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On minimal Poincaré 4-complexes

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Abstract: We consider 2 types of minimal Poincaré 4-complexes. One is defined with respect to the degree 1-map order. This idea was already present in our previous papers, and more systematically studied later by Hillman. The second type of minimal Poincaré 4-complexes was introduced by Hambleton, Kreck, and Teichner. It is not based on an order relation. In the present paper we study existence and uniqueness questions.

Key words: Poincaré 4-complex, equivariant intersection form, degree 1-map, k-invariant, homotopy type, obstruction theory, homology with local coefficients, Whitehead's quadratic functor, Whitehead's exact sequence

1. Introduction

Minimal objects are usually defined with respect to a partial order. We consider oriented Poincaré 4-complexes (in short, PD_4-complexes). If X and Y are 2 PD_4-complexes, we define $X \succ Y$ if there is a degree 1-map $f : X \to Y$ inducing an isomorphism on the fundamental groups. If also $Y \succ X$, well-known theorems imply that $f : X \to Y$ is a homotopy equivalence. So "$\succ$" defines a symmetric partial order on the set of homotopy types of PD_4-complexes. A PD_4-complex P is said to be *minimal* for X if $X \succ P$ and whenever $P \succ Q$, Q is homotopy equivalent to P. We also consider special minimal objects called *strongly minimal*. In this paper we study existence and uniqueness questions. It is an interesting problem to calculate homotopy equivalences of X relative to a minimal P: that is, if $f : X \to P$ is as above, then calculate

$$\mathrm{Aut}(X \succ P) = \{h : X \to X : h \text{ homotopy equivalence such that } f \circ h \text{ is homotopic to } f\}.$$

Self-homotopy equivalences were studied by various authors (see [12] and references there). Pamuk's method can be used to calculate $\mathrm{Aut}(X \succ P)$.

Constructions of minimal objects were indicated by Hegenbarth, Repovš, and Spaggiari in [6] and more recently by Hillman in [8] and [9]. Degree 1-maps can be constructed from Λ-submodules $G \subset H_2(X, \Lambda)$. More precisely, we have the following (cf. Proposition 2.4 below):

Proposition 1.1 *Suppose X is a Poincaré 4-complex, and $G \subset H_2(X, \Lambda)$ is a stably free Λ-submodule such that the intersection form λ_X restricted to G is nonsingular. Then one can construct a Poincaré 4-complex Y*

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and a degree 1*-map* $f : X \to Y$. *Moreover, there is an isomorphism*

$$K_2(f,\Lambda) = \mathrm{Ker}(H_2(X,\Lambda) \xrightarrow{f_*} H_2(Y,\Lambda)) \cong G$$

and λ_X *restricted to* $K_2(f,\Lambda)$ *coincides with* λ_X *on* G *via this isomorphism.*

Corollary 1.2 *Given any Poincaré* 4*-complex* X, *there exists a minimal Poincaré* 4*-complex* P *for* X.

The above proposition is useful to answer the following 2 basic questions about the minimal objects:

(1) Existence; and

(2) Uniqueness.

A Poincaré 4-complex P is called *strongly minimal* for π if the adjoint map $\hat{\lambda}_P : H_2(P,\Lambda) \to \mathrm{Hom}_\Lambda(H_2(P,\Lambda),\Lambda)$ of the intersection form λ_P vanishes [8]. Proposition 1.1 implies that P is minimal. The same questions arise if we consider the originally defined minimal objects in [5].

Existence of strongly minimal models P is known only for few fundamental groups π (see [5] and [8]). All these examples satisfy $H^3(B\pi,\Lambda) \cong 0$, and hence $\mathrm{Hom}_\Lambda(H_2(P,\Lambda),\Lambda) \cong 0$ (see below). So all are "trivial" in the sense that λ_P is zero because its adjoint $\hat{\lambda}_P : H_2(P,\Lambda) \to \mathrm{Hom}_\Lambda(H_2(P,\Lambda),\Lambda)$ maps to the trivial Λ-module. An interesting question is therefore: *Do there exist strongly minimal models* P *such that* $H^3(B\pi_1(P),\Lambda) \neq 0$?

We prove the following:

Theorem 1.3 *Let* π *be a finitely presented group such that* $H^2(B\pi,\Lambda)$ *is not a torsion group. Let* P *and* P' *be strongly minimal models for* π. *Then* P *and* P' *are homotopy equivalent if the map* $G : H_4(D,\mathbb{Z}) \to \mathrm{Hom}_\Lambda(H^2(D,\Lambda),\overline{H}_2(D,\Lambda))$ *is injective, and if the* k*-invariants of* P *and* P' *correspond appropriately.*

Here D is a 2-stage Postnikov space and G is defined via cap-products. Apart from the k-invariant, the injectivity of the map G is an essential condition for uniqueness of strongly minimal models. In Section 4 we consider groups π such that $B\pi$ is homotopy equivalent to a 2-complex and prove that for any element of $\mathrm{Ker}\, G$ one can construct a strongly minimal model. More precisely, we obtain:

Theorem 1.4 *Suppose* $B\pi$ *is homotopy equivalent to a* 2*-complex, and* $\pi_2 = H^2(B\pi,\Lambda)$ *is not a torsion group. Then* $\mathrm{Ker}\, G \cong \Gamma(\pi_2) \otimes_\Lambda \mathbb{Z}$. *Moreover, for any strongly minimal model* P *and any* $\xi \in \Gamma(\pi_2)$, *another strongly minimal model* X *can be constructed.*

Examples are given by solvable Baumslag–Solitar groups (see [5]), or by surface fundamental groups. In Section 5 we construct non-homotopy equivalent strongly minimal models for these fundamental groups.

2. Construction of degree 1-maps

In this section we are going to prove Proposition 1.1 announced in Section 1. First we mention a result of Wall [14].

Lemma 2.1 *Let* $f : X \to Y$ *be a degree* 1*-map between Poincaré* 4*-complexes and suppose that* $f_* : \pi_1(X) \to \pi_1(Y)$ *is an isomorphism. Then* $K_2(f,\Lambda) = \mathrm{Ker}(H_2(X,\Lambda) \to H_2(Y,\Lambda))$ *is a stably* Λ*-free submodule of* $H_2(X,\Lambda)$ *and* λ_X *restricted to* $K_2(f,\Lambda)$ *is nonsingular. Also,* $K_2(f,\Lambda) \subset H_2(X,\Lambda)$ *is a direct summand.*

This section is devoted to proving a converse statement to Lemma 2.1.

First we will show Proposition 2.2. Before that, let us note that Λ has an anti-involution that permits a switch from Λ-left to Λ-right modules and to introduce compatible Λ-module structures on Hom-duals, etc. We follow Wall's convention and consider Λ-right modules.

Proposition 2.2 *Let X be a Poincaré 4-complex and $G \subset H_2(X,\Lambda)$ a Λ-free submodule so that λ_X restricts to a nonsingular Hermitian pairing on G. Then there exist a Poincaré 4-complex P and a degree 1-map $f : X \to P$ such that $f_* : \pi_1(X) \to \pi_1(P)$ is an isomorphism and $K_2(f,\Lambda) \cong G$.*

Proof We recall that λ_X is defined as the composite map

$$\begin{array}{ccc} H^2(X,\Lambda)\times H^2(X,\Lambda) & \xrightarrow{\ \cup\ } & H^4(X,\Lambda\otimes_{\mathbb{Z}}\Lambda)\cong H_0(X,\Lambda\otimes_{\mathbb{Z}}\Lambda)\cong \mathbb{Z}\otimes_{\Lambda}(\Lambda\otimes_{\mathbb{Z}}\Lambda) \\ \cong\uparrow & & \uparrow\cong \\ H_2(X,\Lambda)\times H_2(X,\Lambda) & \xrightarrow{\ \lambda_X\ } & \Lambda\cong\Lambda\otimes_{\Lambda}\Lambda \end{array}$$

and

$$\hat{\lambda}_X : H_2(X,\Lambda) \to \mathrm{Hom}_{\Lambda}(H_2(X,\Lambda),\Lambda)$$

is the adjoint map of λ_X.

To construct P, we consider a Λ-base $a_1, \ldots, a_r$ of $G \subset H_2(X,\Lambda) \cong \pi_2(X)$, and

$$\varphi_1, \ldots, \varphi_r : \mathbb{S}^2 \to X$$

representatives of $a_1, \ldots, a_r$, respectively. Then P is obtained from X by adjoining 3-cells along $\varphi_1, \ldots, \varphi_r$. So $X \subset P$, and

$$H_p(P,X,\Lambda) \cong \begin{cases} G & p=3 \\ 0 & \text{otherwise} \end{cases} \qquad H^p(P,X,\Lambda) \cong \begin{cases} G^* = \mathrm{Hom}_{\Lambda}(G,\Lambda) & p=3 \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, the sequence

$$0 \longrightarrow H_3(P,X,\Lambda) \xrightarrow{\ \partial_*\ } H_2(X,\Lambda) \longrightarrow H_2(P,\Lambda) \longrightarrow 0$$

is exact because $\partial_* : H_3(P,G,\Lambda) \to G \subset H_2(X,\Lambda)$ is an isomorphism.

