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Homological dimensions of complexes related to cotorsion pairs

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Abstract: Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in R-Mod. We define and study notions of $\mathcal{A}$ dimension and $\mathcal{B}$ dimension of unbounded complexes, which is given by means of dg-projective resolution and dg-injective resolution, respectively. As an application, we extend the Gorenstein flat dimension of complexes, which was defined by Iacob. Gorenstein cotorsion, FP-projective, FP-injective, Ding projective, and Ding injective dimension are also extended from modules to complexes. Moreover, we characterize Noetherian rings, von Neumann regular rings, and QF rings by the FP-projective, FP-injective, and Ding projective (injective) dimension of complexes, respectively.

Key words: Cotorsion pairs, $\mathcal{A}$ dimension of complexes, $\mathcal{B}$ dimension of complexes

1. Introduction and preliminaries

In the classical book by Cartan and Eilenberg [4], concepts of projective, injective, and weak (flat) dimensions were defined for left R-modules over arbitrary rings. The extension of homological algebra from modules to complexes of modules, which had started already in the last chapter of [4], has produced a theory of homological dimensions. In that chapter, the projective (or flat) dimension was only defined for complexes that are homologically bounded below, while the injective dimension was introduced only for those that are homologically bounded above. However, in [2], Avramov and Foxby defined injective, projective, and flat dimensions for arbitrary complexes of left R-modules over associative rings in terms of dg-injective, dg-projective, and dg-flat complexes, respectively. For complexes homologically bounded below over commutative rings, Yassemi [31] and Christensen [5] introduced a notion of Gorenstein projective dimension. In [6] Christensen et al. gave generalizations of the Gorenstein projective, Gorenstein injective, and Gorenstein flat dimensions. In [27], Veliche extended the concept of the Gorenstein projective dimension to the setting of unbounded complexes over associative rings. In [1], Asadollahi and Salarian defined the dual notion, that of Gorenstein injective dimension of complexes over arbitrary associative rings. The main purpose of this paper is to give a general study of homological dimensions of complexes of R-modules related to a cotorsion pair $(\mathcal{A}, \mathcal{B})$ in the category R-Mod of R-modules.

The notion of cotorsion pairs was introduced by Salce in [25], and it provides a good setting for investigating relative homological dimensions. Given a cotorsion pair $(\mathcal{A}, \mathcal{B})$ in R-Mod, Gillespie introduced the notions of dg-$\mathcal{A}$ and dg-$\mathcal{B}$ complexes in [18]. We can thus define the $\mathcal{A}$ dimension and $\mathcal{B}$ dimension

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of unbounded complexes, which are given by means of dg-projective resolution and dg-injective resolution, respectively. Next we describe our construction and some results in more detail.

In Section 2, we define and study the $\mathcal{A}$ dimension and $\mathcal{B}$ dimension of complexes. In Theorem 2.7 and Theorem 2.8, we give characterizations of these new notions. In Theorem 2.9 we also prove that if $(\mathcal{A},\mathcal{B})$ is a hereditary cotorsion pair in R-Mod, and X, Y are R-complexes, then $\mathcal{B}\text{-dim}_R(X) = \inf\{\sup\{l \in \mathbb{Z} \mid V_{-l} \neq 0\} \mid X \simeq V \in dg\widetilde{\mathcal{B}}\}$, and $\mathcal{A}\text{-dim}_R(Y) = \inf\{\sup\{l \in \mathbb{Z} \mid T_l \neq 0\} \mid Y \simeq T \in dg\widetilde{\mathcal{A}}\}$. It is well known that the global dimension $\mathrm{gD}(R)$ of R plays an important role in classical homological algebra. Similarly, we define and study the $\mathcal{A}$ dimension and $\mathcal{B}$ dimension of R, denoted by $\mathcal{A}\text{-dim}(R)$ and $\mathcal{B}\text{-dim}(R)$, as the supremum of $\mathcal{A}$ dimension and $\mathcal{B}$ dimension of all R-modules, respectively. Furthermore, we prove that $\mathrm{gD}(R) \leq \mathcal{A}\text{-dim}(R) + \mathcal{B}\text{-dim}(R)$.

We give some applications of our main results in Section 3. We show that the projective, injective dimensions of complexes are also obtained in the present framework. We also extend the Gorenstein flat dimension of complexes, which was defined by Iacob. Gorenstein cotorsion, FP-projective, FP-injective, Ding projective, and Ding injective dimensions are also extended from modules to complexes. Moreover, we characterize Noetherian rings, von Neumann regular rings, and QF rings by FP-projective, FP-injective, and Ding projective (injective) dimensions of complexes, respectively.

We next recall some known notions and facts needed in the sequel.

In this paper, R denotes a ring with unity, R-Mod the category of left R-modules, and $\mathrm{Ch}(R)$ the category of complexes of left R-modules. A complex

$$\cdots \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} C_0 \xrightarrow{\delta_0} C_{-1} \xrightarrow{\delta_{-1}} \cdots$$

of left R-modules will be denoted (C,δ) or C. Given a left R-module M, we will denote by $D^n(M)$ the complex

$$\cdots \longrightarrow 0 \longrightarrow M \xrightarrow{\mathrm{id}} M \longrightarrow 0 \longrightarrow \cdots$$

with the M in the n and $(n-1)$-th position. We mean by $S^n(M)$ the complex with M in the n-th place and 0 in the other places. Given a complex C and an integer i, $\Sigma^i C$ denotes the complex such that $(\Sigma^i C)_n = C_{n-i}$ and whose boundary operators are $(-1)^i \delta^C_{n-i}$. Given a complex C, the n-th homology module of C is the module $\mathrm{H}_n(C) = \mathrm{Z}_n(C)/\mathrm{B}_n(C)$, where $\mathrm{Z}_n(C) = \mathrm{Ker}(\delta^C_n)$, $\mathrm{B}_n(C) = \mathrm{Im}(\delta^C_{n+1})$, and we set $\mathrm{H}^n(C) = \mathrm{H}_{-n}(C)$, $\mathrm{C}_n(C) = \mathrm{Coker}(\delta^C_{n+1})$.

A homomorphism $\varphi : C \longrightarrow D$ of degree n is a family $(\varphi_i)_{i\in\mathbb{Z}}$ of homomorphisms of R-modules $\varphi_i : C_i \longrightarrow D_{n+i}$. All such homomorphisms form an abelian group, denoted $\mathcal{H}\mathrm{om}_R(C,D)_n$, and it is clearly isomorphic to $\prod_{i\in\mathbb{Z}} \mathrm{Hom}_R(C_i, D_{n+i})$. We let $\mathcal{H}\mathrm{om}_R(C,D)$ denote the complex of abelian groups with n-th component $\mathcal{H}\mathrm{om}_R(C,D)_n$ and boundary operator

$$\delta_n((\varphi_i)_{i\in\mathbb{Z}}) = (\delta^D_{n+i}\varphi_i - (-1)^n \varphi_{i-1}\delta^C_i)_{i\in\mathbb{Z}}.$$

A homomorphism $\varphi \in \mathcal{H}\mathrm{om}_R(C,D)_n$ is called a chain map if $\delta(\varphi) = 0$, that is, if $\delta^D_{n+i}\varphi_i = (-1)^n\varphi_{i-1}\delta^C_i$ for all $i \in \mathbb{Z}$. A chain map of degree 0 is called a morphism. $\mathrm{Ext}^i_R(C,D)$ for $i \geq 1$ will denote the groups we get from the right derived functor of $\mathcal{H}\mathrm{om}_R(C,D)$. A morphism $\varphi : C \longrightarrow D$ is called a quasi-isomorphism if the induced morphisms $\mathrm{H}_n(\varphi) : \mathrm{H}_n(C) \longrightarrow \mathrm{H}_n(D)$ are isomorphisms for all $n \in \mathbb{Z}$.

