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# Gonality of curves with a singular model on an elliptic quadric surface

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**Abstract:** Let $W \subset \mathbb{P}^3$ be a smooth quadric surface defined over a perfect field $K$ and with no line defined over $K$ (e.g., an elliptic quadric surface over a finite field). In this note we study the gonality over $K$ of smooth curves with a singular model contained in $W$ and with mild singularities.

**Key words:** Gonality, curve over a perfect field, $K$-gonality, elliptic quadric surface

## 1. Introduction

Let $K$ be a perfect field such that there is a degree 2 extension $L$ of $K$. Let $f(x_0, x_1) \in K[x_0, x_1]$ denote any degree 2 homogeneous polynomial such that $L = K(\alpha)$ with $\alpha$ a root of $f(1,t)$, i.e. take as $f$ any degree 2 homogeneous polynomial that is irreducible over $K$ but reducible over $L$. The main examples are the case $K = \mathbb{R}$, $L = \mathbb{C}$ and the case $K = \mathbb{F}_q$ and $L = \mathbb{F}_{q^2}$. Take homogeneous coordinates $x_0, x_1, x_2, x_3$ of $\mathbb{P}^3$ (over $K$ and hence over $\overline{K}$). Let $W \subset \mathbb{P}^3$ denote the smooth quadric surface with $x_2x_3 + f(x_0, x_1)$ as its equation. If $K = \mathbb{R}$, then these types of surfaces are just ellipsoids. If $K = \mathbb{F}_q$, then $W$ is an elliptic quadric surface [4]. In this paper we study the $K$-gonality of smooth curves $C$ either contained in $W$ or with a singular model $Y \subset W$, but with a small number of singularities. We prove the following result.

**Corollary 1** *Let $Y \subset W$ be a geometrically integral curve defined over $K$ and let $u : C \to Y$ be the normalization of $Y$. Let $a > 0$ be the positive integer such that $Y \in |\mathcal{O}_W(a)|$. Assume that $Y(\overline{K})$ has only ordinary nodes and ordinary cusps as singularities and set $J := \mathrm{Sing}(Y(\overline{K}))$. Assume $\sharp(J) \le a - 5$ and that no line of $W(\overline{K})$ contains at least 2 points of $J$. Let $R \in \mathrm{Pic}^y(C)(K)$ be a spanned line bundle on $C$ defined over $K$ and with minimal positive degree. Then $2a - 4 \le y \le 2a$ and $R$ is induced by a subseries of $|\mathcal{O}_W(1)|$.*

*We have $y = 2a - 4$ if and only if there is a degree 2 extension $K'$ of $K$ such that $\sharp(J(K')) \ge 2$.*

*We have $y = 2a$ if and only if $Y(K') = \emptyset$ for each degree 2 extension $K'$ of $K$.*

See Theorem 1 for spelling out the possible cases of $y$. For the foundational results on the gonality of curves over algebraically closed fields, see [8], [5], [9].

Since we work in arbitrary characteristic we cannot use some of the strongest tools in the literature. In our opinion in characteristic zero the best results are still obtained using [7] or the case $e = 0$ of [10] and [6],

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Remark 2 on page 351. To get Corollary 1 and related results we need first to work over an algebraically closed field $\mathbb{K}$ and study low degree linear series on smooth models of singular curves on a smooth quadric surface $Q$ (see section 2). As stressed above, in characteristic zero stronger tools are available.

We discuss our method and possible improvements in Subsection 2.1.

Many thanks are due to a referee who improved the exposition.