Note that there is a natural homomorphism

$$\mu : H^2(X,\Lambda) \to \mathrm{Hom}_{\Lambda}(H_2(X,\Lambda),\Lambda)$$

such that the diagram

$$\begin{array}{ccc} H^2(X,\Lambda) & \xrightarrow{\ \mu\ } & \mathrm{Hom}_{\Lambda}(H_2(X,\Lambda),\Lambda) \\ \cap[X]\downarrow & & \| \\ H_2(X,\Lambda) & \xrightarrow{\ \hat{\lambda}_X\ } & \mathrm{Hom}_{\Lambda}(H_2(X,\Lambda),\Lambda) \end{array}$$

commutes. Let $[P] = f_*[X]$, where $f : X \subset P$ is the inclusion. Consider the diagrams

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & H^2(P,\Lambda) & \xrightarrow{\ f^*\ } & H^2(X,\Lambda) & \xrightarrow{\ \delta^*\ } & H^3(P,X,\Lambda) & \xrightarrow[\cong]{\ \mu\ } & \mathrm{Hom}_{\Lambda}(H_3(P,X,\Lambda),\Lambda) = G^* & & \\ & & \cap[P]\downarrow & & \downarrow\cap[X] & & & & \uparrow\hat{\lambda}_G & & \\ 0 & \longleftarrow & H_2(P,\Lambda) & \xleftarrow{\ f_*\ } & H_2(X,\Lambda) & \xleftarrow{\ \partial_*\ } & H_3(P,X,\Lambda) = G & = & G & \longleftarrow & 0 \end{array}$$

and

$$\begin{array}{ccc} H^2(X,\Lambda) & \xrightarrow{\delta^*} & H^3(P,X,\Lambda) \\ \mu\downarrow & & \cong\downarrow\mu \\ \mathrm{Hom}_\Lambda(H_2(X,\Lambda),\Lambda) & \longrightarrow & \mathrm{Hom}_\Lambda(H_3(P,X,\Lambda),\Lambda) = G^* \\ \uparrow\overset{\wedge}{\lambda}_X & & \uparrow\overset{\wedge}{\lambda}_G \\ H_2(X,\Lambda) & \xleftarrow{\partial_*} & H_3(P,X,\Lambda) = G. \end{array}$$

Here $\overset{\wedge}{\lambda}_G = \overset{\wedge}{\lambda}_X|_G$. The left-hand square of the first diagram commutes. Combining the right-hand square of the first diagram with the second diagram gives only

$$\mu \circ \delta^* \circ (\cap[X])^{-1} \circ \partial_* = \overset{\wedge}{\lambda}_G.$$

However, this is sufficient to deduce that $\cap[P] : H^2(P,\Lambda) \to H_2(P,\Lambda)$ is an isomorphism. It follows from the above short exact sequence that

$$f_* : H_3(X,\Lambda) \xrightarrow[\cong]{} H_3(P,\Lambda) \qquad f^* : H^3(P,\Lambda) \xrightarrow[\cong]{} H^3(X,\Lambda)$$

hence we obtain that

$$\cap[P] : H^*(P,\Lambda) \xrightarrow[\cong]{} H_{4-*}(P,\Lambda)$$

for all $*$. The map f is obviously of degree 1. □

In the sequel we shall need another result of Wall about Poincaré complexes (see for instance [14]).

Lemma 2.3 *Any Poincaré 4-complex X is homotopy equivalent to a CW–complex of the form $K \cup_\varphi D^4$, where K is a 3-complex and $\varphi : \mathbb{S}^3 \to K$ is an attaching map of the single 4-cell D^4.*

Proposition 2.2 can be improved so that together with Lemma 2.1, we obtain the following:

Proposition 2.4 *Let X be a Poincaré 4-complex. There exists a degree 1-map $f : X \to Q$ if and only if there exists a stably free Λ-submodule $G \subset H_2(X,\Lambda)$ so that λ_X restricts to a nonsingular Hermitian form on G. In this case, $G \cong K_2(f,\Lambda)$.*

Proof By Lemma 2.3 we can identify $X = K \cup_\varphi D^4$. The submodule G is stably free, so $G \oplus H \cong \oplus_1^\ell \Lambda$, where H is Λ-free. We may assume $H = \oplus_1^{2m}\Lambda$. Let $Z = X\#(\#_1^m(\mathbb{S}^2 \times \mathbb{S}^2))$ be the Poincaré 4-complex formed from X by connected sum inside the 4-cell with $\#_1^m(\mathbb{S}^2 \times \mathbb{S}^2)$.Then $G \oplus H \subset H_2(Z,\Lambda)$ and λ_Z restricted to H is the canonical hyperbolic form. If $a_1, \ldots, a_\ell \in G \oplus H$ is a Λ-base, we attach 3-cells to Z along representatives $\varphi_1, \ldots, \varphi_\ell : \mathbb{S}^2 \to X$ as in Proposition 2.2. We obtain a Poincaré 4-complex Q and a degree 1-map $g : Z \to Q$ with $K_2(g,\Lambda) = G \oplus H$. We are going to show that g factors over the collapsing map

$$c : Z = X\#(\#_1^m(\mathbb{S}^2 \times \mathbb{S}^2)) \to X$$

giving a degree 1-map $f : X \to Q$. Note that

$$X\#(\#_1^m(\mathbb{S}^2 \times \mathbb{S}^2))\backslash 4\text{–cell} \simeq K \vee \{\vee_1^m(\mathbb{S}^2 \vee \mathbb{S}^2)\}$$

and the attaching map of the 4-cell of Z is of the following type

$$a \oplus b \in \pi_3(K) \oplus [\pi_3(\vee_1^m(\mathbb{S}^2 \vee \mathbb{S}^2)) \otimes \Lambda] \subset \pi_3(Z \backslash (4\breve{g}\text{cell})),$$

where $a = [\varphi]$ and $b = [\psi] \otimes 1$ with $\psi : \mathbb{S}^3 \to \vee_1^m(\mathbb{S}^2 \vee \mathbb{S}^2)$ the attaching map of the 4-cell of $\#_1^m(\mathbb{S}^2 \times \mathbb{S}^2)$. Obviously, $a \oplus b$ maps to zero in $\pi_3(Q)$.

Now we apply Whitehead's Γ-functor to

$$\pi_2(Z) \cong \pi_2(K) \oplus H \cong \pi_2(Z \backslash (4\text{–cell})) : \Gamma(\pi_2(Z)) \cong \Gamma(\pi_2(K)) \oplus \Gamma(H) \oplus \pi_2(K) \otimes H.$$

The Γ-functor fits into a certain Whitehead's exact sequence (see [1] and [15]) and by naturality one has the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma(\pi_2(K) \oplus H) & \longrightarrow & \pi_3(Z\backslash(4\text{–cell})) & \longrightarrow & H_3(Z\backslash(4\text{–cell}), \Lambda) & \longrightarrow & 0 \\
 & & \| & & & & \| & & \\
0 & \longrightarrow & \Gamma(\pi_2(K)) \oplus \Gamma(H) \oplus \pi_2(K) \otimes H & \longrightarrow & \pi_3(Z\backslash(4\text{–cell})) & \longrightarrow & H_3(K, \Lambda) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
H_4(Q, \Lambda) & \longrightarrow & \Gamma(\pi_2(Q)) & \longrightarrow & \pi_3(Q) & \longrightarrow & H_3(Q, \Lambda) & \longrightarrow & 0.
\end{array}$$

Obviously, $b \in \Gamma(H) \subset \pi_3(Z \backslash (4\text{–cell}))$, and hence $b = \sum \lambda_{ij}[e_i, e_j]$, where the set $\{e_1, \ldots, e_{2m}\} \subset H$ is the standard base and $[\cdot, \cdot]$ denotes the Whitehead product. Now $H \subset G \oplus H \subset \pi_2(Z)$ maps to zero under $g_* : \pi_2(Z) \to \pi_2(Q)$, so $b \in \Gamma(\pi_2(K) \oplus H)$ maps to zero in $\Gamma(\pi_2(Q))$, and hence it is zero in $\pi_3(Q)$. Because $a \oplus b$ is zero in $\pi_3(Q)$, $a \in \pi_3(K)$ also maps to zero under $\pi_3(K) \to \pi_3(Q)$. Therefore, the inclusion map $K \subset Q$ extends to $f : X \to Q$, and f induces a map

$$(X, K) \to (Q, Q \backslash (4\text{–cell})).$$

We also have

$$g : (Z, Z \backslash (4\text{–cell})) \to (Q, Q \backslash (4\text{–cell}))$$

and a collapsing map

$$c : (Z, Z \backslash (4\text{–cell})) \to (X, K).$$

Since Q is obtained from Z by adding 3-cells attached away from the 4-cell, the following diagram commutes:

$$\begin{array}{ccc}
H_4(Z, Z\backslash(4\text{–cell}), \mathbb{Z}) & \xrightarrow{c_*} & H_4(X, K, \mathbb{Z}) \\
{\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle f_*} \\
H_4(Q, Q\backslash(4\text{–cell}), \mathbb{Z}) & = & H_4(Q, Q\backslash(4\text{–cell}), \mathbb{Z}).
\end{array}$$

Because c_* and g_* map the fundamental class to the fundamental class, the degree of f is 1. □

Proof of Corollary 1.2 We observe that for any degree 1-map $f : X \to Y$ with $f_* : \pi_1(X) \underset{\cong}{\to} \to \pi_1(Y)$, one has

$$K_2(f, \Lambda) \otimes_\Lambda \mathbb{Z} = K_2(f, \mathbb{Z}) = \text{Ker}(H_2(X, \mathbb{Z}) \to H_2(Y, \mathbb{Z})),$$

and that $H_2(X,\mathbb{Z})$ is finitely generated. By Proposition 2.4 we can successively construct degree 1-maps

$$X \xrightarrow{f} Q, \qquad Q_1 \xrightarrow{f_1} Q_2, \quad \cdots$$

if we find nondegenerate stably free nontrivial submodules in $H_2(Q_k,\Lambda)$, and one has

$$K_2(f_k\circ\cdots\circ f_1\circ f,\Lambda)\cong K_2(f_k,\Lambda)\oplus\cdots\oplus K_2(f_1,\Lambda)\oplus K_2(f,\Lambda)\subset H_2(X,\Lambda).$$

Now

$$K_2(f_k\circ\cdots\circ f_1\circ f,\mathbb{Z})\cong K_2(f_k,\mathbb{Z})\oplus\cdots\oplus K_2(f_1,\mathbb{Z})\oplus K_2(f,\mathbb{Z})\subset H_2(X,\mathbb{Z})$$

is finitely generated. Hence, after certain k, we have

$$K_2(f_{k+1},\Lambda)\otimes_\Lambda\mathbb{Z}=K_2(f_{k+1},\mathbb{Z})=\{0\}.$$

Kaplansky's lemma (see remark below) implies $K_2(f_{k+1},\Lambda)\cong 0$. Therefore, $g=f_k\circ\cdots\circ f_1\circ f:X\to Q_k$ is of degree 1, and Q_k is minimal. This completes the proof of Corollary 1.2.

Remark In [10, p.122], the following result is stated:

Lemma *Let $\mathbb{F}$ be a field of characteristic zero, and π an arbitrary group. Let $A=\mathbb{F}[\pi]$ be the group algebra, and let u, $v\in M_n(A)$ be 2 $(n\times n)$ matrices such that the product vu is the identity matrix I_n. Then $uv=I_n$.*

It has the following consequence (referred to above as "Kaplansky's lemma"):

Corollary *If $K_2(f,\Lambda)\otimes_\Lambda\mathbb{Q}\cong 0$, then $K_2(f,\Lambda)\cong 0$.*

Proof We know that $K_2=K_2(f,\Lambda)$ is stably free, i.e. $K_2\oplus\Lambda^a\cong\Lambda^b$, where a and b are positive integers. Tensoring with $\mathbb{Q}$ implies that $a=b$. Let $h:K_2\oplus\Lambda^a\to\Lambda^b$ be an isomorphism, and consider

$$u=h\circ i:\Lambda^a \xrightarrow{\subset} K_2\oplus\Lambda^a \xrightarrow{h} \Lambda^a$$

and

$$v=\mathrm{pr}\circ h^{-1}:\Lambda^a \xrightarrow{\subset} K_2\oplus\Lambda^a \xrightarrow{\mathrm{pr}} \Lambda^a.$$

Obviously $v\circ u=\mathrm{Id}$, and hence $u\circ v=\mathrm{Id}$. This implies that $K_2\subset\mathrm{Ker}(u\circ v)\cong 0$. □

Note also that $K_2\otimes_\Lambda\mathbb{Q}\cong 0$ is equivalent to $K_2\otimes_\Lambda\mathbb{Z}\cong 0$.