Let X be an R-complex and $m,\ n \in \mathbb{Z}$. The soft left-truncation, ${}_{\subset m}X$, of X at m and the soft right-truncation, $X_{\supset -n}$, of X at $-n$ are given by

$$ {}_{\subset m}X : 0 \longrightarrow \mathrm{C}_m(X) \xrightarrow{\delta^X_m} X_{m-1} \xrightarrow{\delta^X_{m-1}} X_{m-2} \xrightarrow{\delta^X_{m-2}} \cdots, $$

$$ X_{\supset -n} : \cdots \longrightarrow X_{-n+2} \xrightarrow{\delta^X_{-n+2}} X_{-n+1} \xrightarrow{\delta^X_{-n+1}} \mathrm{Z}_{-n}(X) \longrightarrow 0. $$

The hard left-truncation, ${}_{\sqsubset m}X$, of X at m and the hard right-truncation, $X_{\sqsupset -n}$, of X at $-n$ are given by

$$ {}_{\sqsubset m}X : 0 \longrightarrow X_m \xrightarrow{\delta^X_m} X_{m-1} \xrightarrow{\delta^X_{m-1}} X_{m-2} \xrightarrow{\delta^X_{m-2}} \cdots, $$

$$ X_{\sqsupset -n} : \cdots \longrightarrow X_{-n+2} \xrightarrow{\delta^X_{-n+2}} X_{-n+1} \xrightarrow{\delta^X_{-n+1}} X_{-n} \longrightarrow 0. $$

To every complex C we associate the numbers

$$ \sup C = \sup\{i \mid C_i \neq 0\}, \quad \inf C = \inf\{i \mid C_i \neq 0\}. $$

The complex C is called bounded above when $\sup C < \infty$, bounded below when $\inf C > -\infty$, and bounded when it is bounded below and above.

For objects C and D of $\mathrm{Ch}(R)$, $\mathrm{Hom}(C,D)$ is the abelian group of morphisms from C to D in $\mathrm{Ch}(R)$ and $\mathrm{Ext}^i(C,D)$ for $i \geq 1$ will denote the groups we get from the right derived functor of $\mathrm{Hom}(C,D)$, and $\mathrm{pd}_R C$ ($\mathrm{id}_R C$) denotes the projective (injective) dimension of C.

Let $\mathcal{A}, \mathcal{B}$ be 2 classes of R-modules. The pair $(\mathcal{A},\mathcal{B})$ is called a cotorsion pair (also called a cotorsion theory) if $\mathcal{A}^{\perp} = \mathcal{B}$ and $\mathcal{A} = {}^{\perp}\mathcal{B}$. Here $\mathcal{A}^{\perp}$ is the class of R-modules C such that $\mathrm{Ext}^1(A,C) = 0$ for all $A \in \mathcal{A}$, and similarly ${}^{\perp}\mathcal{B}$ is the class of R-modules C such that $\mathrm{Ext}^1(C,B) = 0$ for all $B \in \mathcal{B}$. A cotorsion pair $(\mathcal{A},\mathcal{B})$ is said to be hereditary if whenever $0 \to \widetilde{A} \to A \to \widehat{A} \to 0$ is exact with $A, \widehat{A} \in \mathcal{A}$ then $\widetilde{A}$ is also in $\mathcal{A}$, or, equivalently, if $0 \to \widetilde{B} \to B \to \widehat{B} \to 0$ is exact with $\widetilde{B}, B \in \mathcal{B}$ then $\widehat{B}$ is also in $\mathcal{B}$. A cotorsion pair $(\mathcal{A},\mathcal{B})$ is said to have enough injectives (projectives) [15] if for any object M there exists an exact sequence $0 \to M \to B \to A \to 0$ ($0 \to B \to A \to M \to 0$) with $A \in \mathcal{A}$ and $B \in \mathcal{B}$. By [[15], Proposition 1.1.5], a cotorsion pair $(\mathcal{A},\mathcal{B})$ has enough projectives if and only if it has enough injectives. The cotorsion pair $(\mathcal{A},\mathcal{B})$ is called complete if it has enough projectives and injectives.

Given a class $\mathcal{B}$ of objects of $\mathrm{Ch}(R)$, a morphism $\phi : X \to B$ is called a $\mathcal{B}$-preenvelope ([11]) if $B \in \mathcal{B}$ and $\mathrm{Hom}(B,B') \to \mathrm{Hom}(X,B') \to 0$ is exact for all $B' \in \mathcal{B}$. If, moreover, any $f : B \to B$ such that $f\phi = \phi$ is an automorphism of B then $\phi : X \to B$ is called a $\mathcal{B}$-envelope of X. A complex X is said to have a special $\mathcal{B}$-preenvelope [14] if there is an exact sequence $0 \to X \to B \to L \to 0$ with $B \in \mathcal{B}$ and $L \in {}^{\perp}\mathcal{B}$. A precover, cover, and special precover of X are defined dually.

2. Main results

Let $(\mathcal{A},\mathcal{B})$ be a cotorsion pair in R-Mod. In this section we introduce 2 dimensions for complexes of R-modules: the $\mathcal{A}$ dimension is defined by dg-projective resolution and the $\mathcal{B}$ dimension is defined by dg-injective resolution. The $\mathcal{A}$ and $\mathcal{B}$ dimensions of the ring R are also defined and studied.

Recall from [13] that a complex P is said to be dg-projective if each P_m is projective and $\mathcal{H}\mathrm{om}_R(P,E)$ is exact for any exact complex E. A dg-injective complex is defined dually. Gillespie [[18], Definition 3.3] introduced the following definitions, which generalize the notions of dg-projective and dg-injective complexes.

Definition 2.1 ([18], Definition 3.3) *Let* $(\mathcal{A},\mathcal{B})$ *be a cotorsion pair in* R*-Mod and* X *an* R*-complex.*
(1) X *is called an* $\mathcal{A}$ *complex if it is exact and* $\mathrm{Z}_nX \in \mathcal{A}$ *for all* $n \in \mathbb{Z}$.
(2) X *is called a* $\mathcal{B}$ *complex if it is exact and* $\mathrm{Z}_nX \in \mathcal{B}$ *for all* $n \in \mathbb{Z}$.
(3) X *is called a dg-*$\mathcal{A}$ *complex if* $X_n \in \mathcal{A}$ *for each* $n \in \mathbb{Z}$, *and* $\mathcal{H}\mathrm{om}_R(X,B)$ *is exact whenever* B *is a* $\mathcal{B}$ *complex.*
(4) X *is called a dg-*$\mathcal{B}$ *complex if* $X_n \in \mathcal{B}$ *for each* $n \in \mathbb{Z}$, *and* $\mathcal{H}\mathrm{om}_R(A,X)$ *is exact whenever* A *is an* $\mathcal{A}$ *complex.*

We denote the class of $\mathcal{A}$ complexes by $\widetilde{\mathcal{A}}$ and the class of dg-$\mathcal{A}$ complexes by $dg\widetilde{\mathcal{A}}$. Similarly, the class of $\mathcal{B}$ complexes is denoted by $\widetilde{\mathcal{B}}$ and the class of dg-$\mathcal{B}$ complexes is denoted by $dg\widetilde{\mathcal{B}}$.

The next 2 lemmas play an important role in proving our main result.