## 2. Over an algebraically closed field $\mathbb{K}$

Let $Q \subset \mathbb{P}^3$ be a smooth quadric surface defined over an algebraically closed field $\mathbb{K}$. For any coherent sheaf $\mathcal{F}$ on $Q$ and any integer $i \geq 0$ set $H^i(\mathcal{F}) := H^i(Q, \mathcal{F})$ and $h^i(\mathcal{F}) := \dim(H^i(\mathcal{F}))$. For all $(a,b) \in \mathbb{Z}^2$ let $\mathcal{O}_Q(a,b)$ denote the line bundle on $Q$ with bidegree $(a,b)$. We have $h^0(\mathcal{O}_Q(a,b)) = (a+1)(b+1)$ and $h^1(\mathcal{O}_Q(a,b)) = 0$ if $a \geq 0$ and $b \geq 0$, while $h^0(\mathcal{O}_Q(a,b)) = 0$ if either $a < 0$ or $b < 0$. If $a \geq 0$, $b \geq 0$ and $T \in |\mathcal{O}_Q(a,b)|$, then we say that $T$ has type $(a,b)$. The lines contained in $Q$ are the curves $D \subset Q$ with either type $(1,0)$ or type $(0,1)$. For any zero-dimensional scheme $Z \subset Q$ and any $T \in |\mathcal{O}_Q(u,v)|$, let $\mathrm{Res}_T(Z)$ denote the residual scheme of $Z$ with respect to $T$, i.e. the closed subscheme of $Q$ with $\mathcal{I}_Z : \mathcal{I}_T$ as its ideal sheaf. We have $\mathrm{Res}_T(Z) \subseteq Z$, $\deg(Z) = \deg(\mathrm{Res}_T(Z)) + \deg(Z \cap T)$ and for all $(a,b) \in \mathbb{Z}^2$ we have an exact sequence (often called the residual exact sequence)

$$0 \to \mathcal{I}_{\mathrm{Res}_T(Z)}(a-u, b-v) \to \mathcal{I}_Z(a,b) \to \mathcal{I}_{Z \cap T, T}(a,b) \to 0 \tag{1}$$

### 2.1. Outline of the proof and of possible improvements

Take an integral curve $Y \subset Q$ with bidegree $(a,a)$. Let $u : C \to Y$ be the normalization map and $w : C \to Q$ the composition of $u$ with the inclusion $Y \hookrightarrow Q$. Let $\mathcal{J} \subseteq \mathcal{O}_Q$ be the conductor of $w$ and $J \subset Q$ the zero-dimensional subscheme of $Q$ with $\mathcal{J}$ as its ideal sheaf. Let $J_{\mathrm{red}}$ be the support of $J$. We assume for instance $\deg(J) \leq a-5$. Let $\mathcal{F}$ be the set of all irreducible $E \in |\mathcal{O}_Q(1,1)|$ such that $1 \leq \sharp(E \cap J_{\mathrm{red}}) \leq 2$. Let $\mathcal{G}$ be the set of all irreducible $E \in |\mathcal{O}_Q(1,1)|$ such that $\sharp(E \cap J_{\mathrm{red}}) \geq 3$. Let $\mathcal{H}$ be the set of all reducible $E \in |\mathcal{O}_Q(1,1)|$ such that each component of $E$ meets $J_{\mathrm{red}}$. Take $B$ as in the proof of Lemma 5. Since $\mathcal{G} \cup \mathcal{H}$ is finite, while $B$ is general, we have $E \cap B = \emptyset$ for all $E \in (\mathcal{G} \cup \mathcal{H})$. To apply Lemmas 1 and 2 to the scheme $Z = J \cup B$ it is sufficient to assume $\deg(J \cap E) + y \leq 2a-5$ for all $E \in |\mathcal{O}_Q(1,1)|$. With this assumption steps (ii), (iii), (iv) of the proof of Lemma 5 carry over, because $\deg(J \cap E) \leq 2a-5-y$ for all $E \in \mathcal{F}$ and $\deg(D \cap B) \leq 2$ if $D \in |\mathcal{O}_Q(1,1)|$ is reducible and $b_1 = b_2 = 1$. Step (i) of the proof of Lemma 5 requires the following modifications for arbitrary singularities. For each $P \in J_{\mathrm{red}}$ let $u_P$ be the degree of the effective divisor $w^{-1}(P) \subset C$. For each connected degree 2 zero-dimensional scheme $Z \subset Q$ whose support is a point $P \in J_{\mathrm{red}}$ let $u_{Z,P}$ be the degree of the effective divisor $w^{-1}(Z) \subset C$. We say that $Y$ has either an ordinary node or an ordinary cusp at $P$ if $u_P = 2$ and for each connected degree 2 scheme $Z \subset Q$ with $P$ as its support either $u_{Z,P} = 3$ (if and only if in the plane $T_PQ$ the line through $Z$ is in the tangent cone of $Y$ at $P$) or $u_{Z,P} = 2$. In the description of step (i) of the proof of Lemma 5 we use the integers $u_P$ (with $u_P = 2$ for double points) and $u_{Z,P}$ (which are 2 or 3 for ordinary nodes and cusps with 3 if and only if $Z$ corresponds to a branch of $Y$ at $P$. See for instance [1], [2], [3] for the formal theory of plane and space curves.