Of course, starting with X one cannot in general assume that there is only one minimal P and degree 1-map $f:X\to P$ with $f_*:\pi_1(X)\underset{\cong}{\to}\pi_1(P)$.

Problem 2.5 *Construct examples of X that admit several minimal Poincaré 4-complexes P_i and degree 1-maps $f_i:X\to P_i$ satisfying $f_{i*}:\pi_1(X)\underset{\cong}{\to}\to\pi_1(P_i)$.*

The next proposition completes the description of the correspondence between stably free Λ-modules with nondegenerate Hermitian forms and degree 1-maps of Poincaré 4-complexes. However, we have to assume that $\pi_1(X)$ does not contain elements of order 2.

Proposition 2.6 *Let X be a Poincaré 4-complex and G a stably free Λ-module with nondegenerate Hermitian form. Then there are a Poincaré 4-complex Y and a degree 1-map $f:Y\to X$ such that $K_2(f,\Lambda)\cong G$, λ_Y restricted to $K_2(f,\Lambda)$ coincides with λ on G under the isomorphism. Moreover, $f_*:\pi_1(Y)\to\pi_1(X)$ is an isomorphism.*

Proof Let first G be free of rank m. The proof procedes as in [7]. Here we begin with $Y' = X\#(\#_1^m \mathbb{C}P^2)$ and the Hermitian form λ, and continue as in Section 3 of [7] to construct $f : Y \to X$. If G is stably free, that is, $G \oplus H \cong \Lambda^m$, where $H = \Lambda^t$, we begin with $Y' = X\#(\#_1^m \mathbb{C}P^2)$ and the Hermitian form $\lambda' = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$ on $G \oplus H$, and construct a degree 1-map $f^{''} : Y^{''} \to X$ with $K_2(f^{''}, \Lambda) = G \oplus H$, and $\lambda_{Y''}$ restricted to $K_2(f^{''}, \Lambda)$ is equal to λ'. Now $H \subset H_2(Y^{''}, \Lambda)$ is Λ-free, and $\lambda_{Y''}$ restricted to H is non-singular. As in the proof of Proposition 2.4 we can construct a degree 1-map $f : Y \to X$ with $K_2(f, \Lambda) \cong G$. □

3. A general result on the uniqueness of strongly minimal models

Let π be a finitely presented group. Suppose we are given strongly minimal Poincaré 4-complexes P and P' with $\pi_1(P) \cong \pi \cong \pi_1(P')$. For simplicity, we denote $\pi_1' = \pi_1(P')$, $\pi_1 = \pi_1(P)$, $\Lambda' = \mathbb{Z}[\pi_1']$, and $\Lambda = \mathbb{Z}[\pi_1]$. Then we have

$$p^* : H^2(B\pi_1, \Lambda) \xrightarrow[\cong]{} H^2(P, \Lambda)$$
$$p^{'*} : H^2(B\pi_1', \Lambda') \xrightarrow[\cong]{} H^2(P', \Lambda')$$

where $p : P \to B\pi_1$ and $p' : P' \to B\pi_1'$ are the classifying maps. We denote by $\chi : D \to B\pi_1$ and $\chi' : D' \to B\pi_1'$ the 2-stage Postnikov fibrations with fibers $K(\pi_2(P), 2)$ and $K(\pi_2(P'), 2)$, respectively. Spaces D and D' are obtained from P and P', respectively, by adding cells of dimension ≥ 4 so that $\pi_q(D) \cong 0 \cong \pi_q(D')$ for every $q \geq 3$, and the inclusions $f : P \to D$ and $f' : P' \to D'$ induce isomorphisms $f_* : \pi_i(P) \to \pi_i(D)$ and $f'_* : \pi_i(P') \to \pi_i(D')$, for every $i < 3$. We shall often write it as diagrams

$$\begin{array}{ccc} P & \xrightarrow{f} & D \\ \| & & \downarrow \chi \\ P & \xrightarrow{p} & B\pi_1 \end{array} \qquad \begin{array}{ccc} P' & \xrightarrow{f'} & D' \\ \| & & \downarrow \chi' \\ P' & \xrightarrow{p'} & B\pi_1'. \end{array}$$

We choose an isomorphism $\alpha : \pi_1 \to \pi_1'$. It determines an isomorphism $\Lambda \to \Lambda'$ of rings. For the sake of simplicity we shall identify Λ' with Λ via this isomorphism when we use it as coefficients in (co)homology groups. We define

$$\beta : H_2(P, \Lambda) \to H_2(P', \Lambda)$$

by the following diagram

$$\begin{array}{ccccc} H^2(B\pi_1, \Lambda) & \xrightarrow[\cong]{p^*} & H^2(P, \Lambda) & \xrightarrow[\cong]{\cap[P]} & H_2(P, \Lambda) \\ {\scriptstyle (B\alpha)^*}\uparrow & & & & \downarrow \beta \\ H^2(B\pi_1', \Lambda) & \xrightarrow[\cong]{p^{'*}} & H^2(P', \Lambda) & \xrightarrow[\cong]{\cap[P']} & H_2(P', \Lambda). \end{array} \qquad (3.1)$$

The next diagram explains the compatibility of the k-invariants $k_P^3 \in H^3(B\pi_1, \pi_2(P))$ and $k_{P'}^3 \in H^3(B\pi_1', \pi_2(P'))$:

$$\begin{array}{ccc}
\mathrm{Hom}_\Lambda(H_2(P,\Lambda),H_2(P,\Lambda)) & \longrightarrow & H^3(B\pi_1,H_2(P,\Lambda)) \\
\Big\downarrow \beta_\# & & \Big\downarrow \beta_\# \\
\mathrm{Hom}_\Lambda(H_2(P,\Lambda),H_2(P',\Lambda)) & \longrightarrow & H^3(B\pi_1,H_2(P',\Lambda)) \\
\Big\uparrow \beta^\# & & \Big\uparrow (B\alpha)^* \\
\mathrm{Hom}_\Lambda(H_2(P',\Lambda),H_2(P',\Lambda)) & \longrightarrow & H^3(B\pi_1',H_2(P',\Lambda))
\end{array} \tag{3.2}$$

where the top (resp. bottom) horizontal map sends Id into k^3_P (resp. $k^3_{P'}$), and on the left (resp. right) vertical side we have $\beta_\#(\mathrm{Id}) = \beta = \beta^\#(\mathrm{Id})$ (resp. $\beta_\#(k^3_P) = (B\alpha)^*(k^3_{P'})$). Therefore, there is a homotopy equivalence $h : D \to D'$ such that the diagram

$$\begin{array}{ccc}
D & \xrightarrow{h} & D' \\
\Big\downarrow \chi & & \Big\downarrow \\
B\pi_1 & \xrightarrow{B\alpha} & B\pi_1'
\end{array}$$

commutes (up to homotopy). Furthermore, Diagram (3.1) can be completed to the following diagram

$$\begin{array}{ccccccccc}
H^2(D,\Lambda) & \xleftarrow[\cong]{\chi^*} & H^2(B\pi_1,\Lambda) & \xrightarrow{p^*} & H^2(P,\Lambda) & \xrightarrow{\cap[P]} & H_2(P,\Lambda) & \xrightarrow{f_*} & H_2(D,\Lambda) \\
h^*\Big\uparrow & & \Big\uparrow (B\alpha)^* & & & & \beta\Big\downarrow & & \Big\downarrow h_* \\
H^2(D',\Lambda) & \xleftarrow[\cong]{\chi'^*} & H^2(B\pi_1',\Lambda) & \xrightarrow{p'^*} & H^2(P',\Lambda) & \xrightarrow{\cap[P']} & H_2(P',\Lambda) & \xrightarrow{f'_*} & H_2(D',\Lambda)
\end{array} \tag{3.3}$$

where

$$\begin{array}{ccc}
H^2(D,\Lambda) & \xrightarrow{f^*} & H^2(P,\Lambda) \\
\Big\| & & \Big\uparrow p^* \\
H^2(D,\Lambda) & \xleftarrow[\cong]{\chi^*} & H^2(B\pi_1,\Lambda)
\end{array}
\qquad
\begin{array}{ccc}
H^2(D',\Lambda) & \xrightarrow{f'^*} & H^2(P',\Lambda) \\
\Big\| & & \Big\uparrow p'^* \\
H^2(D',\Lambda) & \xleftarrow[\cong]{\chi'^*} & H^2(B\pi_1',\Lambda).
\end{array}$$

Note that all the maps are Λ-isomorphisms.

At this point it is convenient to introduce the map

$$G : H_4(D,\mathbb{Z}) \to \mathrm{Hom}_\Lambda(H^2(D,\Lambda),\overline{H}_2(D,\Lambda))$$

using the equivariant cap-product construction, and similarly G' for D'. From Diagram (3.1) we summarize as follows:

Corollary 3.1 *Diagram (3.1) commutes, and the composed horizontal homomorphisms (from left to right) are* $G(f_*[P])$ *and* $G'(f'_*[P'])$.

We again invoke Wall's theorem (Lemma 2.3) and identify

$$P = K \cup_\varphi D^4 \qquad P' = K' \cup_{\varphi'} D'^4$$

where K and K' are 3-complexes, and $\varphi : \mathbb{S}^3 \to K$ and $\varphi' : \mathbb{S}^3 \to K'$ are the attaching maps of the 4-cells D^4 and D'^4, respectively. Hence, (D, K) and (D', K') are relative CW–complexes with cells in dimensions $k \geq 4$, that is, $D^{(3)} = K$ and $D'^{(3)} = K'$. Approximate $h : D \to D'$ by a cellular map (again denoted by h). Then

$$h^{(3)} = h|_K : K \to K'$$

and

$$\begin{array}{ccc} D & \xrightarrow{h} & D' \\ i\uparrow & & \uparrow i' \\ K & \xrightarrow{h^{(3)}} & K' \end{array}$$

commutes, where $i : K \subset D$ and $i' : K' \subset D'$ are the inclusion maps.