Lemma 2.2 *Let* $(\mathcal{A},\mathcal{B})$ *be a cotorsion pair in* R*-Mod.*
(1) $(\widetilde{\mathcal{A}}, dg\widetilde{\mathcal{B}})$ *and* $(dg\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}})$ *are cotorsion pairs in* $\mathrm{Ch}(R)$ *(see [[18], Proposition 3.6]).*
(2) *If* $(\mathcal{A},\mathcal{B})$ *is hereditary, then* $(\widetilde{\mathcal{A}}, \mathrm{dg}\widetilde{\mathcal{B}})$ *and* $(\mathrm{dg}\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}})$ *are hereditary in* $\mathrm{Ch}(R)$ *(see [[18], Corollary 3.13]).*
(3) *If* $(\mathcal{A},\mathcal{B})$ *is hereditary, then* $\mathrm{dg}\widetilde{\mathcal{A}} \cap \mathcal{E} = \widetilde{\mathcal{A}}$, *and* $\mathrm{dg}\widetilde{\mathcal{B}} \cap \mathcal{E} = \widetilde{\mathcal{B}}$, *where* $\mathcal{E}$ *denotes the class of exact complexes (see [[18], Theorem 3.12]).*
(4) *If* $(\mathcal{A},\mathcal{B})$ *is complete and hereditary, then* $(\widetilde{\mathcal{A}}, \mathrm{dg}\widetilde{\mathcal{B}})$ *and* $(\mathrm{dg}\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}})$ *in* $\mathrm{Ch}(R)$ *are both complete (see [[29], Theorem 3.5]).*

Lemma 2.3 ([18], Lemma 3.1) *For any* R*-module* C *and* R*-complex* X, *we have the following natural isomorphism:*

$$\mathrm{Hom}_R(C, \mathrm{Z}_n(X)) \cong \mathrm{Hom}_{\mathrm{Ch}(R)}(S^n(C), X).$$

Definition 2.4 *Let* $(\mathcal{A},\mathcal{B})$ *be a cotorsion pair in* R*-Mod and* X *an* R*-complex. A morphism* $A \longrightarrow X$ *is called a dg-*$\mathcal{A}$ *resolution of* X *if* $A \longrightarrow X$ *is a quasi-isomorphism and* A *is a dg-*$\mathcal{A}$ *complex. Dually, a morphism* $X \longrightarrow B$ *is called a dg-*$\mathcal{B}$ *resolution of* X *if* $X \longrightarrow B$ *is a quasi-isomorphism and* B *is a dg-*$\mathcal{B}$ *complex.*

Since every dg-projective complex is a dg-$\mathcal{A}$ complex, and every complex has a surjective dg-projective resolution, every complex has a surjective dg-$\mathcal{A}$ resolution. Dually, every complex has an injective dg-$\mathcal{B}$ resolution.

Definition 2.5 *Let* $(\mathcal{A},\mathcal{B})$ *be a cotorsion pair in* R*-Mod and* $n \in \mathbb{Z}$, X *be a complex of left* R*-modules. The* $\mathcal{A}$ *dimension of* X *is defined by* $\mathcal{A}\text{-}\dim_R(X) \leq n$ *if there is a dg-projective resolution* $P \longrightarrow X$ *such that* $\sup \mathrm{H}(P) \leq n$ *and* $\mathrm{C}_n(P) \in \mathcal{A}$. *If* $\mathcal{A}\text{-}\dim_R(X) \leq n$ *but* $\mathcal{A}\text{-}\dim_R(X) \leq n-1$ *does not hold, then* $\mathcal{A}\text{-}\dim_R(X) = n$. *Dually, the* $\mathcal{B}$ *dimension of* X *is defined by* $\mathcal{B}\text{-}\dim_R(X) \leq n$ *if there is a dg-injective resolution* $X \longrightarrow I$ *such that* $\inf \mathrm{H}(I) \geq -n$ *and* $\mathrm{Z}_{-n}(I) \in \mathcal{B}$. *If* $\mathcal{B}\text{-}\dim_R(X) \leq n$ *but* $\mathcal{B}\text{-}\dim_R(X) \leq n-1$ *does not hold, then* $\mathcal{B}\text{-}\dim_R(X) = n$. *If* $\mathcal{A}\text{-}\dim_R(X) \leq n$ *for each* n, *then* $\mathcal{A}\text{-}\dim_R(X) = -\infty$. *If* $\mathcal{A}\text{-}\dim_R(X) \leq n$ *does not hold for any* n, *then* $\mathcal{A}\text{-}\dim_R(X) = \infty$. *Similar statements for the* $\mathcal{B}$ *dimension hold.*

We prove that the $\mathcal{A}$ dimension of X is well defined. The case of $\mathcal{B}$ dimension is dual.

Assume that $\widetilde{P} \longrightarrow X$ is another dg-projective resolution of X. Then $\sup \mathrm{H}(\widetilde{P}) = \sup \mathrm{H}(X) = \sup \mathrm{H}(P) \leq n$. We can assume that $P \longrightarrow X$ is a surjective dg-projective resolution (if not, let $\overline{P} \longrightarrow X$ be surjective with $\overline{P}$ a projective complex, then $\overline{P} \oplus P \longrightarrow X$ is a surjective dg-projective resolution and $\mathrm{C}_n(P) \oplus \mathrm{C}_n(\overline{P}) \in \mathcal{A}$). Then there exists an exact sequence

$$0 \longrightarrow U \longrightarrow P \longrightarrow X \longrightarrow 0$$

with U exact. This yields an exact sequence

$$0 \longrightarrow \mathrm{Hom}(\widetilde{P}, U) \longrightarrow \mathrm{Hom}(\widetilde{P}, P) \longrightarrow \mathrm{Hom}(\widetilde{P}, X) \longrightarrow \mathrm{Ext}^1(\widetilde{P}, U) = 0$$

[[13], Proposition 3.6]. Thus, there is a morphism of complexes $\widetilde{P} \longrightarrow P$ such that the diagram

$$\begin{array}{ccc} & & \widetilde{P} \\ & \swarrow & \downarrow \\ P & \longrightarrow & X \end{array}$$

commutes. Since both $P \longrightarrow X$ and $\widetilde{P} \longrightarrow X$ are quasi-isomorphisms, so is $\widetilde{P} \longrightarrow P$. We can assume that $\widetilde{P} \longrightarrow P$ is a surjective quasi-isomorphism (if not, let $\widehat{P} \longrightarrow P$ be surjective with $\widehat{P}$ a projective complex, then $\widehat{P} \oplus \widetilde{P} \longrightarrow P$ is a surjective quasi-isomorphism). Then there exists an exact sequence

$$0 \longrightarrow U' \longrightarrow \widetilde{P} \longrightarrow P \longrightarrow 0$$

with U' an exact complex. Both $\widetilde{P}$ and P are dg-projective complexes, so U' is a dg-projective complex. Thus U' is exact and dg-projective complex, and so U' is a projective complex. On the other hand, we have an exact sequence

$$0 \longrightarrow \mathrm{C}_n(U') \longrightarrow \mathrm{C}_n(\widetilde{P}) \longrightarrow \mathrm{C}_n(P) \longrightarrow 0$$

with $\mathrm{C}_n(P) \in \mathcal{A}$ and $\mathrm{C}_n(U') \in \mathcal{A}$. It follows that $\mathrm{C}_n(\widetilde{P}) \in \mathcal{A}$.

Remark 2.6 $\mathcal{A}$-$\dim_R(X) = -\infty$ *if and only if* X *is exact. For each* $k \in \mathbb{Z}$,

$$\mathcal{A}\text{-}\dim_R(\Sigma^k X) = \mathcal{A}\text{-}\dim_R(X) + k \textit{ and } \mathcal{A}\text{-}\dim_R(X) \leq \mathrm{pd}_R(X).$$

Dually, $\mathcal{B}$-$\dim_R(X) = -\infty$ *if and only if* X *is exact. For each* $k \in \mathbb{Z}$,

$$\mathcal{B}\text{-}\dim_R(\Sigma^k X) = \mathcal{B}\text{-}\dim_R(X) - k \textit{ and } \mathcal{B}\text{-}\dim_R(X) \leq \mathrm{id}_R(X).$$

The following 2 results give characterizations of the $\mathcal{A}$ and $\mathcal{B}$ dimensions of complexes.