Now assume $Y \subset W$ and that $Y$ is defined over $K$. To extend Theorem 1 one needs to know the integers $u_P$, $P \in J_{\mathrm{red}}(K')$ for any degree 2 extension $K'$ of $K$ and the integers $u_{Z,P}$ with $P \in J_{\mathrm{red}}(K)$ and $Z$ defined over $K$. The tools work for all spanned $R \in \mathrm{Pic}^y(C)(K)$ with $\deg(J) + y \le 3a - 5$, without assuming that $y$ is the $K$-gonality of $C$.

### 2.2. Proofs over $\mathbb{K}$

**Lemma 1** *Fix an integer $c \ge 2$ and a zero-dimensional scheme $Z \subset Q$. Assume $\deg(Z \cap L) \le 1$ for each line $L \subset Q$, $h^1(\mathcal{I}_Z(c,c)) > 0$ and $\deg(Z) \le 3c+1$. Then there is an integral $D \in |\mathcal{O}_Q(1,1)|$ such that $\deg(D \cap Z) \ge 2c+2$.*

**Proof** Set $Z_0 := Z$. Let $T_1 \subset Q$ be any element of $|\mathcal{O}_Q(1,1)|$ such that $e_1 := \deg(T_1 \cap Z)$ is maximal. Set $Z_1 := \mathrm{Res}_{T_1}(Z_0)$. For each integer $i \ge 2$ define recursively the integer $e_i$, the curve $T_i \in |\mathcal{O}_Q(1,1)|$, and the scheme $Z_i \subseteq Z_{i-1}$ in the following way. Let $T_i \subset Q$ be any element of $|\mathcal{O}_Q(1,1)|$ such that $e_i := \deg(T_i \cap Z_{i-1})$ is maximal. Set $Z_i := \mathrm{Res}_{T_1}(Z_{i-1})$. The sequence $\{e_i\}_{i \ge 1}$ is nonincreasing. Since $h^0(\mathcal{O}_Q(1,1)) = 4$, we have $e_{i+1} = 0$ and $Z_i = \emptyset$ if $e_i \le 2$. Since $\deg(Z \cap L) \le 1$ for each line $L \subset Q$, we may take $T_i$ as above and with the additional restriction that each $T_i$ is irreducible. Since $\deg(Z) \le 3c+1$, we get $e_{c+1} \le 1$ and $Z_{c+1} = \emptyset$. From (1) for each $i \in \{1, \dots, c\}$ we get the exact sequences

$$0 \to \mathcal{I}_{Z_i}(c-i, c-i) \to \mathcal{I}_{Z_{i-1}}(c-i+1, c-i+1) \to \mathcal{I}_{Z_{i-1}, T_i}(c-i+1, c-i+1) \to 0 \tag{2}$$