Proposition 3.2 *(a)* $h^{(3)} : K \to K'$ *extends to* $\phi : P \to P'$ *if* $h_* f_*[P] = \ell f'_*[P'] \in H_4(D', \mathbb{Z})$ *for some* $\ell \in \mathbb{Z}$*; and*

(b) If $f'_* : H_4(P', \mathbb{Z}) \to H_4(D', \mathbb{Z})$ *is injective and* $\ell = \pm 1$*, then* ϕ *is of degree* ± 1*; hence, it is a homotopy equivalence.*

Proof (a) The obstruction to extending $h^{(3)}$ belongs to

$$\begin{aligned} H^4(P, \pi_3(P')) &\cong H_0(P, \pi_3(P')) \cong \mathbb{Z} \otimes_\Lambda \pi_3(P') \cong \mathbb{Z} \otimes_\Lambda \pi_4(D', P') \\ &\cong \mathbb{Z} \otimes_\Lambda H_4(D', P', \Lambda) = H_4(D', P', \mathbb{Z}) \end{aligned}$$

(one applies among others: $\pi_3(D') = \pi_3(D) = 0$ and the Hurewicz theorem). The obstruction in $\mathbb{Z} \otimes_\Lambda \pi_3(P')$ is given by the image of $[h^{(3)} \circ \varphi] \in \pi_3(K')$ under the composite map

$$\pi_3(K') \longrightarrow \pi_3(P') \longrightarrow \pi_3(P') \otimes_\Lambda \mathbb{Z}.$$

The obstruction in $H_4(D', P', \mathbb{Z})$ is given by the induced map of the composition

$$(D^4, \mathbb{S}^3) \xrightarrow{\varphi} (P, K) \subset (D, K) \xrightarrow{h} (D', K') \subset (D', P')$$

and hence it is the image of $[P] \in H_4(P, \mathbb{Z})$ under the composition on the bottom horizontal row in the following diagram:

$$\begin{array}{ccccccc} H_4(P,K,\mathbb{Z}) & \longrightarrow & H_4(D,K,\mathbb{Z}) & \xrightarrow{h_*} & H_4(D',K',\mathbb{Z}) & = & H_4(D',K',\mathbb{Z}) \\ \cong\uparrow & & \uparrow & & \uparrow & & \downarrow \\ H_4(P,\mathbb{Z}) & \xrightarrow{f_*} & H_4(D,\mathbb{Z}) & \xrightarrow{h_*} & H_4(D',\mathbb{Z}) & \longrightarrow & H_4(D',P',\mathbb{Z}). \end{array} \qquad (3.4)$$

Hence, the obstruction vanishes if and only if $h_* f_*[P] = \ell f'_*[P']$ for some $\ell \in \mathbb{Z}$.

(b) If $\phi : P \to P'$ exists, then it is such that the diagram

$$\begin{array}{ccccc}
H_4(P,\mathbb{Z}) & \xrightarrow{\phi_*} & H_4(P',\mathbb{Z}) & \xrightarrow{f'_*} & H_4(D',\mathbb{Z}) \\
\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} & & \| \\
H_4(P,K,\mathbb{Z}) & \xrightarrow{\phi_*} & H_4(P',K',\mathbb{Z}) & & H_4(D',\mathbb{Z}) \\
\downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f'_*} & & \| \\
H_4(D,K,\mathbb{Z}) & \xrightarrow{h_*} & H_4(D',K',\mathbb{Z}) & \longleftarrow & H_4(D',\mathbb{Z})
\end{array}$$

commutes. Hence, $f'_*\phi_*[P] = h_*f_*[P] = \pm f'_*[P']$ implies $\phi_*[P] = \pm[P']$ since f'_* is injective. Using the Poincaré duality one obtains

$$\phi_* : H_*(P,\Lambda) \xrightarrow[\cong]{} H_*(P',\Lambda).$$

Because $\phi_* : \pi_1(P) \to \pi_1(P')$ is an isomorphism, the map $\phi : P \to P'$ is a homotopy equivalence by the Hurewicz–Whitehead theorem. □

Proof of Theorem 1.3 We have a commutative diagram (up to homotopy)

$$\begin{array}{ccccccc}
D & = & D & \xrightarrow{h} & D' & = & D' \\
{\scriptstyle f}\uparrow & & \downarrow & & \downarrow & & \uparrow{\scriptstyle f'} \\
P & \xrightarrow{p} & B\pi_1 & \xrightarrow{B\alpha} & B\pi'_1 & \xleftarrow{p'} & P'
\end{array}$$

where $h : D \to D'$ is a homotopy equivalence. Consider the diagram

$$\begin{array}{ccc}
H_4(D,\mathbb{Z}) & \xrightarrow{G} & \mathrm{Hom}_\Lambda(H^2(D,\Lambda), \overline{H}_2(D,\Lambda)) \\
{\scriptstyle h_*}\downarrow & & \downarrow{\scriptstyle T} \\
H_4(D',\mathbb{Z}) & \xrightarrow{G'} & \mathrm{Hom}_\Lambda(H^2(D',\Lambda), \overline{H}_2(D',\Lambda))
\end{array} \tag{3.5}$$

where $\cap z$ is the cap product with $z \in H_4(D,\mathbb{Z})$. Similarly, $\cap'$. The map T is defined by $T(\xi) = h_* \circ \xi \circ h^*$. Note that T is an isomorphism. □

Lemma 3.3 *Diagram (3.5) commutes.*

Proof Given $x \in H_4(D,\mathbb{Z})$ and $u' \in H^2(D',\mathbb{Z})$, then we have

$$TG(x)(u') = h_*(h^*(u') \cap x) = u' \cap h_*(x) = G'h_*(x)$$

as required. □

Now consider the diagram

$$\begin{array}{ccccc}
H_4(P,\mathbb{Z}) & \xrightarrow{f_*} & H_4(D,\mathbb{Z}) & \xrightarrow{G} & \mathrm{Hom}_\Lambda(H^2(D,\Lambda), \overline{H}_2(D,\Lambda)) \\
 & & \downarrow{\scriptstyle h_*} & & \downarrow{\scriptstyle T} \\
H_4(P',\mathbb{Z}) & \xrightarrow{f'_*} & H_4(D',\mathbb{Z}) & \xrightarrow{G'} & \mathrm{Hom}_\Lambda(H^2(D',\Lambda), \overline{H}_2(D',\Lambda)).
\end{array}$$

It follows from Corollary 3.1 that

$$TGf_*[P] = G'f'_*[P'],$$

and from $TG = G'h_*$ we get $G'h_*f_*[P] = G'f'_*[P']$; hence, $h_*f_*[P] = f'_*[P']$. So Proposition 3.2 (a) holds with $\ell = 1$.

A similar diagram as (5) holds for the space P':

$$\begin{array}{ccc} H_4(P',\mathbb{Z}) & \xrightarrow{G''} & \mathrm{Hom}_\Lambda(H^2(P',\Lambda),\overline{H}_2(P',\Lambda)) \cong \mathrm{Hom}_\Lambda(\overline{H}_2(P',\Lambda),\overline{H}_2(P',\Lambda)) \\ {\scriptstyle f'_*}\downarrow & & \downarrow{\scriptstyle T} \\ H_4(D',\mathbb{Z}) & \xrightarrow{G'} & \mathrm{Hom}_\Lambda(H^2(D',\Lambda),\overline{H}_2(D',\Lambda)) \end{array}$$

with $T(\xi) = f_* \circ \xi \circ f^*$. Since T is an isomorphism, f'_* is injective if and only if the map G'' is injective. Now observe that under the maps the generator $[P']$ goes to Id. The upper right isomorphism is induced by Poincaré duality. Hence G'' is injective if and only if Id is not of finite order. Now $H_2(P',\Lambda) \cong H^2(B\pi'_1,\Lambda) \cong H^2(B\pi_1,\Lambda)$. The claim now follows from Proposition 3.2(b).

4. Construction of strongly minimal models

The principal examples of fundamental groups π admitting a strongly minimal model P are discussed in [5]. These are groups of geometric dimension equal to 2, i.e. $B\pi$ is a 2-dimensional aspherical complex. It is easy to see that the boundary of a regular neighborhood N of an embedding $B\pi \subset \mathbb{R}^5$ is a strongly minimal model for π (see [5]). Here we show that the map G is not injective, and hence we cannot expect uniqueness up to homotopy equivalence. In fact, we are going to classify all strongly minimal models fixing π by elements of the kernel of G. Note that all k-invariants vanish since $B\pi$ is a 2-complex. We assume $H_4(P,\Lambda) \cong 0$, i.e. that π is infinite (which holds for the known examples).

4.1. Computation of $\mathrm{Ker}\, G$

We fix π as above, and for convenience also one strongly minimal model P, say $P = \partial N$. We have the following 2-stage Postnikov system.

$$\begin{array}{ccc} D & \xrightarrow{\chi} & B\pi \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle p} \\ P & = & P \end{array}$$

Lemma 4.1 *There is an exact sequence*

$$0 \longrightarrow \Gamma(\pi_2)\otimes_\Lambda \mathbb{Z} \longrightarrow H_4(D,\mathbb{Z}) \longrightarrow H_2(B\pi, H_2(D,\Lambda)) \longrightarrow 0$$

where $\pi_2 = \pi_2(P) \cong \pi_2(D)$.

Proof This follows from the spectral sequence

$$E^2_{pq} = H_p(B\pi, H_q(D,\Lambda)) \underset{p+q=n}{\Longrightarrow} H_n(D,\mathbb{Z}).$$

Taking $n=4$, we have $E^2_{pq} = E^\infty_{pq} = [F_pH_4(D,\mathbb{Z})]/[F_{p-1}H_4(D,\mathbb{Z})]$ with filtration

$$0 \cong F_{-1}H_4 \subset F_0H_4 \subset F_1H_4 \subset F_2H_4 \subset F_3H_4 \subset F_4H_4(D,\mathbb{Z}) = H_4(D,\mathbb{Z}).$$

The result follows since $E^2_{22} = H_2(B\pi, H_2(D,\Lambda))$, $E^2_{04} = H_0(B\pi, H_4(D,\Lambda)) = H_4(D,\Lambda)\otimes_\Lambda \mathbb{Z}$, and $E^2_{pq} \cong 0$ else for $p+q=4$. □

Remark Similarly one gets the exact sequence

$$0 \longrightarrow H_1(P, H_3(P,\Lambda)) \longrightarrow H_4(P,\mathbb{Z}) \longrightarrow H_2(B\pi, H_2(P,\Lambda)) \longrightarrow 0.$$

In particular, $H_2(B\pi, H_2(D,\Lambda))$ is a quotient of $\mathbb{Z}$ because $H_2(D,\Lambda)\cong H_2(P,\Lambda)$ and $H_4(P,\mathbb{Z})\cong\mathbb{Z}$.