Theorem 2.7 *Let* $(\mathcal{A}, \mathcal{B})$ *be a hereditary cotorsion pair in* R*-Mod and* X *an* R*-complex. Then the following assertions are equivalent:*

(1) $\mathcal{B}$-$\dim_R(X) \leq n$.

(2) $\mathrm{Ext}^i_R(A, X) = 0$ *for any module* $A \in \mathcal{A}$ *and* $i > n$.

(3) $\inf \mathrm{H}(X) \geq -n$ *and* $\mathrm{Z}_{-n}(I) \in \mathcal{B}$ *for any* dg*-injective resolution* $X \longrightarrow I$.

(4) *There exists a* dg-$\mathcal{B}$ *resolution* $X \longrightarrow B$ *such that* $\inf \mathrm{H}(X) \geq -n$ *and* $\mathrm{Z}_{-n}(B) \in \mathcal{B}$.

Moreover, if $(\mathcal{A}, \mathcal{B})$ *is complete, then the above conditions are also equivalent to:*

(5) *For every* dg-$\mathcal{B}$ *resolution* $X \longrightarrow B'$*, we have* $\inf \mathrm{H}(B') \geq -n$ *and* $\mathrm{Z}_{-n}(B') \in \mathcal{B}$.

Proof (1) $\Rightarrow$ (2). Let $X \longrightarrow I$ be a dg-injective resolution, such that $\inf \mathrm{H}(I) \geq -n$ and $\mathrm{Z}_{-n}(I) \in \mathcal{B}$. Then we have

$$\mathrm{Ext}_R^i(A, X) = \mathrm{H}^i\mathrm{Hom}_R(A, I) = \mathrm{H}_{-i}\mathrm{Hom}_R(A, I) = 0$$

for any module $A \in \mathcal{A}$ and $i > n$.

(2) $\Rightarrow$ (3). Since every projective module is in $\mathcal{A}$, we have $\mathrm{H}_i(X) = 0$ for any $i < -n$ by choosing A to be R especially. It remains to prove that the R-module $\mathrm{Z}_{-n}(B)$ is in $\mathcal{B}$. It is sufficient to prove that $\mathrm{Ext}_R^1(A, \mathrm{Z}_{-n}(B)) = 0$ for any module $A \in \mathcal{A}$. This follows from

$$\mathrm{Ext}_R^1(A, \mathrm{Z}_{-n}(B)) = \mathrm{H}_{-1}\mathrm{Hom}_R(A, \Sigma^n I_{\sqsubset -n}) = \mathrm{H}_{-(n+1)}\mathrm{Hom}_R(A, I) = \mathrm{Ext}_R^{n+1}(A, X) = 0.$$

(3) $\Rightarrow$ (4). Since every dg-injective resolution is a dg-$\mathcal{B}$ resolution, it holds by definition.

(4) $\Rightarrow$ (1). If $X \longrightarrow I$ is a dg-injective resolution, then $\inf \mathrm{H}(I) = \inf \mathrm{H}(X) = \inf \mathrm{H}(B) \geq -n$. We can assume without loss of generality that $X \longrightarrow B$ is an injective dg-$\mathcal{B}$ resolution. Then there exists an exact sequence

$$0 \longrightarrow X \longrightarrow B \longrightarrow L' \longrightarrow 0$$

with L' exact. This yields an exact sequence

$$0 \longrightarrow \mathrm{Hom}(L', I) \longrightarrow \mathrm{Hom}(B, I) \longrightarrow \mathrm{Hom}(X, I) \longrightarrow \mathrm{Ext}^1(L', I) = 0.$$

Therefore, there is a morphism of complexes $B \longrightarrow I$ such that the diagram

$$\begin{array}{ccc} X & \longrightarrow & B \\ \downarrow & \swarrow & \\ I & & \end{array}$$

commutes. Since both $X \longrightarrow B$ and $X \longrightarrow I$ are quasi-isomorphisms, so is $B \longrightarrow I$. We can assume that $B \longrightarrow I$ is an injective quasi-isomorphism (if not, let $B \longrightarrow \overline{I}$ be injective with $\overline{I}$ an injective complex, then $B \longrightarrow I \oplus \overline{I}$ is an injective quasi-isomorphism). Then there exists an exact sequence $0 \longrightarrow B \longrightarrow I \longrightarrow Q \longrightarrow 0$ with Q an exact complex. Both B and I are dg-$\mathcal{B}$ complexes, so Q is a dg-$\mathcal{B}$ complex. Since Q is exact and a dg-$\mathcal{B}$ complex, Q is also a $\mathcal{B}$ complex. On the other hand, we have an exact sequence $0 \longrightarrow \mathrm{Z}_{-n}(B) \longrightarrow \mathrm{Z}_{-n}(I) \longrightarrow \mathrm{Z}_{-n}(Q) \longrightarrow 0$ with $\mathrm{Z}_{-n}(Q) \in \mathcal{B}$ and $\mathrm{Z}_{-n}(B) \in \mathcal{B}$. It follows that $\mathrm{Z}_{-n}(I) \in \mathcal{B}$.

(4) $\Rightarrow$ (5). If $X \longrightarrow B'$ is a dg-$\mathcal{B}$ resolution, then $\inf \mathrm{H}(B') = \inf \mathrm{H}(X) = \inf \mathrm{H}(B) \geq -n$. We can assume that $X \longrightarrow B$ is a special dg-$\mathcal{B}$ preenvelope. Then there exists an exact sequence

$$0 \longrightarrow X \longrightarrow B \longrightarrow L \longrightarrow 0$$

with $L \in \widetilde{\mathcal{A}}$. This yields an exact sequence

$$0 \longrightarrow \mathrm{Hom}(L, B') \longrightarrow \mathrm{Hom}(B, B') \longrightarrow \mathrm{Hom}(X, B') \longrightarrow \mathrm{Ext}^1(L, B') = 0$$

by Lemma 2.2. Therefore, there is a morphism of complexes $B \longrightarrow B'$ such that the diagram

$$\begin{array}{ccc} X & \longrightarrow & B \\ \downarrow & \swarrow & \\ B' & & \end{array}$$

commutes. Since both $X \longrightarrow B$ and $X \longrightarrow B'$ are quasi-isomorphisms, so is $B \longrightarrow B'$. We can assume that $B \longrightarrow B'$ is an injective quasi-isomorphism (if not, let $B \longrightarrow \overline{I}$ be injective with $\overline{I}$ an injective complex, then $B \longrightarrow B' \oplus \overline{I}$ is an injective quasi-isomorphism). Then there exists an exact sequence

$$0 \longrightarrow B \longrightarrow B' \longrightarrow W \longrightarrow 0$$

with W an exact complex. Both B and B' are dg-$\mathcal{B}$ complexes, so W is a dg-$\mathcal{B}$ complex. Thus W is exact and dg-$\mathcal{B}$ complex, and so W is a $\mathcal{B}$ complex. On the other hand, we have an exact sequence

$$0 \longrightarrow \mathrm{Z}_{-n}(B) \longrightarrow \mathrm{Z}_{-n}(B') \longrightarrow \mathrm{Z}_{-n}(W) \longrightarrow 0$$

with $\mathrm{Z}_{-n}(W) \in \mathcal{B}$ and $\mathrm{Z}_{-n}(B) \in \mathcal{B}$. It follows that $\mathrm{Z}_{-n}(B') \in \mathcal{B}$.

$(5) \Rightarrow (1)$. Obviously. □

The following result is the dual version of Theorem 2.7.

Theorem 2.8 *Let $(\mathcal{A}, \mathcal{B})$ be a hereditary cotorsion pair in R-Mod and Y an R-complex. Then the following assertions are equivalent:*

(1) $\mathcal{A}\text{-}\dim_R(Y) \leq n$.