Since $\deg(Z_c) \le 1$, we have $h^1(\mathcal{I}_{Z_c}) = 0$. Since $h^1(\mathcal{I}_Z(c,c)) > 0$, we get the existence of an integer $i \in \{1, \dots, c\}$ such that $h^1(T_i, \mathcal{I}_{Z_{i-1}, T_i}(c-i+1, c-i+1)) > 0$. Let $f$ be the minimal such integer. Since $T_f$ is irreducible, we have $T_f \cong \mathbb{P}^1$. Since $\deg(\mathcal{O}_{T_f}(c-f+1, c-f+1)) = 2c-2f+2$, we have $h^1(T_f, \mathcal{I}_{Z_{f-1}, T_f}(c-f+1, c-f+1)) > 0$ if and only if $e_f \ge 2c - 2f + 4$. If $f = 1$, then we may take $D := T_1$. Now assume $f \ge 2$. Since $e_i \ge e_f$ for all $i < f$, we get $\deg(Z) \ge 2f(c-f+2)$. The function $\psi(t) := 2t(c+2-t)$ is increasing in the interval $[2, (c+2)/2]$ and decreasing for $t > (c+2)/2$. Since $\psi(2) = \psi(c) = 4c$, we get $\deg(Z) \ge 4c$, a contradiction. □

**Lemma 2** *Fix integers $k \ge c \ge 0$ and a zero-dimensional scheme $Z \subset Q$ such that $\deg(Z) \le k+c+1$ and $\deg(Z \cap L) \le 1$ for each line $L \subset Q$. Then $h^1(\mathcal{I}_Z(k,c)) = 0$.*

**Proof** If $c = 0$, then one may use $k-c$ residual exact sequences, each time with respect to some $L \in |\mathcal{O}_Q(1,0)|$. If $k = c = 1$, then the lemma is obvious. If $k = c \ge 2$, then we may apply Lemma 1. Now assume $k > c > 0$. By the case $c = 0$ we may assume $\deg(Z) \ge k - c$. Since $h^0(Q, \mathcal{O}_Q(k-c, 0)) = k-c+1$, there is $F \in |\mathcal{O}_Q(k-c, 0)|$ such that $\deg(F \cap Z) \ge k-c$. Since $\deg(L \cap Z) \le 1$ for each $L \in |\mathcal{O}_Q(1,0)|$, we have $\deg(F \cap Z) = k-c$. Hence $\deg(\mathrm{Res}_F(Z)) = \deg(Z) - k + c \le 2c+1$. Lemma 1 gives $h^1(\mathcal{I}_{\mathrm{Res}_F(Z)}(c,c)) = 0$. We saw that $h^1(\mathcal{I}_{F \cap Z}(k,0)) = 0$ and hence $h^1(\mathcal{I}_{F \cap Z}(k,c)) = 0$. Therefore $h^1(F, \mathcal{I}_{F \cap Z, F}(k,c)) = 0$. A residual exact sequence gives $h^1(\mathcal{I}_Z(k,c)) = 0$. □

**Lemma 3** *Let $T \subset Q$ be an integral element of $|\mathcal{O}_Q(a,a)|$ and $u : C \to T$ its normalization. Let $\mathcal{J} \subset \mathcal{O}_Q$ be the conductor of $u$ and $J \subset Q$ the closed subscheme with $\mathcal{J}$ as its ideal sheaf. Fix integers $x \in \{0, \dots, a-2\}$ and $y \in \{0, \dots, a-2\}$. We have $h^0(C, u^*(\mathcal{O}_T(x,y))) = (x+1)(y+1)$ if and only if $h^1(\mathcal{I}_J(a-2-x, b-2-y)) = 0$.*

**Proof** Since $a > x$, $a > y$ and $T$ has type $(a,a)$, we have $h^0(\mathcal{I}_T(x,y)) = 0$. Since $h^1(Q, \mathcal{O}_Q(a-x, b-y)) = 0$, the exact sequence (1) for $Z = \emptyset$ gives $h^0(T, \mathcal{O}_T(x,y)) = (x+1)(y+1)$. Hence we have

