(1+x^3)(1+5x^2)\sum_{k=0}^{\infty} C_{6k+5}x^{6k}. \end{aligned}$$

We have

$$\sum_{k=0}^{\infty} C_{6k+3}x^{6k} = \frac{6}{31}\sum_{k=0}^{\infty}(5^{3k+2}+6)x^{6k} = \frac{6}{31}\left(\frac{25}{1-(5x^2)^3} + \frac{6}{1-x^6}\right)$$

$$\sum_{k=0}^{\infty} C_{6k+5}x^{6k} = \frac{6}{31}\sum_{k=0}^{\infty}(5^{3k+3}-1)x^{6k} = \frac{6}{31}\left(\frac{125}{1-(5x^2)^3} - \frac{1}{1-x^6}\right)$$

It follows that $f_3(x)$ can be expressed by the formula given in the lemma. □

Lemma 3.6 *Let E_n denote the number of cyclically reduced words in $\mathcal{C}(a, bc; n) \backslash (\mathcal{C}(a; n) \cup \mathcal{C}(bc; n))$. Then for $n \geq 0$ we have*

$$E_{n+3} = E_{n+2} + E_{n+1} + 3E_n + 8 + (-1)^n 4. \tag{3.1}$$

Proof Let us define some subsets of $\mathcal{C}(a, bc; n)$:
$\mathcal{E}_n$ – the set of cyclically reduced words in $\mathcal{C}(a, bc; n) \backslash (\mathcal{C}(a; n) \cup \mathcal{C}(bc; n))$,
$\mathcal{X}_n$ – the set of words of length n, of the form $a^{\varepsilon_1}u(bc)^{\varepsilon_2}$,
$\overline{\mathcal{X}_n}$ – the set of words of length n, of the form $(bc)^{\varepsilon_1}ua^{\varepsilon_2}$,
$\mathcal{Y}_n$ – the set of words of length n, of the form $a^{\varepsilon_1}ua^{\varepsilon_2}$,
where $\varepsilon_i \in \{-1, 1\}$ for $i = 1, 2$ and $u \in \langle a, bc \rangle$. Note that $\mathcal{X}_n$ and $\overline{\mathcal{X}_n}$ are subsets of $\mathcal{E}_n$, but $\mathcal{Y}_n$ is not, as it contains words that are not cyclically reduced, and powers of a. The mapping $w \mapsto w^{-1}$ defines a bijection $\mathcal{X}_n \to \overline{\mathcal{X}_n}$. We define $X_n = \#\mathcal{X}_n = \#\overline{\mathcal{X}_n}$, $Y_n = \#\mathcal{Y}_n$.

Every element of $\mathcal{X}_{n+2}$ is of the form $w(bc)^\varepsilon$ for $w \in \mathcal{X}_n \cup \mathcal{Y}_n$. Conversely, if $n > 0$, then for $w \in \mathcal{X}_n$ there is 1 element of the form $w(bc)^\varepsilon$ in $\mathcal{X}_{n+2}$, while for $w \in \mathcal{Y}_n$ there are 2 such elements. Thus $X_{n+2} = X_n + 2Y_n$. Similarly we have $Y_{n+1} = Y_n + 2X_n$. Now we can obtain a recursive equation for X_n as follows: $X_{n+3} - X_{n+1} = 2Y_{n+1} = 2Y_n + 4X_n = X_{n+2} - X_n + 4X_n$. Thus for $n \geq 1$ we have

$$X_{n+3} = X_{n+2} + X_{n+1} + 3X_n. \tag{3.2}$$

For $n \geq 1$ we define a mapping $\iota\colon \mathcal{E}_n \to \mathcal{E}_{n+2}$. Let $w \in \mathcal{E}_n$. By the definition of $\mathcal{E}_n$ and Theorem 3.1, w is a word of length n in $\langle a, bc\rangle$, possibly cyclically permuted, that is neither a power of a nor a power of bc. We set