(2) $\mathrm{Ext}^i_R(Y, B) = 0$ *for any module* $B \in \mathcal{B}$ *and* $i > n$.

(3) $\sup \mathrm{H}(Y) \leq n$ *and* $\mathrm{C}_n(P) \in \mathcal{A}$ *for any dg-projective resolution* $P \longrightarrow Y$.

(4) *There exists a dg-$\mathcal{A}$ resolution $A \longrightarrow Y$ such that* $\sup \mathrm{H}(Y) \leq n$ *and* $\mathrm{C}_n(A) \in \mathcal{A}$.

Moreover, if $(\mathcal{A}, \mathcal{B})$ is complete, then the above conditions are also equivalent to:

(5) *For every dg-$\mathcal{A}$ resolution $A' \longrightarrow Y$, we have* $\sup \mathrm{H}(Y) \leq n$ *and* $\mathrm{C}_n(A') \in \mathcal{A}$.

The following theorem will prove that $\mathcal{B}\text{-}\dim_R(X)$ can be expressed in the form $\inf\{\sup\{l \in \mathbb{Z} \mid B_{-l} \neq 0\} \mid X \simeq B \in dg\widetilde{\mathcal{B}}\}$ (where $\simeq$ is the equivalence relation defined by quasi-isomorphisms).

Theorem 2.9 *Let $(\mathcal{A}, \mathcal{B})$ be a hereditary cotorsion pair in R-Mod, X and Y are R-complexes. Then*

$$\mathcal{B}\text{-}\dim_R(X) = \inf\{\sup\{l \in \mathbb{Z} \mid V_{-l} \neq 0\} \mid X \simeq V \in dg\widetilde{\mathcal{B}}\},$$

and

$$\mathcal{A}\text{-}\dim_R(Y) = \inf\{\sup\{l \in \mathbb{Z} \mid T_l \neq 0\} \mid Y \simeq T \in dg\widetilde{\mathcal{A}}\}.$$

Proof Let $X \longrightarrow I$ be a dg-injective resolution of X. Set

$$\Lambda = \inf\{\sup\{l \in \mathbb{Z} \mid V_{-l} \neq 0\} \mid X \simeq V \in dg\widetilde{\mathcal{B}}\}.$$

If $\mathcal{B}\text{-}\dim_R(X) = n < \infty$, then $\mathrm{Z}_{-n}(I) \in \mathcal{B}$ and $\inf \mathrm{H}(X) \geq -n$ by Definition 2.5. There thus exists an injective quasi-isomorphism $I_{\supset -n} \to I$. We have an exact sequence of complexes

$$0 \longrightarrow I_{\supset -n} \longrightarrow I \longrightarrow I/I_{\supset -n} \longrightarrow 0.$$

Choosing the n-th degree of the above exact sequence, there is an exact sequence of R-modules

$$0 \longrightarrow \mathrm{Z}_{-n}(I) \longrightarrow I_{-n} \longrightarrow I_{-n}/\mathrm{Z}_{-n}(I) \longrightarrow 0.$$

Since $\mathrm{Z}_{-n}(I)$ and I_{-n} are in $\mathcal{B}$ and $(\mathcal{A},\mathcal{B})$ is hereditary, $I_{-n}/\mathrm{Z}_{-n}(I) \in \mathcal{B}$. Therefore, $I/I_{\supset -n}$ is a bounded above complex with all entries in $\mathcal{B}$. By [[18], Lemma 3.4], $I/I_{\supset -n} \in \widetilde{\mathcal{B}}$.

Because all of the modules $I_{\supset -n}$ are in $\mathcal{B}$, the sequence of complexes of $\mathbb{Z}$-modules

$$0 \longrightarrow \mathcal{H}\mathrm{om}_R(A, I_{\supset -n}) \longrightarrow \mathcal{H}\mathrm{om}_R(A, I) \longrightarrow \mathcal{H}\mathrm{om}_R(A, I/I_{\supset -n}) \longrightarrow 0$$

is exact for any complex $A \in \mathcal{A}$. Since $\mathcal{H}\mathrm{om}_R(A, I)$ and $\mathcal{H}\mathrm{om}_R(A, I/I_{\supset -n})$ are acyclic, it yields that $\mathcal{H}\mathrm{om}_R(A, I_{\supset -n})$ is also acyclic. Then it follows from Definition 2.1 that $I_{\supset -n} \in dg\widetilde{\mathcal{B}}$.

Since $X \longrightarrow I$ is a quasi-isomorphism, $X_{\supset -n} \longrightarrow I_{\supset -n}$ is a quasi-isomorphism. But $X \simeq X_{\supset -n}$, and we get $X \simeq I_{\supset -n}$. $I_{\supset -n} \in dg\widetilde{\mathcal{B}}$, which implies that $\Lambda \leq n$.

Now suppose that $\Lambda = n < \infty$. We will show that $\mathcal{B}\text{-}\dim_R(X) \leq n$. By the hypothesis, there exists a complex

$$V = \cdots \longrightarrow V_0 \longrightarrow V_{-1} \longrightarrow \cdots \longrightarrow V_{-n+1} \longrightarrow V_{-n} \longrightarrow 0$$

such that $V \in dg\widetilde{\mathcal{B}}$ and $X \simeq V$. Since $V \simeq X \simeq I$, and I is a dg-injective complex, there is a quasi-isomorphism $V \longrightarrow I$. In addition, there is an injective morphism $V \longrightarrow I^*$ with I^* injective. Then $V \longrightarrow I \oplus I^*$ is an injective quasi-isomorphism. Thus we have an exact sequence

$$0 \longrightarrow V \longrightarrow I \oplus I^* \longrightarrow W \longrightarrow 0$$

with W exact. Since V and $I \oplus I^*$ are dg-$\mathcal{B}$ complexes and $(\mathcal{A},\mathcal{B})$ is hereditary, W is a dg-$\mathcal{B}$ complex. Thus W is a $\mathcal{B}$ complex, and so $\mathrm{Z}_j(W) \in \mathcal{B}$. In the exact sequence

$$0 \longrightarrow \mathrm{Z}_{-n}(V) \longrightarrow \mathrm{Z}_{-n}(I) \oplus \mathrm{Z}_{-n}(I^*) \longrightarrow \mathrm{Z}_{-n}(W) \longrightarrow 0,$$

$\mathrm{Z}_{-n}(V) = V_{-n} \in \mathcal{B}$ and $\mathrm{Z}_{-n}(W) \in \mathcal{B}$, so $\mathrm{Z}_{-n}(I) \oplus \mathrm{Z}_{-n}(I^*) \in \mathcal{B}$, which implies that $\mathrm{Z}_{-n}(I) \in \mathcal{B}$. Since $X \longrightarrow I$ is a dg-injective resolution with $\inf \mathrm{H}(X) \geq -n$ and $\mathrm{Z}_{-n}(I) \in \mathcal{B}$, it follows that $\mathcal{B}\text{-}\dim_R(X) \leq n$.

By the above, $\mathcal{B}\text{-}\dim_R(X) = \infty$ if and only if $\Lambda = \infty$. In addition, $\mathcal{B}\text{-}\dim_R(X) = -\infty$ if and only if X is exact if and only if $\Lambda = -\infty$.

The proof of $\mathcal{A}\text{-}\dim_R(Y)$ is dual. $\square$

Lemma 2.10 ([27], Proposition 1.3.8) (Horseshoe Lemma) *For every exact sequence of complexes* $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$*, there exists a commutative diagram with exact rows*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I^X & \longrightarrow & I^Y & \longrightarrow & I^Z & \longrightarrow & 0
\end{array}$$

in which the columns are injective dg-injective resolution.