$h^0(C, u^*(\mathcal{O}_T(x,y))) = (x+1)(y+1)$ if and only if $h^1(C, u^*(\mathcal{O}_T(x,y))) = h^1(T, \mathcal{O}_T(x,y)) - \deg(J)$. Since $\omega_Q \cong \mathcal{O}_Q(-2,-2)$, we have $\omega_T \cong \mathcal{O}_T(a-2, a-2)$. Since $h^i(\mathcal{O}_Q(-2,-2)) = 0$, $i = 0, 1$, the restriction map $H^0(Q, \mathcal{I}_J(a-2, a-2)) \to H^0(T, \omega_T)$ is bijective. Hence $h^1(C, u^*(\mathcal{O}_T(x,y))) = h^1(T, \mathcal{O}_T(x,y)) - \deg(J)$ if and only if $h^1(\mathcal{I}_J(a-2-x, b-2-y)) = 0$. □

**Corollary 2** *Let $T \subset Q$ be an integral element of $|\mathcal{O}_Q(a,a)|$ with only ordinary nodes or ordinary cusps as its singularities. Let $u : C \to T$ be the normalization map. Set $J := \mathrm{Sing}(T)$ and assume $\deg(J \cap L) \le 1$ for every line $L \subset Q$. If $\sharp(J) \le 3(a-3)+1$, then $h^0(C, u^*(\mathcal{O}_T(0,1))) = h^0(C, u^*(\mathcal{O}_T(1,0))) = 2$ and $h^0(C, u^*(\mathcal{O}_T(1,1))) = 4$.*

**Proof** Since $T$ has only ordinary nodes and ordinary cusps as singularities, the set $J$ is the conductor scheme used in Lemma 3. Apply Lemmas 1 and 3. □

**Lemma 4** *Fix positive integers $c, b_1, b_2$ such that $\max\{b_1, b_2\} \le c+1$. Fix a zero-dimensional scheme $J \subset Q$ and a finite set $B \subset Q$ such that $B \cap J = \emptyset$, $\deg(J \cap I) \le 1$ for every line $I \subset Q$, no line of $Q$ intersects both $J$ and $B$, either $I \cap B = \emptyset$ or $I \cap B = b_1$ for each $I \in |\mathcal{O}_Q(1,0)|$ and either $I \cap B = \emptyset$ or $I \cap B = b_2$ for each $I \in |\mathcal{O}_Q(0,1)|$. Assume $h^1(\mathcal{I}_{J \cup B}(c,c)) > 0$.*

*(a) If $b_1 = b_2 = 1$ and $\deg(J \cup B) \le 3c+1$, then there is an integral $D \in |\mathcal{O}_Q(1,1)|$ such that $\sharp(D \cap (J \cup B)) \ge 2c+2$.*

*(b) If $\delta := \max\{b_1, b_2\} \ge 2$, then $\deg(J) \ge 2c+2-\sharp(B)/\delta$.*

**Proof** Set $Z = J \cup B$. The case $b_1 = b_2 = 1$ is true by Lemma 1. Hence we may assume $b_1 \ge 2$. We have $\sharp(B) = xb_1 = yb_2$ for some positive integers $x, y$. Without losing generality we may assume $b_1 \ge b_2$. Let $F \in |\mathcal{O}_Q(x,0)|$ be the union of all lines containing at least one point of $B$. By assumption $F \cap J = \emptyset$. Since $\sharp(B \cap I) = b_1 \le c+1$ for each component $I$ of $F$, we have $h^1(F, \mathcal{I}_{Z \cap F}(c,c)) = 0$. Hence the exact sequence

$$0 \to \mathcal{I}_J(c-x, c) \to \mathcal{I}_Z(c,c) \to \mathcal{I}_{F \cap Z, F}(c,c) \to 0$$

gives $h^1(\mathcal{I}_J(c-x, c)) > 0$. Lemma 2 gives $\deg(J) \ge 2c - x + 2$. □

**Remark 1** *In the next lemma the integers $b_1$ and $b_2$ are positive integers dividing $y$ (they may be $1$). In the applications to $W$ (Corollary 1 and Theorem 1) $b_1 = b_2$ and $b_1$ divides $a$. Hence when one needs to apply Lemma 5 to curves in $W$ there is a very small number of possible pairs $(b_1, b_2) \neq (1,1)$.*