$$\iota(w) = \begin{cases} a^{\varepsilon}ua^{2\varepsilon} & \text{if } w = a^{\varepsilon}u \\ (bc)^{\varepsilon}u(bc)^{\varepsilon} & \text{if } w = (bc)^{\varepsilon}u \\ cubcb & \text{if } w = cub \\ b^{-1}u(bc)^{-1}c^{-1} & \text{if } w = b^{-1}uc^{-1}, \end{cases}$$

where $\varepsilon \in \{-1, 1\}$. Note that ι is injective and

$$\mathcal{E}_{n+2} = \iota(\mathcal{E}_n) \cup \mathcal{X}_{n+2} \cup \overline{\mathcal{X}_{n+2}} \cup \mathcal{Z} \cup \mathcal{U},$$

where $\mathcal{Z}$ is the set of words of the form $a^{\varepsilon_1}u(bc)^{\varepsilon_2}a^{\varepsilon_1}$, and $\mathcal{U}$ is the set of words of the form $cua^{\varepsilon_1}b$ or $b^{-1}ua^{\varepsilon_1}c^{-1}$, where $\varepsilon_i \in \{-1, 1\}$ for $i = 1, 2$ and $u \in \langle a, bc\rangle$. There are bijections $\mathcal{X}_{n+1} \to \mathcal{Z}$ given by $a^{\varepsilon_1}u(bc)^{\varepsilon_2} \mapsto a^{\varepsilon_1}u(bc)^{\varepsilon_2}a^{\varepsilon_1}$, and $\overline{\mathcal{X}_{n+2}} \to \mathcal{U}$ given by $bcua^{\varepsilon} \mapsto cua^{\varepsilon}b$, $(bc)^{-1}ua^{\varepsilon} \mapsto b^{-1}ua^{\varepsilon}c^{-1}$. Thus $\#\mathcal{Z} = X_{n+1}$, $\#\mathcal{U} = X_{n+2}$ and

$$E_{n+2} = 3X_{n+2} + X_{n+1} + E_n. \tag{3.3}$$

We have $E_n = X_n = 0$ for $n \leq 2$, $\mathcal{X}_3 = \{a^{\varepsilon_1}(bc)^{\varepsilon_2} \mid \varepsilon_1, \varepsilon_2 \in \{-1, 1\}\}$, $\mathcal{X}_4 = \{a^{2\varepsilon_1}(bc)^{\varepsilon_2} \mid \varepsilon_1, \varepsilon_2 \in \{-1, 1\}\}$; thus $X_3 = X_4 = 4$, $E_3 = 12$ and $E_4 = 16$. Thus (3.1) holds for $n = 0$ and $n = 1$. It is now routine to prove that (3.1) holds for all $n \geq 0$ by induction, using (3.3) and (3.2). □

Lemma 3.7 *For $n \geq 0$ we have*

$$D_{n+3} = D_{n+2} + D_{n+1} + 3D_n + \varphi(n), \tag{3.4}$$

where $\varphi(2k+1) = 4 \cdot 5^k$, $\varphi(2k) = 12 \cdot 5^k$ for $k \geq 0$. The growth function of $\mathcal{C}(a, bc)\backslash(\mathcal{C}(a) \cup \mathcal{C}(bc))$ with respect to the generators a, b, c is

$$f_4(x) = \frac{4x^3(3+x)}{(1-5x^2)(1-x-x^2-3x^3)}.$$

Proof Let $\mathcal{D}_n = \mathcal{C}(a, bc; n)\backslash(\mathcal{C}(a; n) \cup \mathcal{C}(bc; n))$. Every element of $\mathcal{D}_{n+2}$ that is not cyclically reduced is of the form $\alpha u \alpha^{-1}$, where α is a letter and $u \in \mathcal{D}_n$. Conversely, if $n \geq 1$, then for every $u \in \mathcal{D}_n$ there are 5 elements of the form $\alpha u \alpha^{-1}$ in $\mathcal{D}_{n+2}$ if u is not cyclically reduced, or 4 such words if u is cyclically reduced. Thus $D_{n+2} - E_{n+2} = 5(D_n - E_n) + 4E_n$, which gives, for $n \geq 0$,

$$D_{n+2} = E_{n+2} - E_n + 5D_n. \tag{3.5}$$

We have $D_n = E_n = 0$ for $n \leq 2$, $D_3 = E_3 = 12$ and $D_4 = E_4 = 16$. Thus (3.4) holds for $n = 0$ and $n = 1$. It is now routine to prove that (3.4) holds for all $n \geq 0$ by induction, using (3.5) and (3.1) from Lemma 3.6.