Lemma 4.2 *The kernel of*

$$G : H_4(D,\mathbb{Z}) \to \mathrm{Hom}_{\Lambda-\Lambda}(H^2(D,\Lambda), \overline{H}_2(D,\Lambda))$$

is $\Gamma(\pi_2)\otimes_\Lambda \mathbb{Z}$.

Proof The map $\chi^* : H^2(B\pi,\Lambda) \to H^2(D,\Lambda)$ is an isomorphism, and $H^2(B\pi,\Lambda) \cong [\mathrm{Hom}_\Lambda(C_2(\widetilde{B\pi}),\Lambda)]/[\mathrm{Im}\,\delta^1]$, where

$$\delta^1 : \mathrm{Hom}_\Lambda(C_1(\widetilde{B\pi}),\Lambda) \to \mathrm{Hom}_\Lambda(C_2(\widetilde{B\pi}),\Lambda)$$

is the co-boundary map. The composition

$$\begin{array}{ccc} \mathrm{Hom}_{\Lambda-\Lambda}(H^2(B\pi,\Lambda), \overline{H}_2(D,\Lambda)) & \longrightarrow & \mathrm{Hom}_\Lambda(\mathrm{Hom}_\Lambda(C_2(\widetilde{B\pi}),\Lambda), \overline{H}_2(D,\Lambda)) \\ \cong\Big\uparrow & & \\ \mathrm{Hom}_\Lambda(H^2(D,\Lambda), H_2(D,\Lambda)) & & \end{array}$$

is obviously injective. Because $C_2(\widetilde{B\pi})$ is Λ-free, there is a canonical isomorphism

$$\mathrm{Hom}_{\Lambda-\Lambda}(\mathrm{Hom}_\Lambda(C_2(\widetilde{B\pi}),\Lambda), \overline{H}_2(D,\Lambda)) \cong C_2(\widetilde{B\pi})\otimes_\Lambda H_2(D,\Lambda).$$

Composing all these maps gives an injective map

$$\mathrm{Hom}_{\Lambda-\Lambda}(H^2(D,\Lambda), \overline{H}_2(D,\Lambda)) \to C_2(\widetilde{B\pi})\otimes_\Lambda H_2(D,\Lambda).$$

The composition with G gives a map $H_4(D,\mathbb{Z}) \to C_2(\widetilde{B\pi})\otimes_\Lambda H_2(D,\Lambda)$ with image of the 2-cycle subgroup of the complex $C_*(\widetilde{B\pi})\otimes_\Lambda H_2(D,\Lambda)$, i.e. $H_2(B\pi, H_2(D,\Lambda))$. This is the map $H_4(D,\mathbb{Z}) \to H_2(B\pi, H_2(D,\Lambda))$ of Lemma 4.1. In other words, we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}_{\Lambda-\Lambda}(H^2(D,\Lambda), \overline{H}_2(D,\Lambda)) & \longrightarrow & H_2(B\pi, H_2(D,\Lambda)) \\ {\scriptstyle G}\Big\uparrow & & \Big\uparrow \\ H_4(D,\mathbb{Z}) & = & H_4(D,\mathbb{Z}) \end{array}$$

where the horizontal map is injective. The result now follows from Lemma 4.1. □

Supplement to Lemma 4.1. If P and P' are 2 strongly minimal models for π, let

$$\begin{array}{ccc} P & \xrightarrow{f} & D \\ \| & & \downarrow \chi \\ P & \xrightarrow[p]{} & B\pi \end{array} \qquad \begin{array}{ccc} P & \xrightarrow{f'} & D' \\ \| & & \downarrow \chi' \\ P' & \xrightarrow[p']{} & B\pi \end{array}$$

be the 2 associated 2-stage Postnikov systems. Let $h : D \to D'$ be the homotopy equivalence constructed in Section 3. Then the diagram

$$\begin{array}{ccc} H_4(D, \mathbb{Z}) & \longrightarrow & H_2(B\pi, H_2(D, \Lambda)) \\ h_* \downarrow & & \downarrow \\ H_4(D', \mathbb{Z}) & \longrightarrow & H_2(B\pi, H_2(D', \Lambda)) \end{array}$$

commutes. The right vertical map is induced by $h_* : H_2(D, \Lambda) \to H_2(D', \Lambda)$.

4.2. Construction of strongly minimal models

We choose a strongly minimal model P for π. By Wall's theorem [13], P is homotopy equivalent to $K \cup_{\varphi_1} D^4$, where K is a 3-complex, and $\varphi_1 : \mathbb{S}^3 \to K$ is the attaching map of the only 4-cell. This representation is unique, i.e. given a homotopy equivalence

$$K_1 \cup_{\varphi_1} D^4 \xrightarrow{h} K_2 \cup_{\varphi_2} D^4$$

there is a homotopy equivalence of pairs $(K_1, \varphi_1(\mathbb{S}^3)) \to (K_2, \varphi_2(\mathbb{S}^3))$ (see [13, p.222]). We simply write $P = K \cup_{\varphi_1} D^4$ and change the attaching map $[\varphi_1] \in \pi_3(K)$ by an element $[\varphi] \in \Gamma(\pi_2)$, i.e. $[\varphi] \in \Gamma(\pi_2) = \mathrm{Im}(\pi_3(K^{(2)}) \to \pi_3(K))$, and we consider $X = K \cup_{\varphi_2} D^4$, where $\varphi_2 = \varphi_1 + \varphi$ and $\varphi : \mathbb{S}^3 \to K^{(2)}$. Let $q : X \to B\pi$ be the classifying map. It follows that $q^* : H^2(B\pi, \Lambda) \to H^2(X, \Lambda)$ is an isomorphism. If X is a Poincaré 4-complex, then X is a strongly minimal model for π.

4.3. Proof of the Poincaré duality

(I) We have an isomorphism $\pi_4(X, K) \to H_4(X, K, \Lambda) \cong \Lambda$. Let us consider the diagram of Whitehead's sequences:

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & \pi_4(X, K) & \xrightarrow{\cong} & H_4(X, K, \Lambda) & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Gamma(\pi_2) & \longrightarrow & \pi_3(K) & \longrightarrow & H_3(K, \Lambda) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ & & \Gamma(\pi_2) & \longrightarrow & \pi_3(X) & \longrightarrow & H_3(X, \Lambda) & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

One has a similar diagram if we replace X by P. Under the Hurewicz map, $[\varphi_1]$ and $[\varphi_2]$ go to the same element in $H_3(K,\Lambda)$, which coincides with the images of the generators of $H_4(P,K,\Lambda)$ resp. $H_4(X,K,\Lambda)$ under the connecting homomorphism, and hence $H_3(X,\Lambda) \cong H_3(P,\Lambda)$. Moreover, this gives us the following:

Lemma 4.3 $H_4(X,\mathbb{Z}) \cong \mathbb{Z}$

Proof Tensoring with $\otimes_\Lambda \mathbb{Z}$ the upper part of the above diagram gives

$$\begin{array}{ccccccc} & & \pi_4(X,K)\otimes_\Lambda \mathbb{Z} & \xrightarrow[\cong]{} & H_4(X,K,\Lambda)\otimes_\Lambda \mathbb{Z} & \xrightarrow[\cong]{} & H_4(X,K,\mathbb{Z}) \\ & & \downarrow & & \downarrow & & \downarrow \\ \Gamma(\pi_2)\otimes_\Lambda \mathbb{Z} & \longrightarrow & \pi_3(K)\otimes_\Lambda \mathbb{Z} & \longrightarrow & H_3(K,\Lambda)\otimes_\Lambda \mathbb{Z} & \longrightarrow & H_3(K,\mathbb{Z}) \end{array}$$

and similarly for X replaced by P (we do not claim the exactness of the lower row). Now $H_4(P,K,\mathbb{Z}) \to H_3(K,\mathbb{Z})$ is the zero map. By the argument above, $[\varphi_1]\otimes_\Lambda 1$ and $[\varphi_2]\otimes_\Lambda 1$ map to the same element in $H_3(K,\Lambda)\otimes_\Lambda \mathbb{Z}$, and hence the generators of $H_4(X,K,\mathbb{Z})$ resp. $H_4(P,K,\mathbb{Z})$ map to the same element in $H_3(K,\mathbb{Z})$ under the connecting homomorphisms. Thus, $H_4(X,K,\mathbb{Z}) \to H_3(K,\mathbb{Z})$ is the zero map. Therefore, there is an isomorphism $H_4(X,\mathbb{Z}) \to H_4(X,K,\mathbb{Z}) \cong \mathbb{Z}$. □

Let $[X] \in H_4(X,\mathbb{Z})$ be a generator. We have to study

$$\cap[X] : H^p(X,\Lambda) \to H_{4-p}(X,\Lambda).$$

To examine the cases $p=1$ and $p=3$, we introduce an auxiliary space $Y = K \cup_{\varphi_1,\varphi} \{D^4, D^4\}$, obtained from K by attaching two 4-cells with attaching maps φ_1 and φ. Note that $Y = P \cup_\varphi D^4$.