**Lemma 5** *Let $T \subset Q$ be an integral element of $|\mathcal{O}_Q(a, a')|$, $a' \ge a \ge 2$, and $u : C \to T$ its normalization. Let $w : C \to Q$ be the composition of $u$ with the inclusion $T \hookrightarrow Q$. Assume that $T$ has only ordinary nodes and ordinary cusps as singularities and set $J := \mathrm{Sing}(T)$. Assume $\deg(J \cap L) \le 1$ for each line $L \subset Q$. Fix $R \in \mathrm{Pic}^y(C)$, $y > 0$, such that $R$ has no base points and $R$ is neither $u^*(\mathcal{O}_C(1,0))$ nor $u^*(\mathcal{O}_C(0,1))$. Let $h : C \to \mathbb{P}^1$ be the morphism associated to a general $2$-dimensional linear subspace of $H^0(C, R)$. Let $u_1 : C \to \mathbb{P}^1$ and $u_2 : C \to \mathbb{P}^1$ be the morphisms associated to the 2 projections $Q \to \mathbb{P}^1$. Let $b_i$ be the degree of the morphism $(h, u_i)$.*

*(a) Assume* $b_1 = b_2 = 1$ *and* $y + \sharp(J) \le 2a + a' - 5$*. There is a zero-dimensional scheme* $\Gamma \subset Q$ *with* $0 \le \deg(\Gamma) \le 2$ *such that* $h^0(R) = 4 - \deg(\Gamma)$ *and* $R$ *is induced by the linear system* $|\mathcal{I}_\Gamma(1,1)|$*. We have* $\deg(R) = a + a' - \deg(\Gamma')$*, where* $\Gamma' := w^{-1}(\Gamma)$*.*

*(b) Assume* $(b_1, b_2) \neq (1,1)$ *and set* $\delta := \max\{b_1, b_2\}$*. We have* $\sharp(J) \ge a' + a - 2 - y/\delta$*.*

**Proof** Set $R' := u^*(\mathcal{O}_Q(1,1))$. Lemma 3 gives $h^0(C, R') = 4$. Hence $|R'|$ is induced by $|\mathcal{O}_Q(1,1)|$.

(i) Assume for the moment that $|R|$ is induced by a linear subseries $M$ of $|\mathcal{O}_Q(1,1)|$, after deleting a base locus. Let $\Gamma \subset Q$ be the base locus of $M$. Since $R$ is neither $u^*(\mathcal{O}_C(1,0))$ nor $u^*(\mathcal{O}_C(0,1))$, $\Gamma$ is not a line. Hence $\Gamma$ is a zero-dimensional scheme (it may be empty). Set $\Gamma' := w^{-1}(\Gamma)$. Since $\mathcal{O}_Q(1,1)$ is very ample, we have $h^0(\mathcal{I}_E(1,1)) = 4 - \deg(E)$ for all zero-dimensional schemes $E \subset Q$ with $\deg(E) \le 2$. Notice that $h^0(\mathcal{I}_E(1,1)) = 1$ for each degree 3 scheme $E \subset Q$ not contained in a line of $Q$. Since every line $L \subset \mathbb{P}^3$ with $\deg(L \cap Q) \ge 3$ is contained in $Q$, we get $\deg(\Gamma) \le 2$ and $h^0(R) = 4 - \deg(\Gamma)$. Moreover, $\mathcal{I}_\Gamma(1,1)$ is spanned, unless $\deg(\Gamma) = 2$ and $\Gamma$ is contained in a line of $Q$. The latter case does not occur for $R$, because the line would be in the base locus $\Gamma$, while $\dim(\Gamma) = 0$. Hence $\mathcal{I}_\Gamma(1,1)$ is spanned. Since $\mathcal{I}_\Gamma(1,1)$ and $R$ are spanned, we have $R \cong R'(-\Gamma')$.

(ii) Fix a general $A \in |R|$ and set $B := u(A)$. Let $f : C \to \mathbb{P}^1$ be the degree $y$ morphism induced by $|R|$. Since $f$ is induced by a general pencil of the complete linear system $|R|$, it cannot factor through the Frobenius of order $p$. Since $\mathbb{K}$ is perfect, we get that $f$ is separable. Since $A$ is general, $A$ is a reduced set of $y$ points. Since $|R|$ is spanned, we may also assume $A \cap u^{-1}(\mathrm{Sing}(T)) = \emptyset$. Hence $B \cap J = \emptyset$ and $\sharp(B) = y$.