Proposition 2.11 *Let* $(\mathcal{A},\mathcal{B})$ *be a hereditary cotorsion pair in* R*-Mod and* $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ *be an exact sequence of complexes of* R*-modules. If 2 complexes of* X, Y, Z *have finite* $\mathcal{B}$ *dimension, then so does the third.*

Proof By Lemma 2.10, there is an exact sequence of complexes $0 \longrightarrow I^X \longrightarrow I^Y \longrightarrow I^Z \longrightarrow 0$ with $X \longrightarrow I^X$, $Y \longrightarrow I^Y$ and $Z \longrightarrow I^Z$ dg-injective resolutions. If 2 of the complexes X, Y, Z have finite $\mathcal{B}$ dimension, then there is $n \in \mathbb{Z}$ such that $\mathrm{H}_j(I^X) = \mathrm{H}_j(I^Y) = \mathrm{H}_j(I^Z) = 0$ for all $j \leq -n$. For each $j \leq -n$, we have an exact sequence

$$0 \longrightarrow \mathrm{Z}_j(I^X) \longrightarrow \mathrm{Z}_j(I^Y) \longrightarrow \mathrm{Z}_j(I^Z) \longrightarrow 0$$

in R-Mod. If $\mathrm{Z}_j(I^X) \in \mathcal{B}$, then $\mathrm{Z}_j(I^Y) \in \mathcal{B}$ if and only if $\mathrm{Z}_j(I^Z) \in \mathcal{B}$. If both $\mathrm{Z}_j(I^Y) \in \mathcal{B}$ and $\mathrm{Z}_j(I^Z) \in \mathcal{B}$, then $\mathcal{B}$-dim$(\mathrm{Z}_j(I^X)) \leq 1$, and so $\mathrm{Z}_{j-1}(I^X) \in \mathcal{B}$. □

Dually we have:

Proposition 2.12 *Let* $(\mathcal{A},\mathcal{B})$ *be a hereditary cotorsion pair in* R*-Mod,* $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ *be an exact sequence of complexes of* R*-modules. If 2 complexes of* X, Y, Z *have finite* $\mathcal{A}$ *dimension, then so does the third.*

Let $(\mathcal{A},\mathcal{B})$ be a cotorsion pair in R-Mod. Given an integer $n \geq 0$ and an R-module M, we say $\mathcal{B}$-dim$(M) \leq n$, if there exists an exact sequence

$$0 \to M \to B_0 \to B_{-1} \to \cdots \to B_{-n+1} \to B_{-n} \to 0$$

with each module $B_{-i} \in \mathcal{B} (0 \leq i \leq n)$. Then the deleted complex

$$B = 0 \to B_0 \to B_{-1} \to \cdots \to B_{-n+1} \to B_{-n} \to 0$$

is a dg-$\mathcal{B}$ complex and there is a quasi-isomorphism $S^0(M) \simeq B$. If no integer $n \geq 0$ exists with $\mathcal{B}$-dim$(M) \leq n$, then we set $\mathcal{B}$-dim$(M) = \infty$. Dual statements for the $\mathcal{A}$ dimension of module hold, as well.

The following results show that the $\mathcal{B}$ $(\mathcal{A})$ dimension of complexes is a generalization for the $\mathcal{B}$ $(\mathcal{A})$ dimension of modules.

Proposition 2.13 *Let* $(\mathcal{A},\mathcal{B})$ *be a hereditary cotorsion pair and* M *an* R*-module. Then* $\mathcal{B}$-dim$(M) =$ $\mathcal{B}$-dim$_R(S^0(M))$.

Proof Let

$$0 \longrightarrow M \longrightarrow B_0 \longrightarrow B_{-1} \longrightarrow \cdots$$

be an injective resolution of M. Then $S^0(M) \longrightarrow B$ is a dg-injective resolution, where

$$B = \cdots \longrightarrow 0 \longrightarrow B_0 \longrightarrow B_{-1} \longrightarrow \cdots .$$

If $\mathcal{B}$-dim$(M) = \infty$ and $\mathcal{B}$-dim$_R(S^0(M)) = n < \infty$, then $\mathrm{Z}_j(B) \in \mathcal{B}$ for every $j \leq -n$ by Theorem 2.7. Since

$$0 \longrightarrow M \longrightarrow B_0 \longrightarrow \cdots \longrightarrow B_{-n+1} \longrightarrow \mathrm{Z}_{-n}(B) \longrightarrow 0$$

is exact, where $\mathrm{Z}_{-n}(B) \in \mathcal{B}$ and $B_j \in \mathcal{B}$, it follows that $\mathcal{B}$-dim$(M) \leq n$. This contradicts the fact that $\mathcal{B}$-dim$(M) = \infty$. So $\mathcal{B}$-dim$_R(S^0(M)) = \infty$.

If $\mathcal{B}$-dim$(M) = n < \infty$, then $\mathrm{Z}_{-n}(B) \in \mathcal{B}$, and so $\mathrm{Z}_j(B) \in \mathcal{B}$ for every $j \leq -n$. Thus $S^0(M) \longrightarrow B$ is a *dg*-$\mathcal{B}$ resolution with $\mathrm{Z}_j(B) \in \mathcal{B}$ for all $j \leq -n$ and $\mathrm{H}_j(B) = 0$ for every $j \leq -n-1$. By Theorem 2.7, we get $\mathcal{B}$-$\dim_R(S^0(M)) \leq n$. Suppose that $\mathcal{B}$-$\dim_R(S^0(M)) \leq n-1$; then $\mathcal{B}$-dim$(M) \leq n-1$. This contradicts the fact that $\mathcal{B}$-dim$(M) = n$. Therefore, $\mathcal{B}$-$\dim_R(S^0(M)) = n$. □

Proposition 2.14 *Let* $(\mathcal{A}, \mathcal{B})$ *be a hereditary cotorsion pair and* N *an* R*-module. Then* $\mathcal{A}$-dim$(N) = \mathcal{A}$-$\dim_R(S^0(N))$.

It is easily seen that:

Corollary 2.15 *Let* $(\mathcal{A}, \mathcal{B})$ *be a hereditary cotorsion pair in the category of* R*-modules. For an* R*-module* M *and a nonnegative integer* n*, the following are equivalent:*
(1) $\mathcal{B}$-dim$(M) \leq n$.
(2) $\mathrm{Ext}_R^{n+1}(A, M) = 0$ *for any module* $A \in \mathcal{A}$.
(3) $\mathrm{Ext}_R^{n+j}(A, M) = 0$ *for any module* $A \in \mathcal{A}$ *and* $j > 1$.
(4) *If the sequence* $0 \longrightarrow M \longrightarrow B_0 \longrightarrow B_{-1} \longrightarrow \cdots \longrightarrow B_{-n+1} \longrightarrow B_{-n} \longrightarrow 0$ *is exact with* $B_0, B_{-1}, \ldots, B_{-n+1} \in \mathcal{B}$*, then* B_{-n} *is also in* $\mathcal{B}$.

Corollary 2.16 *Let* $(\mathcal{A}, \mathcal{B})$ *be a hereditary cotorsion pair in the category of* R*-modules. For an* R*-module* N *and a nonnegative integer* n*, the following are equivalent:*
(1) $\mathcal{A}$-dim$(N) \leq n$.
(2) $\mathrm{Ext}_R^{n+1}(N, B) = 0$ *for any module* $B \in \mathcal{B}$.
(3) $\mathrm{Ext}_R^{n+j}(N, B) = 0$ *for any module* $B \in \mathcal{B}$ *and* $j > 1$.
(4) *If the sequence* $0 \longrightarrow A_n \longrightarrow A_{n-1} \longrightarrow \cdots \longrightarrow A_1 \longrightarrow A_0 \longrightarrow N \longrightarrow 0$ *is exact with* $A_0, A_1, \cdots, A_{n-1} \in \mathcal{A}$*, then* A_n *is also in* $\mathcal{A}$.