Now we can compute the growth function.

$$
\begin{aligned}
f_4(x) &= \sum_{n=0}^{\infty} D_n x^n = x^3 \sum_{n=0}^{\infty} D_{n+3} x^n \\
&= x^3 \sum_{n=0}^{\infty} (D_{n+2} + D_{n+1} + 3D_n + \varphi(n)) x^n \\
&= x f_4(x) + x^2 f_4(x) + 3x^3 f_4(x) + x^3 \sum_{k=0}^{\infty} 5^k (12x^{2k} + 4x^{2k+1}) \\
&= (x + x^2 + 3x^3) f_4(x) + \frac{4x^3(3+x)}{1-5x^2},
\end{aligned}
$$

and the lemma is proved. □

4. Growth functions and density of reducible and pseudo-Anosov elements

In this section we prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Let $f(x)$ and $g(x)$ denote the growth functions of the sets of reducible and pseudo-Anosov elements respectively. Since $f(x) + g(x)$ is the growth function of $\mathcal{PM}(S)$, we have

$$
f(x) + g(x) = 1 + 6 \sum_{n=1}^{\infty} 5^{n-1} x^n = \frac{1+x}{1-5x}.
$$

Let $f_1(x), f_2(x), f_3(x), f_4(x)$ be the growth functions computed in Lemmas 3.3, 3.4, 3.5, 3.7. By Lemma 3.2 we have

$$
\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} R_n x^n = 1 + \sum_{n=1}^{\infty} (2B_n + 2D_n - A_n - C_n) x^n \\
&= 1 + 2f_2(x) + 2f_4(x) - f_1(x) - f_3(x),
\end{aligned}
$$

which is a rational function. Since $f(x)$ and $f(x) + g(x)$ are rational, so is $g(x)$. □

Let $f(n)$ and $g(n)$ be 2 sequences of nonnegative numbers. We write $f(n) = \Theta(g(n))$ if there exist 2 positive numbers c_1, c_2 such that $c_1 g(n) \leq f(n) \leq c_2 g(n)$ for all but finitely many n.

Lemma 4.1 *Let $\mathcal{R}$ be the set of reducible elements in $\mathcal{PM}(S)$. Then $\#(\mathcal{B}(n) \cap \mathcal{R}) = \Theta(3^n)$.*

Proof Since we have the isometry $\rho\colon \mathcal{F} \to \mathcal{PM}(S)$,

$$
\#(\mathcal{B}(n) \cap \mathcal{R}) = \sum_{k=0}^{n} R_k.
$$

Clearly it suffices to show that $R_n = \Theta(3^n)$. We have $R_n > B_n$ and, by Lemma 3.2, $R_n < 2(B_n + D_n)$. Since $B_n = \Theta(3^n)$ by Lemma 3.4, it suffices to show that $D_n < 3^n$. That is easily proved by induction, using (3.4) from Lemma 3.7 and the inequality $\varphi(n) \leq 12 \cdot 3^n$. □

Proof of Theorem 1.2. By Lemma 4.1 we have $\#(\mathcal{B}(n) \cap \mathcal{R}) = \Theta(3^n)$, and since

$$\#\mathcal{B}(n) = 1 + 6\sum_{k=0}^{n-1} 5^k = \frac{3 \cdot 5^n - 1}{2},$$

thus $d(\mathcal{R}) = 0$. The result follows, because $d(\mathcal{P}) = 1 - d(\mathcal{R})$. □

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