*Claim:* We have $h^1(\mathcal{I}_{J \cup B}(a-2, a'-2)) > 0$.

*Proof of the Claim:* Fix $O \in A$. Since $R$ is spanned, we have $h^0(R(-O)) = h^0(R) - 1$, i.e. $h^0(\omega_C(-(A \setminus \{O\}))) = h^0(\omega_C(-A))$ (Riemann–Roch and Serre duality). Hence $h^1(\omega_C(-A)) > 0$. We have $\omega_Q \cong \mathcal{O}_Q(-2,-2)$. Hence the adjunction formula gives $\omega_T \cong \mathcal{O}_T(a-2, a'-2)$. Since $h^i(\mathcal{O}_Q(-2,-2)) = 0$, $i = 0,1$, the restriction map $H^0(\mathcal{O}_Q(a-2, a'-2)) \to H^0(T, \omega_T)$ is bijective. Since $T$ has only ordinary nodes and ordinary cusps as singularities, we have $H^0(C, \omega_C) \cong H^0(\mathcal{I}_J(a-2, a'-2))$. Hence $h^1(\mathcal{I}_{J \cup B}(a-2, a'-2)) > 0$.

(iii) In this step we assume $a' = a$ and $h^0(R) = 2$. We first prove that $R$ is a subsheaf of $u^*(\mathcal{O}_T(1,1))$.

(a) Assume $b_1 = b_2 = 1$. Since $y + \sharp(J) \le 3a - 5$ and $h^1(\mathcal{I}_{J \cup B}(a-2, a-2)) > 0$ by the Claim, Lemma 4 gives the existence of a divisor $D \in |\mathcal{O}_Q(1,1)|$ such that $\deg(D \cap (J \cup B)) \ge 2a - 2$. Since $R$ has no base points and $h^0(R) = 2$, we get $B = B \cap D$. Moving $A \in |R|$ the set $B$ moves and hence $D$ moves, but $Y$ and the set $J \cap D$ are the same for all general $A$. Hence $|R|$ is induced by a subseries $M$ of the linear system $|\mathcal{O}_Q(1,1)|$. Let $\Gamma \subset Q$ be the base locus of $M$. Since $h^0(R) = 2$, step (i) gives $\deg(\Gamma) = 2$. Step (i) gives $y = 2a - \deg(\Gamma')$.

(b) Assume $\delta \ge 2$ and say $b_1 \ge b_2$. Since $B$ is general, either $I \cap B = \emptyset$ or $\sharp(I \cap B) = b_1$ for each $I \in |\mathcal{O}_Q(1,0)|$ and either $I \cap B = \emptyset$ or $\sharp(I \cap B) = b_2$ for each $I \in |\mathcal{O}_Q(0,1)|$. Since $R \neq u^*(\mathcal{O}_T(1,0))$, we have $\delta < a$. Lemma 4 gives $\sharp(J) \ge 2a - 2 - y/\delta$.

(iv) Assume $a' > a$ and $h^0(R) = 2$. Let $F \subset Q$ be a union of $a' - a$ lines of type $(0,1)$, each of them meeting $B$. Notice that $F \cap J = \emptyset$ and $\sharp(L \cap B) = b_1$ for each component $L$ of $F$. Since $b_1 \le a + 1$, we have $h^1(F, \mathcal{I}_{F \cap (B \cup J), F}(a, a')) = 0$. Hence $h^1(\mathcal{I}_{J \cup B}(a, a')) \le h^1(\mathcal{I}_{J \cup (B \setminus B \cap F)}(a, a))$ by a residual exact sequence like (1). Apply step (iii).