The global dimension of R plays an important role in describing its homological properties. In the following, we will define relative homological dimensions of R and discuss the relations between them and the dimensions of complexes.

Definition 2.17 *Let* R *be a ring and* $(\mathcal{A}, \mathcal{B})$ *be a cotorsion pair in* R*-Mod. We define the* $\mathcal{A}$ *dimension and* $\mathcal{B}$ *dimension of* R*, denoted by* $\mathcal{A}$-dim(R) *and* $\mathcal{B}$-dim(R)*, respectively, as follows:*

$$\mathcal{A}\text{-dim}(R) = \sup\{\mathcal{A}\text{-dim}(M) \mid \textit{for any } R\textit{-module } M\}.$$

$$\mathcal{B}\text{-dim}(R) = \sup\{\mathcal{B}\text{-dim}(N) \mid \textit{for any } R\textit{-module } N\}.$$

Theorem 2.18 *Let* $(\mathcal{A}, \mathcal{B})$ *be a hereditary cotorsion pair in* R*-Mod. Then the following are equivalent for a nonnegative integer* n.

(1) $\mathcal{A}$-dim$(R) \leq n$.

(2) $\mathcal{A}$-dim$(Y) \leq \sup \mathrm{H}(Y) + n$ *for every complex of* R*-modules* Y.

(3) $\mathrm{Ext}_R^i(Y, B) = 0$ *for every complex of* R*-modules* Y*, any module* $B \in \mathcal{B}$*, and* $i > n + \sup \mathrm{H}(Y)$.

Proof (1) $\Rightarrow$ (2). True if $\sup \mathrm{H}(Y) = \infty$.

Assume $\sup \mathrm{H}(Y) = l < \infty$. Let $A \to Y$ be a dg-projective resolution. Then $\mathrm{H}_j(A) = 0$ for any $j > l$. So we have an exact complex $\cdots \to A_{l+1} \to A_l \to \mathrm{C}_l(A) \to 0$. Since $\mathcal{A}\text{-dim}(R) \leq n$, $\mathcal{A}\text{-dim}(\mathrm{C}_l(A)) \leq n$. Thus $\mathrm{C}_j(A) \in \mathcal{A}$ for any $j \geq n + l$. Therefore, $\mathcal{A}\text{-dim}(Y) \leq l + n$ by Theorem 2.8.

(2) $\Rightarrow$ (1). If Y is a left R-module then by Proposition 2.14, $\mathcal{A}\text{-dim}(Y) \leq n$. The conclusion is obvious.

(2) $\Leftrightarrow$ (3). By Theorem 2.8 it is clear. □

The case of $\mathcal{B}$-dim is as follows.

Theorem 2.19 *Let $(\mathcal{A}, \mathcal{B})$ be a hereditary cotorsion pair in R-Mod. Then the following are equivalent for a nonnegative integer m.*

(1) $\mathcal{B}\text{-dim}(R) \leq m$.

(2) $\mathcal{B}\text{-dim}(X) \leq -\inf \mathrm{H}(X) + m$ *for every complex of R-modules X.*

(3) $\mathrm{Ext}_R^i(A, X) = 0$ *for every complex of R-modules X, any module $A \in \mathcal{A}$, and $i > m - \inf \mathrm{H}(X)$.*

Theorem 2.20 *Let $(\mathcal{A}, \mathcal{B})$ be a hereditary cotorsion pair in R-Mod. Then*

$$\mathrm{gD}(R) \leq \mathcal{A}\text{-dim}(R) + \mathcal{B}\text{-dim}(R).$$

Proof If $\mathcal{A}\text{-dim}(R) = \infty$ or $\mathcal{B}\text{-dim}(R) = \infty$, the conclusion is obvious. So we assume that $\mathcal{A}\text{-dim}(R) = n$ and $\mathcal{B}\text{-dim}(R) = m$ are finite. For any complex Y, it follows from Theorem 2.18 that $\mathcal{A}\text{-dim}(Y) \leq \sup \mathrm{H}(Y) + n$. Then by Theorem 2.8, we have $\mathrm{Ext}_R^i(Y, B) = 0$ for any module $B \in \mathcal{B}$ and $\mathrm{H}_i(Y) = 0$ for $i > \sup \mathrm{H}(Y) + n$. For any R-module M, there is an exact sequence $0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_{-1} \longrightarrow \cdots \longrightarrow I_{-m+1} \longrightarrow J \longrightarrow 0$ with $I_0, I_{-1}, \cdots, I_{-m+1}$ injective. Since $\mathcal{B}\text{-dim}(R) = m$ and every injective module is in $\mathcal{B}$, we have $J \in \mathcal{B}$ by Corollary 2.15. Set $j = m + n + \sup \mathrm{H}(Y) + 1$. Applying the dimension shifting, one gets $\mathrm{Ext}_R^j(Y, M) \cong \mathrm{Ext}_R^{j-m}(Y, J)$. Since $\mathrm{Ext}_R^{j-m}(Y, J) = 0$ from the above argument, then $\mathrm{Ext}_R^j(Y, M)$ vanishes. This implies, by [[2], Theorem 2.4.P.], that $\mathrm{pd}_R(Y) \leq m + n + \sup \mathrm{H}(Y)$. Hence, it follows from [[2], Proposition 3.1] that $\mathrm{gD}(R) \leq m + n$. □

3. Applications

In this section, we give some applications of our main results.

3.1. Projective, injective, flat, and cotorsion dimensions of R-complexes

We use $\mathcal{P}$, $\mathcal{I}$, $\mathcal{M}$ to denote the classes of projective and injective R-modules and all R-modules, respectively. It is trivial that $(\mathcal{P}, \mathcal{M})$ and $(\mathcal{M}, \mathcal{I})$ are complete and hereditary cotorsion pairs. A complex of R-modules P is called π-projective (π-injective) if $\mathrm{Hom}_R(P, -)$ ($\mathrm{Hom}_R(-, I)$) preserves homology isomorphisms. Let $n \in \mathbb{Z}$. A complex of R-modules M is said to have π-projective dimension of at most n (denoted $\pi\text{-pd}_R(M) \leq n$), if there exists an equivalence $P \simeq M$, with P a π-projective complex of R-modules with $P_i = 0$ for $i > n$. If $\pi\text{-pd}_R(M) \leq n$ holds, but $\pi\text{-pd}_R(M) \leq n - 1$ does not, we write $\pi\text{-pd}_R(M) = n$; if $\pi\text{-pd}_R(M) \leq n$ for all $n \in \mathbb{Z}$, we write $\pi\text{-pd}_R(M) = -\infty$; and if $\pi\text{-pd}_R(M) \leq n$ for no $n \in \mathbb{Z}$ we write $\pi\text{-pd}_R(M) = \infty$ [2]. Then we have the characterizations of projective and injective dimensions of complexes as follows.