(II) *Case* $p=1$

Let $i : P \to Y$ be the inclusion, and $j : X \to Y$ be the map induced by $K \subset Y$ and

$$\varphi_2 = \varphi_1 + \varphi : \mathbb{S}^3 \longrightarrow \mathbb{S}^3 \vee \mathbb{S}^3 \xrightarrow{\varphi_1 \vee \varphi} K.$$

We have the following maps of pairs:

$$\begin{array}{ccc} (D^4,\mathbb{S}^3) & \xrightarrow{\bar{i}\circ\bar{\varphi}_1} & (Y,K) \\ {\scriptstyle \bar{\varphi}_1}\downarrow & & \uparrow{\scriptstyle \bar{i}} \\ (P,K) & = & (P,K) \end{array} \qquad \begin{array}{ccc} (D^4,\mathbb{S}^3) & \xrightarrow{\bar{j}\circ\bar{\varphi}_2} & (Y,K) \\ {\scriptstyle \bar{\varphi}_2}\downarrow & & \uparrow{\scriptstyle \bar{j}} \\ (X,K) & = & (X,K) \end{array}$$

and $\bar{\varphi} : (D^4,\mathbb{S}^3) \to (Y,K)$. Obviously, $\bar{\varphi}_2 = \bar{\varphi}_1 + \bar{\varphi} : (D^4,\mathbb{S}^3) \to (Y,K)$ is the 4-cell $[\varphi_1]$ "slided" over $[\varphi]$. Since $[\bar{\varphi}] \in \Gamma(\pi_2)$, $\bar{\varphi}$ factors as follows:

$$\begin{array}{ccc} (D^4,\mathbb{S}^3) & \xrightarrow{\bar{k}\circ\bar{\varphi}} & (Y,K) \\ {\scriptstyle \bar{\varphi}}\downarrow & & \uparrow{\scriptstyle \bar{k}} \\ (K^{(2)} \cup_\varphi D^4, K^{(2)}) & = & (K^{(2)} \cup_\varphi D^4, K^{(2)}) \end{array}$$

From this one sees that $\bar{j}_*[\bar{\varphi}_2] - \bar{i}_*[\bar{\varphi}_1]$ belongs to

$$\mathrm{Im}(H_4(K^{(2)} \cup_\varphi D^4, K^{(2)}) \to H_4(Y, K)).$$

The diagram

$$\begin{array}{ccccc}
H_4(X) & \xrightarrow{j_*} & H_4(Y) & \xleftarrow{i_*} & H_4(P) \\
\cong\downarrow & & \downarrow & & \downarrow\cong \\
H_4(X,K) & \xrightarrow{\bar{j}_*} & H_4(Y,K) & \xleftarrow{\bar{i}_*} & H_4(P,K)
\end{array}$$

as well as injectivity of $H_4(Y) \to H_4(Y,K)$ and the isomorphism

$$H_4(K^{(2)} \cup_\varphi D^4) \to H_4(K^{(2)} \cup_\varphi D^4, K^{(2)})$$

prove the following:

Lemma 4.4 $j_*[X] - i_*[P]$ *belongs to* $\mathrm{Im}(H_4(K^{(2)} \cup_\varphi D^4) \to H_4(Y))$.

Corollary 4.5 *Taking cap-products with* $i_*[P]$ *and* $j_*[X] : H^1(Y,\Lambda) \to H_3(Y,\Lambda)$ *gives the same map.*

Proof Let $\theta \in H_4(K^{(2)} \cup_\varphi D^4)$ map to $j_*[X] - i_*[P]$. Then the diagram

$$\begin{array}{ccc}
H^1(Y,\Lambda) & \xrightarrow{\cap j_*[X] - \cap i_*[P]} & H_3(Y,\Lambda) \\
\cong\downarrow & & \uparrow \\
H^1(K^{(2)} \cup_\varphi D^4, \Lambda) & \xrightarrow{\cap\theta} & H_3(K^{(2)} \cup_\varphi D^4, \Lambda) \cong 0
\end{array}$$

commutes. □

Lemma 4.6 $i_* : H_3(P,\Lambda) \to H_3(Y,\Lambda)$ *is an isomorphism.*

Proof Since $Y = P \cup_\varphi D^4$, i_* is surjective. Let us consider the diagram

$$\begin{array}{ccccc}
H_4(K^{(2)} \cup_\varphi D^4, K^{(2)}, \Lambda) & \longrightarrow & H_4(Y,P,\Lambda) & \longrightarrow & H_3(P,\Lambda) \\
\cong\uparrow & & \cong\uparrow & & \uparrow \\
H_4(K^{(2)} \cup_\varphi D^4, \Lambda) & \xrightarrow[\cong]{} & H_4(K^{(2)} \cup_\varphi D^4, K^{(2)}, \Lambda) & \longrightarrow & H_3(K^{(2)} \cup_\varphi D^4, \Lambda),
\end{array}$$

which shows that $H_4(Y,P,\Lambda) \to H_3(P,\Lambda)$ is the zero map. □

Lemma 4.7 $j_* : H_3(X,\Lambda) \to H_3(Y,\Lambda)$ *is an isomorphism.*

Proof The map j_* is surjective because $Y^{(3)} = K = X^{(3)}$. We identify $H_4(Y,K,\Lambda) \equiv \Lambda \oplus \Lambda$ according to the diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \uparrow & & \\
H_4(D^4,\mathbb{S}^3,\Lambda) & \xrightarrow[\cong]{\bar{\varphi}_{1*}} & H_4(P,K,\Lambda) & & H_4(Y,\Lambda) & & H_4(Y,X,\Lambda) & & \\
 & & \bar{i}_*\downarrow & & \downarrow & & \uparrow & & \\
 & & H_4(Y,K,\Lambda) & = & H_4(Y,K,\Lambda) & = & H_4(Y,K,\Lambda) & \xleftarrow{\bar{k}_*\bar{\varphi}_*} & H_4(D^4,\mathbb{S}^3,\Lambda) \\
 & & \bar{j}_*\uparrow & & & & \downarrow & & \downarrow\cong \\
H_4(D^4,\mathbb{S}^3,\Lambda) & \xrightarrow[\cong]{\bar{\varphi}_{1*}+\bar{\varphi}_*} & H_4(X,K,\Lambda) & & & & H_4(Y,P,\Lambda) & = & H_4(Y,P,\Lambda) \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}
$$

where $\bar{i}_*[\bar{\varphi}_1] = (1,0) \in \Lambda \oplus \Lambda$ and $\bar{k}_*[\bar{\varphi}] = (0,1) \in \Lambda \oplus \Lambda$. The map $\bar{k}_*\bar{\varphi}_*$ defines a splitting of $H_4(Y,P,\Lambda) \to H_4(Y,K,\Lambda)$. Since: $H_4(Y,\Lambda) \to H_4(Y,P,\Lambda)$ is an isomorphism (here we use our assumption $H_4(P,\Lambda) \cong 0$ and Lemma 4.6), the image of $H_4(Y,\Lambda)$ in $H_4(Y,K,\Lambda) \equiv \Lambda \oplus \Lambda$ is generated by $(0,1)$. Thus, we can write the following diagram.

$$
\begin{array}{ccccc}
 & & H_4(Y,\Lambda) & & \\
 & & \downarrow & & \\
\Lambda & \longrightarrow & \Lambda \oplus \Lambda & \longrightarrow & (\Lambda \oplus \Lambda)/\Lambda(1,1) \\
\| & & \| & & \\
\Lambda & \longrightarrow & \Lambda \oplus \Lambda & \longrightarrow & \Lambda
\end{array}
$$

The map $\bar{j}_*$ corresponds to $\Lambda \to \Lambda \oplus \Lambda$ defined by $1 \to (1,1)$. Hence, the map $H_4(Y,\Lambda) \to H_4(Y,X,\Lambda)$ corresponds to the isomorphism $\Lambda \to (\Lambda \oplus \Lambda)/\Lambda(1,1)$ defined by $1 \to [(0,1)]$, the class of $(0,1)$ in the quotient. Therefore, we have an isomorphism $H_3(X,\Lambda) \to H_3(Y,\Lambda)$. □

Lemma 4.8 *The map* $\cap[X] : H^1(X,\Lambda) \to H_3(X,\Lambda)$ *is an isomorphism.*

Proof This follows from the diagram

$$
\begin{array}{ccc}
H^1(X,\Lambda) & \xrightarrow{\cap[X]} & H_3(X,\Lambda) \\
j^*\uparrow\cong & & \cong\downarrow j_* \\
H^1(Y,\Lambda) & \xrightarrow{\cap j_*[X]} & H_3(Y,\Lambda) \\
i^*\downarrow\cong & & \cong\uparrow i_* \\
H^1(P,\Lambda) & \xrightarrow[\cong]{\cap[P]} & H_3(P,\Lambda)
\end{array}
$$

and $\cap j_*[X] = \cap i_*[P] : H^1(Y,\Lambda) \to H_3(Y,\Lambda)$. □

(III) *Case* $p = 3$

Now we look N at the case $\cap[X] : H^3(X,\Lambda) \to H_1(X,\Lambda) \cong 0$, i.e. we have to show that $H^3(X,\Lambda) \cong 0$. Note that the sequence

$$0 \longrightarrow H^3(K,\Lambda) \longrightarrow H^4(P,K,\Lambda) \longrightarrow H^4(P,\Lambda) \longrightarrow 0$$

is exact. Since $H^4(P,\Lambda) \cong H_0(P,\Lambda) \cong \mathbb{Z}$, this sequence coincides with

$$0 \longrightarrow I(\Lambda) \longrightarrow \Lambda \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

where ϵ is the augmentation, and $I(\Lambda) = \operatorname{Ker}\epsilon$. Let us consider the following diagram.

$$\begin{array}{ccccc}
H^3(K,\Lambda) & \longrightarrow & H^4(Y,P,\Lambda) & \xrightarrow[\cong]{\bar{\varphi}^*} & H^4(D^4,\mathbb{S}^3,\Lambda) \\
\| & & \uparrow & & \\
H^3(K,\Lambda) & \longrightarrow & H^4(Y,K,\Lambda) & \xrightarrow{\bar{j}^*} & H^4(X,K,\Lambda) \\
\| & & \downarrow{\bar{i}_*} & & \\
H^3(K,\Lambda) & \longrightarrow & H^4(P,K,\Lambda) & \xrightarrow[\cong]{\bar{\varphi}_1^*} & H^4(D^4,\mathbb{S}^3,\Lambda)
\end{array}$$

The 2 vertical maps split $H^4(Y,K,\Lambda) \cong \Lambda \oplus \Lambda$ so that

$$\bar{i}^* : H^4(Y,K,\Lambda) \cong \Lambda \oplus \Lambda \to \Lambda \cong H^4(P,K,\Lambda)$$

projects onto the first component and $H^4(Y,K,\Lambda) \to H^4(Y,P,\Lambda) \cong \Lambda$ projects onto the second component. Since the composition $H^3(K,\Lambda) \to H^4(Y,P,\Lambda)$ is the zero map, we can identify the image of $H^3(K,\Lambda) \to H^4(Y,K,\Lambda)$ with $(I(\Lambda),0) \subset \Lambda \oplus \Lambda$. The map $\bar{j}^*$ is the sum $\Lambda \oplus \Lambda \to \Lambda$ since the generator of $H^4(X,K,\Lambda) \cong \Lambda$ maps under

$$\bar{\varphi}_1^* + \bar{\varphi}^* : H^4(X,K,\Lambda) \to H^4(D^4,\mathbb{S}^3,\Lambda) \cong \Lambda$$

to a generator. Hence, the image of

$$H^3(K,\Lambda) \longrightarrow H^4(Y,K,\Lambda) \xrightarrow{\bar{j}^*} H^4(X,K,\Lambda)$$

is $I(\Lambda) \subset \Lambda$, i.e. $H^3(K,\Lambda) \to H^4(X,K,\Lambda)$ is injective. The long exact sequence of the pair (X,K) implies $H^3(X,\Lambda) \cong 0$.