(v) Assume $h^0(R) > 2$. By steps (iii) and (iv) a general pencil of $R$ is induced by a 2-dimensional linear subspace of $|\mathcal{O}_Q(1,1)|$. Hence $R$ is induced by a subseries of $|\mathcal{O}_Q(1,1)|$ after deleting the base points. Use step (i). □

**Corollary 3** *In the set-up of Lemma 5 assume $a = a'$. Then $y \ge 2a - 2 - \min\{2, \deg(J)\}$ and for each $y$ with $2a - 2 - \min\{2, \sharp(J)\} \le y \le 2a$ there is a spanned $R \in \mathrm{Pic}^y(C)$ with $|R|$ induced by a linear subspace of $|\mathcal{O}_Q(1,1)|$.*

## 3. The quadric surface $W$

Let $K$ be a perfect field having a quadratic extension. Fix homogeneous coordinates $x_0, x_1, x_2, x_3$ on $\mathbb{P}^3$. Fix $f \in K[x_0, x_1]$ with $f$ homogeneous of degree 2 and with no nontrivial zero in $K$. Set $W := \{x_2x_3 + f(x_0, x_1) = 0\} \subset \mathbb{P}^3$. $W$ is a geometrically smooth quadric surface containing no line defined over $K$. Hence $\mathrm{Pic}(W)(K)$ is freely generated by $\mathcal{O}_W(1)$. Let $Y \subset W$ be a geometrically irreducible curve defined over $K$ and $u : C \to Y$ the normalization map. $C$ is a geometrically connected smooth curve and $C$ and $u$ are defined over $K$. Let $a$ be the only integer such that $Y \in |\mathcal{O}_W(a,a)|$. Set $Q := W(\overline{K})$.

In the set-up of Remark 1 and Corollary 3 the curve $Y(\overline{K})$ has $b_1 = b_2$. For any field $K' \supseteq K$ let $J(K')$ denote the set of all $P \in J$ defined over $K'$.

The following statement implies Corollary 1.

**Theorem 1** *Take the set-up of Corollary 1.*

*(a) If $\sharp(J(K')) \ge 2$ for some quadratic extension $K'$ of $K$, then $y = 2a - 4$.*

*(b) If $\sharp(J(K)) = 1$, $J(K) = J(K')$ for every quadratic extension $K'$ of $K$ and $Y(K) \setminus J(K) \ne \emptyset$, then $y = 2a - 3$.*

*(c) Assume $\sharp(J(K)) = 1$, $J(K) = J(K')$ for every quadratic extension $K'$ of $K$ and $Y(K) = J(K)$. Set $\{P\} := J(K)$. If $Y$ has an ordinary node at $P$ and the formal branches of $Y$ at $P$ are not defined over $K$, then $y = 2a - 2$; otherwise, $y = 2a - 3$.*

*(d) If $J(K'') = \emptyset$ for every quadratic extension $K''$ of $K$ and there is a quadratic extension $K'$ of $K$ with $\sharp(Y(K')) \ge 2$, then $y = 2a - 2$.*

*(e) If $Y(K)$ has a unique point $P$, $P \notin J$ and $Y(K') = \{P\}$ for every quadratic extension $K'$ of $K$, then $y = 2a - 1$.*

*(f) If $J(K') = Y(K') = \emptyset$ for every quadratic extension $K'$ of $K$, then $y = 2a$.*

*In case (e) the only line bundle evincing $y$ is the pull-back of $\mathcal{O}_Y(1)(-P)$ and we have $h^0(R) = 3$.*

*In case (f) the only line bundle $R$ evincing $y$ is the one induced by the pull-back of $\mathcal{O}_W(1)$ and we have $h^0(R) = 4$.*

**Proof** Since $\mathcal{O}_W(1)$ is spanned, we have $y \le 2a$. Part (b) of Lemma 5 shows that $b_1 = b_2 = 1$. Theorem 1 follows from Corollary 3 and step (i) of the proof of Lemma 5. □

Notice that if $J(K') \supsetneq J(K)$ for some quadratic extension $K'$ of $K$, then $J(K') \setminus J(K)$ contains at least 2 elements and hence we are in case (a) with $y = 2a - 4$.

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