(IV) *Case* $p = 4$

Remark The last argument also implies $H^4(X,\Lambda) \cong \Lambda/I(\Lambda) \cong \mathbb{Z}$. We have proven the first part of the following:

Lemma 4.9 $H^3(X,\Lambda) \cong 0$, $H^4(X,\Lambda) \cong \mathbb{Z}$, *and* $\cap[X] : H^4(X,\Lambda) \to H_0(X,\Lambda)$ *is an isomorphism.*

Proof The second part follows from the well-known property of cap-products indicated in the following diagram:

$$\begin{array}{ccc} \mathbb{Z} \cong H^4(X,\Lambda) & \xrightarrow{\cap[X]} & H_0(X,\Lambda) \cong \mathbb{Z} \\ {\scriptstyle\epsilon}\uparrow & & \uparrow{\scriptstyle\epsilon} \\ \Lambda \cong H^4(X,K,\Lambda) = \mathrm{Hom}_\Lambda(C_4(\widetilde{X},\widetilde{K}),\Lambda) & \underset{\cong}{\xrightarrow{A}} & C_0(\widetilde{X}) \cong \Lambda \end{array}$$

Here $A(\alpha)=\alpha(1)$, $1 \in C_4(\widetilde{X},\widetilde{K})$ being the generator. Observe that $H_0(X,\Lambda) = C_0(\widetilde{X})/\partial_1 C_1(\widetilde{X})$, so ϵ corresponds to the canonical map $C_0(\widetilde{X}) \to C_0(\widetilde{X})/\partial_1 C_1(\widetilde{X})$ (we may assume that X has one 0-cell). □

(V) *Case* $p=2$

Recall the 2-stage Postnikov system for P:

$$\begin{array}{ccc} P & \xrightarrow{f} & D \\ \| & & \downarrow{\scriptstyle\chi} \\ P & \xrightarrow{p} & B\pi. \end{array}$$

Let $f_0 = f|_K$. Given any $\psi : \mathbb{S}^3 \to K$, a canonical map $g : K \cup_\psi D^4 \to D$ can be constructed as follows: Let $H : \mathbb{S}^3 \times I \to D$ be the zero homotopy of the composition $f_0 \circ \psi : \mathbb{S}^3 \to D$. It factors over

$$D^4 = (\mathbb{S}^3 \times I)/\mathbb{S}^3 \times \{1\} \xrightarrow{\hat{H}} D.$$

Then $g = f_0 \cup \hat{H} : K \cup_\psi D^4 \to D$. Since $\pi_q(D) \cong 0$ for $q \geq 3$, g is unique up to homotopy. In our case, we have $\psi = \varphi_2 = \varphi_1 + \varphi$ with $[\varphi] \in \Gamma(\pi_2)$, where $\varphi : \mathbb{S}^3 \to K^{(2)}$, i.e. we need the zero homotopy of the composition

$$\mathbb{S}^3 \longrightarrow \mathbb{S}^3 \vee \mathbb{S}^3 \xrightarrow{\varphi_1 \vee \varphi} K \vee K^{(2)} \xrightarrow{f_0 \vee f_0} D \vee D \longrightarrow D.$$

We take the wedge of the zero homotopies $H : \mathbb{S}^3 \times I \to D$ for $f_0 \circ \varphi_1$ and $H_0 : \mathbb{S}^3 \times I \to D$ for $f_0 \circ \varphi$. This gives us the following:

Lemma 4.10 *Let* $g_0 = f_0 \cup \hat{H}_0 : K^{(2)} \cup_\varphi D^4 \to D$ *denote the canonical extension and* $\theta \in H_4(K^{(2)} \cup_\varphi D^4, \mathbb{Z})$ *the canonical generator. Then we have*

$$g_*[X] = f_*[P] + (g_0)_*(\theta).$$

Corollary 4.11 $(g_0)_*(\theta) \in \operatorname{Ker} G \subset H_4(D,\mathbb{Z})$. *In particular,*

$$\cap f_*[P] = \cap g_*[X] : H^2(D,\Lambda) \to H_2(D,\Lambda);$$

that is, the map $\cap[X] : H^2(X,\Lambda) \to H_2(X,\Lambda)$ *is an isomorphism.*

Proof The above spectral sequence applied to $K^{(2)} \cup_\varphi D^4$ gives

$$\begin{array}{l} 0 \longrightarrow \mathbb{Z} \otimes_\Lambda H_4(K^{(2)} \cup_\varphi D^4, \Lambda) \longrightarrow H_4(K^{(2)} \cup_\varphi D^4, \mathbb{Z}) \\ \qquad \longrightarrow H_2(B\pi, H_2(K^{(2)} \cup_\varphi D^4, \Lambda)) \longrightarrow 0. \end{array}$$

The first map is an isomorphism, so $H_2(B\pi, H_2(K^{(2)} \cup_\varphi D^4, \Lambda)) \cong 0$. Comparison with the exact sequence for D:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z} \otimes_\Lambda H_4(K^{(2)} \cup_\varphi D^4, \Lambda) & \longrightarrow & H_4(K^{(2)} \cup_\varphi D^4, \mathbb{Z}) & \longrightarrow & 0 \\ & & & & \downarrow (g_0)_* & & \downarrow \\ & & & & G : H_4(D, \mathbb{Z}) & \longrightarrow & \mathrm{Hom}_\Lambda(H^2(D, \Lambda), H_2(D, \Lambda)) \end{array}$$

gives the result. □

Theorem 4.12 *Suppose $B\pi$ is homotopy equivalent to a 2-dimensional complex. Let $\pi_2 = H^2(B\pi, \Lambda)$. Then, if we fix one model P, we obtain all models by the above construction.*

Proof Fixing P, we constructed for any $[\varphi] \in \pi_2$ a strongly minimal model. Conversely, let $X = K \cup_\psi D^4$ be a minimal model, where $\psi : \mathbb{S}^3 \to K$ is the attaching map. The map $f : X \to D$ into the 2-stage Postnikov space D is given by the zero homotopy of

$$\mathbb{S}^3 \xrightarrow{\psi} K \xrightarrow{f_0} D;$$

that is,

$$\begin{array}{ccc} \mathbb{S}^3 \times I & \xrightarrow{H} & D \\ \downarrow & & \uparrow \hat{H} \\ D^4 = (\mathbb{S}^3 \times I)/\mathbb{S}^3 \times \{1\} & = & D^4 \end{array}$$

with $f = f_0 \cup \hat{H}$. Let us consider $\hat{H} : (D^4, \mathbb{S}^3) \to (D, K)$ and let

$$\bar{\psi} : (D^4, \mathbb{S}^3) \to (X, K)$$

be the top cell. The diagram

$$\begin{array}{ccccc} H_4(X, \mathbb{Z}) & \xrightarrow[\cong]{} & H_4(X, K, \mathbb{Z}) & \xleftarrow[\cong]{\bar{\psi}} & H_4(D^4, \mathbb{S}^3, \mathbb{Z}) \\ f_* \downarrow & & & & \| \\ H_4(D, \mathbb{Z}) & \longrightarrow & H_4(D, K, \mathbb{Z}) & \xleftarrow{\hat{H}_*} & H_4(D^4, \mathbb{S}^3, \mathbb{Z}) \end{array}$$

shows that $f_*[X]$ depends only on $\psi \otimes_\Lambda 1 \in \pi_3(K) \otimes_\Lambda \mathbb{Z}$. Note that $H_4(D, \mathbb{Z}) \to H_4(D, K, \mathbb{Z})$ is injective. This also demonstrates that the above construction only depends on ξ, not on the choice of $[\varphi] \in \Gamma(\pi_2)$ with $[\varphi] \otimes_\Lambda 1 = \xi$.

It remains to be shown that any minimal model X' is homotopy equivalent to some model X obtained by the above construction. Write

$$X' = K' \cup_\psi D^4 \xrightarrow{f'} D',$$

where D' is the 2-stage Postnikov space, K' is a 3-dimensional complex, and $\psi : \mathbb{S}^3 \to K'$ is the attaching map. Recall our standard model:

$$P = K \cup_{\varphi_1} D^4 \xrightarrow{f} D.$$

In Section 3 we constructed a homotopy equivalence $h : D' \to D$ sending $K' \to K$. Lemma 3.3 implies

$$h_* f'_*[X'] - f_*[P] \in \operatorname{Ker} G = \Gamma(\pi_2) \otimes_\Lambda \mathbb{Z}.$$

By Lemma 4.1 of Section 4 choose $[\varphi] \in \Gamma(\pi_2)$ so that $[\varphi] \otimes_\Lambda 1 = h_* f'_*[X'] - f_*[P]$, and $\varphi : \mathbb{S}^3 \to K^{(2)} \subset K$. As in Part V of Section 4, we build $X = K \cup_{\varphi_2} D^4$, with $\varphi_2 = \varphi_1 + \varphi$, and $g : X \to D$. Let $g_0 : K^{(2)} \cup_\varphi D^4 \to D$ be the canonically defined map from the zero homotopy of $\mathbb{S}^3 \to K^{(2)} \to D$. Then we have (use Lemma 4.10) $g_*[X] = f_*[P] + (g_0)_*(\theta)$, where $\theta \in H_4(K^{(2)} \cup_\varphi D^4, \mathbb{Z})$ is a generator. But $(g_0)_*(\theta) = h_* f'_*[X'] - f_*[P]$, as can be seen from the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_4(K^{(2)} \cup_\varphi D^4, \Lambda) \otimes_\Lambda \mathbb{Z} & \xrightarrow{\cong} & H_4(K^{(2)} \cup_\varphi D^4, \mathbb{Z}) & & & & \\
 & & \big\downarrow {\scriptstyle (g_0)_* \otimes_\Lambda 1} & & \big\downarrow {\scriptstyle (g_0)_*} & & & & \\
0 & \longrightarrow & \Gamma(\pi_2) \otimes_\Lambda \mathbb{Z} = H_4(D, \Lambda) \otimes_\Lambda \mathbb{Z} & \longrightarrow & H_4(D, \mathbb{Z}) & \longrightarrow & H_2(B\pi, H_2(D, \Lambda)) & \longrightarrow & 0.
\end{array}$$

Therefore, $g_*[X] = h_* f'_*[X']$. By Proposition 3.2 and the proof of Theorem 1.3 (where we have to use that π_2 is not a torsion group) we obtain a homotopy equivalence $X' \to X$. □

5. Non-uniqueness of strongly minimal models: examples

In Section 4 we constructed minimal models for all elements of $\Gamma(\pi_2)$. In this section we address the question of uniqueness up to homotopy equivalence. Recall that for 2 models X and X' we have a homotopy equivalence between the 2-stage Postnikov systems (assuming that the first k-invariants are compatible). It is deduced from Diagram (3.2) in Section 3, i.e. we have the diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & D & \xrightarrow{h} & D' & \xleftarrow{f'} & X' \\
 & & \big\downarrow & & \big\downarrow & & \\
 & & B\pi_1 & \longrightarrow & B\pi_1. & &
\end{array}$$

If $X = K \cup_\varphi D^4$ and $X' = K' \cup_\psi D^4$, then D and D' are constructed from the 3-complexes K and K', respectively, by adjoining cells of dimension greater or equal to 4. Proposition 3.2 defines an obstruction to extending the restriction $h^{(3)} : K \to K'$ to a homotopy equivalence $X \to X'$. Also, if this obstruction does not vanish, it could be that X is homotopy equivalent to X'. We use h to identify $D \to B\pi_1$ with $D' \to B\pi_1$. All this makes sense if $B\pi_1$ is an aspherical 2-complex. From now on we shall consider only Baumslag–Solitar groups $B(k)$, $k \neq 0$, and aspherical surface fundamental groups. For any such model X we obtain $H_3(X, \Lambda) \cong H^1(X, \Lambda) \cong H^1(B\pi, \Lambda) \cong 0$ by Lemma 6.2 of [5] (here $\pi = \pi_1$, as usual). Since $H_4(X, \Lambda) \cong 0$, we get an isomorphism from $H_4(X, K, \Lambda)$ onto $H_3(K, \Lambda)$, i.e. $H_3(K, \Lambda) \cong \Lambda$. Furthermore, the canonical generator of $H_4(X, K, \Lambda)$, given by the attaching map φ, defines a generator of $H_3(K, \Lambda)$ and a splitting $s_X : H_3(K, \Lambda) \to \pi_3(K)$ of the Whitehead sequence given by the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma(\pi_2) & \xrightarrow{i_*} & \pi_3(K) & \xrightarrow{H} & H_3(K, \Lambda) & \longrightarrow & 0 \\
 & & & & \big\uparrow & & \big\uparrow {\scriptstyle \cong} & & \\
 & & & & \pi_4(X, K) & \xrightarrow{\cong} & H_4(X, K, \Lambda). & &
\end{array}$$

Then s_X defines a splitting $t_X : \pi_3(K) \to \Gamma(\pi_2)$. From the Whitehead sequence of X, we have an isomorphism from $\Gamma(\pi_2)$ onto $\pi_3(X)$, and t_X can also be defined by the following diagram:

$$\begin{array}{ccc} \Gamma(\pi_2) & \xleftarrow{t_X} & \pi_3(K) \\ \cong\downarrow & & \downarrow j_* \\ \pi_3(X) & = & \pi_3(X). \end{array}$$

Conversely, t_X defines s_X by the well-known procedure using the projection operator $i_* \circ t_X$. If $X = K \cup_\varphi D^4$ and $X' = K \cup_\psi D^4$ are homotopy equivalent models, there is a homotopy equivalence of pairs (see [13], Theorem 2.4)

$$g : (K, \varphi(\mathbb{S}^3)) \to (K, \psi(\mathbb{S}^3))$$

inducing the diagrams

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Gamma(\pi_2) & \xrightarrow{i_*} & \pi_3(K) & \xrightarrow{j_*} & \pi_3(X) \\ & & g_*\downarrow & & g_*\downarrow & & g_*\downarrow \\ 0 & \longrightarrow & \Gamma(\pi_2) & \xrightarrow{i'_*} & \pi_3(K) & \xrightarrow{j'_*} & \pi_3(X') \end{array}$$

and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma(\pi_2) & \xrightarrow{i_*} & \pi_3(K) & \xrightarrow{H} & H_3(K,\Lambda) & \longrightarrow & 0 \\ & & g_*\downarrow & & g_*\downarrow & & g_*\downarrow & & \\ 0 & \longrightarrow & \Gamma(\pi_2) & \xrightarrow{i'_*} & \pi_3(K) & \xrightarrow{H} & H_3(K,\Lambda) & \longrightarrow & 0. \end{array}$$

Hence, all splittings t_X, $t_{X'}$, s_X, and $s_{X'}$ commute with the induced homomorphisms g_*. In the following we fix one model $X = K \cup_\varphi D^4$. We are going to construct models $X' = K \cup_\psi D^4$ that are not homotopy equivalent to X. Let us denote by $1 \in H_3(K,\Lambda)$ the generator defined by X, i.e. $s_X(1) = [\varphi]$. Let $\theta : \Gamma(\pi_2) \to \Gamma(\pi_2)$ be an isomorphism. Then $\theta \circ t_X = t : \pi_3(K) \to \Gamma(\pi_2)$ is a splitting. It defines a splitting $s : H_3(K,\Lambda) \to \pi_3(K)$. Then $s(1) = s_X(1) + i_*(a)$ for some $a \in \Gamma(\pi_2)$. As in Section 4, we construct the model $X' = K \cup_\psi D^4$ with $[\psi] = s(1)$.

Proposition 5.1 *If θ is not induced by an isomorphism $\pi_2 \to \pi_2$, then X' is not homotopy equivalent to X.*

Proof Any homotopy equivalence $g : X \to X'$ induces

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Gamma(\pi_2) & \longrightarrow & \pi_3(K) & \longrightarrow & \pi_3(X) \\ & & g_*\downarrow & & g_*\downarrow & & g_*\downarrow \\ 0 & \longrightarrow & \Gamma(\pi_2) & \longrightarrow & \pi_3(K) & \longrightarrow & \pi_3(X'). \end{array}$$

However, $g_* : \Gamma(\pi_2) \to \Gamma(\pi_2)$ is never θ. □

Examples Let $X = F \times \mathbb{S}^2$, where F is a closed oriented aspherical surface. Then $\pi_2(X) \cong \mathbb{Z}$, $\Gamma(\pi_2) \cong \mathbb{Z}$ and $-\mathrm{Id} : \Gamma(\pi_2) \to \Gamma(\pi_2)$ is not induced by an isomorphism $\pi_2 \to \pi_2$. This easily follows from the Γ-functor property. There are inclusions $\pi_2 \to \Gamma(\pi_2)$ and $\Gamma(\pi_2) \to \pi_2 \otimes \pi_2$ (because π_2 is free abelian) such that the composition $\pi_2 \to \Gamma(\pi_2) \to \pi_2 \otimes \pi_2$ sends x to $x \otimes x$. In the case when $\pi = B(k)$, π_2 is free abelian (see [5], Lemma 6.2 V), one obtains such θ in this case, too. On the other hand, if θ is induced by an isomorphism $\beta : \pi_2 \to \pi_2$, one needs more to construct a homotopy equivalence. By [15], Theorem 3, one gets a map $g : K \to K$, but the induced maps g_* do not necessarily commute with the splittings s_X and $s_{X'}$.

Supplement to the aspherical surface case. In the example $F \times \mathbb{S}^2$ there are 2 models, namely $F \times \mathbb{S}^2$ and the non-trivial $\mathbb{S}^2$-bundle $E \to F$ with the second Stiefel–Whitney class $\neq 0$ (see, for example, [3], Appendix). Here it is also convenient to consider the map

$$F_{\mathbb{Z}} : H_4(D,\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(H^2(D,\mathbb{Z}) \otimes H^2(D,\mathbb{Z}),\mathbb{Z})$$

given by

$$F_{\mathbb{Z}}(x)(u \otimes v) := x \cap (u \cup v),$$

where $D = F \times \mathbb{C}P^\infty$. Then $F_{\mathbb{Z}}$ is injective. If $f_0 : F \times \mathbb{S}^2 \to D$ and $f_1 : E \to D$ are Postnikov maps, then $F_{\mathbb{Z}}(f_{0*}[F \times \mathbb{S}^2])$ and $F_{\mathbb{Z}}(f_{1*}[E])$ are the integral intersection forms of $F \times \mathbb{S}^2$ and E, respectively. Moreover, these forms are respectively given by the matrices

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

(see [3]). It was shown in [9], Section 5, that $F \times \mathbb{S}^2$ and E are the only models up to homotopy equivalence.

6. Final remarks

The following map was defined in [2]:

$$F : H_4(D,\mathbb{Z}) \to \mathrm{Hom}_{\Lambda-\Lambda}(H^2(D,\Lambda) \otimes_{\mathbb{Z}} \overline{H}^2(D,\Lambda),\Lambda),$$

to classify Poincaré 4-complexes X, where $D \to B\pi$ is a 2-stage Postnikov system for X. Here $H^2(D,\Lambda) \otimes_{\mathbb{Z}} \overline{H}^2(D,\Lambda)$ carries the obvious Λ-bimodule structure. It was proven therein that F is injective for free non-abelian groups π. The maps F and G are related by the following diagram:

$$\begin{array}{ccc} H_4(D,\mathbb{Z}) & \xrightarrow{G} & \mathrm{Hom}_{\Lambda-\Lambda}(H^2(D,\Lambda), \overline{H}_2(D,\Lambda)) \\ \| & & \downarrow H \\ H_4(D,\mathbb{Z}) & \xrightarrow{F} & \mathrm{Hom}_{\Lambda-\Lambda}(H^2(D,\Lambda) \otimes_{\mathbb{Z}} \overline{H}^2(D,\Lambda),\Lambda), \end{array}$$

where $H(\varphi)(u \otimes v) = \overline{\hat{u}(\varphi(v))}$, and $\hat{u}$ is the image of u under

$$H^2(D,\Lambda) \to \mathrm{Hom}_\Lambda(H_2(D,\Lambda),\Lambda).$$

Obviously, G is injective if F is injective. If $f : X \to D$ is a map such that $f_* : \pi_q(X) \to \pi_q(D)$ is an isomorphism for $q = 1,2$, then $F(f_*[X]) \circ (f^* \otimes f^*)$ is the equivariant intersection form on X, and $f_*G(f_*[X])f^* : H^2(X,\Lambda) \to \overline{H}_2(X,\Lambda)$ is the Poincaré duality isomorphism. It is convenient to denote $F(f_*[X])$ as the "intersection type" and $G(f_*[X])$ as the "Poincaré duality type" of X. The Poincaré duality type determines the intersection type. In this sense it is a stronger "invariant". For $\mathbb{S}^2$-bundles over aspherical 2-surfaces all intersection types vanish, whereas the Poincaré types are non-trivial.

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