Corollary 3.1 ([2], Theorem 2.4.P.) *For a complex M of R-modules, the following conditions are equivalent:*
(i) $\mathrm{pd}_R(M) \le n$.
(i)′ *M has a dg-projective resolution P with $P_i = 0$ for $i > n$.*
(ii) $\pi\text{-}\mathrm{pd}_R(M) \le n$.
(ii)′ *M has a π-projective resolution P with $P_i = 0$ for $i > n$.*
(iii) $\mathrm{Ext}^i_R(M,N) = 0$ *for $i > n - \inf N$ and any complex of R-modules N.*
(iv) $\mathrm{Ext}^{n+1}_R(M,N) = 0$ *for any R-module N and $\mathrm{H}_i(M) = 0$ for $i > n+1$.*
(v) $\mathrm{H}_i(M) = 0$ *for $i > n$ and for any (respectively, some) dg-projective complex of R-modules P, such that $P \simeq M$, and the R-module $\mathrm{Coker}(\delta^P_{n+1} : P_{n+1} \longrightarrow P_n)$ is projective.*
(vi) *For P as in (v) the truncation $P \longrightarrow_{\subset n} P$ is a homology isomorphism, and ${}_{\subset n}P$ is dg-projective.*
(vi)′ *For P as in (v) , there is an isomorphism $P \cong P' \oplus P''$, with $P'_i = 0$ for $i > n$, and P'' contractible.*

Corollary 3.2 ([2], Theorem 2.4.I.) *] For a complex N of R-modules the following conditions are equivalent:*
(i) $\mathrm{id}_R(N) \le n$.
(i)′ *N has a dg-injective resolution I with $I_i = 0$ for $i < -n$.*
(ii) $\pi\text{-}\mathrm{id}_R(N) \le n$.
(ii)′ *N has a π-injective resolution I with $I_i = 0$ for $i < -n$.*
(iii) $\mathrm{Ext}^i_R(M,N) = 0$ *for $i > n + \sup M$ and any complex of R-modules M.*
(iv) $\mathrm{Ext}^{n+1}_R(R/J,N) = 0$ *for any left ideal J of R and $\mathrm{H}_i(N) = 0$ for $i < -n-1$.*
(v) $\mathrm{H}_i(N) = 0$ *for $i < -n$ and for any (respectively, some) dg-injective complex of R-modules I, such that $I \simeq N$, and the R-module $\mathrm{Ker}(\delta^I_{-n} : I_{-n} \longrightarrow I_{-n-1})$ is injective.*
(vi) *For I as in (v) the truncation $I_{\supset -n} \longrightarrow I$ is a homology isomorphism, and $I_{\supset -n}$ is dg-injective.*
(vi)′ *For I as in (v) , there is an isomorphism $I \cong I' \oplus I''$, with $I'_i = 0$ for $i < -n$, and I'' contractible.*

We recall from [28] that an R-module K is called cotorsion if $\mathrm{Ext}^1_R(F,K) = 0$ for all flat R-modules F. We use $\mathcal{F}$, $\mathcal{C}$ to denote the classes of flat and cotorsion R-modules, respectively. Since $(\mathcal{F},\mathcal{C})$ is a complete and hereditary cotorsion pair, we have the characterizations of cotorsion dimension of complex K, denoted by $\mathcal{C}\text{-dim}(K)$.

Proposition 3.3 *Let R be a ring and K be an R-complex. Then the following are equivalent:*
(1) $\mathcal{C}\text{-dim}_R(K) \le n$.
(2) *There exists a quasi-isomorphism $K \simeq K'$, where $K' \in dg\widetilde{\mathcal{C}}$ with $K'_i = 0$ for $i < -n$.*
(3) $\mathrm{Ext}^i_R(F,K) = 0$ *for any module $F \in \mathcal{F}$ and $i > n$.*
(4) $\inf \mathrm{H}(K) \ge -n$ *and $\mathrm{Z}_{-n}(K')$ is a cotorsion module for each dg-cotorsion resolution $K \longrightarrow K'$.*
(5) $\inf \mathrm{H}(K) \ge -n$ *and $\mathrm{Z}_{-n}(I)$ is a cotorsion module for each dg-injective resolution $K \longrightarrow I$.*
(5)′ $\inf \mathrm{H}(K) \ge -n$ *and $\mathrm{Z}_j(I)$ is a cotorsion module for every $j \le -n$, for each dg-injective resolution $K \longrightarrow I$.*
(6) *There exists a dg-injective resolution $K \longrightarrow I'$ such that $\mathrm{H}_j(I') = 0$ for every $j \le -n-1$, and $\mathrm{Z}_{-n}(I')$ is a cotorsion module.*

(6)′ *There exists a dg-injective resolution* $K \longrightarrow I'$ *such that* $\mathrm{H}_j(I') = 0$ *for every* $j \leq -n-1$, *and* $\mathrm{Z}_j(I')$ *is a cotorsion module for every* $j \leq -n$.

Proof By Theorem 2.7 and Theorem 2.9 the conclusion is obvious. □

We mention that the notion of dg-cotorsion complex used in Proposition 3.3 is not the same as García Rozas' dg-cotorsion complex. In [[17], Definition 4.3.1], a complex C is called dg-cotorsion if it is exact and $\mathrm{Z}_n(C)$ is cotorsion in R-Mod for all $n \in \mathbb{Z}$. With the definitions that we are using in this paper, such a complex is a $\mathcal{C}$-complex where $\mathcal{C}$ is the class of cotorsion modules.

Take K to be an R-module, and the cotorsion dimension of R-module K is denoted by $\mathrm{cd}(K)$; then we have the following:

Corollary 3.4 ([22], Proposition 7.2.1) *For a left* R*-module* K *and a nonnegative integer* n, *the following are equivalent:*

(1) $\mathrm{cd}(K) \leq n$.

(2) $\mathrm{Ext}_R^{n+1}(F, K) = 0$ *for any flat left* R*-module* F.

(3) $\mathrm{Ext}_R^{n+j}(F, K) = 0$ *for any flat left* R*-module* F *and* $j > 1$.

(4) *If the sequence* $0 \longrightarrow M \longrightarrow K_0 \longrightarrow K_{-1} \longrightarrow \cdots \longrightarrow K_{-n+1} \longrightarrow K_{-n} \longrightarrow 0$ *is exact with* $K_0, K_{-1}, \cdots, K_{-n+1}$ *cotorsion, then* K_{-n} *is also cotorsion.*

(5) $\mathrm{cd}(F(K)) \leq n$, *where* $F(K)$ *denotes the flat cover of* M.

We omit the characterization for flat dimension of complex, which can be obtained by specifying Theorem 2.8 and Theorem 2.9 to $(\mathcal{F}, \mathcal{C})$.

3.2. Gorenstein flat and Gorenstein cotorsion dimensions of R-complexes

Bennis [3] proved that if the Gorenstein flat dimension of M is finite ($\mathrm{Gfd}_R(M) < \infty$), then $\mathrm{Gfd}_R(M) = \sup\{i \in \mathbb{N} \mid \mathrm{Tor}_i^R(E, M) \neq 0$ for some E with $\mathrm{id}_R(E) < \infty\} = \sup\{i \in \mathbb{N} \mid \mathrm{Tor}_i^R(I, M) \neq 0$ for some injective right R -module $I\}$ over a new class of rings, which he called left GF-closed. These are the rings for which the class of Gorenstein flat left R-modules is closed under extensions. The class of left GF-closed rings includes strictly that of right coherent rings and that of rings of finite weak dimension.

An R-module G is called Gorenstein flat if there exists an exact sequence of flat R-modules $\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F_{-1} \longrightarrow F_{-2} \longrightarrow \cdots$ with $G = \mathrm{Ker}(F_0 \longrightarrow F_{-1})$ and that remains exact after applying $I \otimes_R -$ for any injective right R-module I [12]. Recall from [16] that an R-module C is called Gorenstein cotorsion if $\mathrm{Ext}_R^1(G, C) = 0$ for all Gorenstein flat R-modules G. The class of Gorenstein flat (cotorsion) R-modules is denoted by $\mathcal{GF}$ ($\mathcal{GC}$). By [[30], Theorem 3.4], $(\mathcal{GF}, \mathcal{GC})$ is a complete hereditary cotorsion pair over a GF-closed ring. Using Theorem 2.8 and Theorem 2.9 to $(\mathcal{GF}, \mathcal{GC})$, we can extend [[20], Theorem 1] as follows.

Proposition 3.5 *Let* R *be a left GF-closed ring and* G *be an* R*-complex. Then the following are equivalent:*

(1) $\mathcal{GF}\text{-}\mathrm{dim}_R(G) \leq n$.

(2) *There exists a quasi-isomorphism* $G' \simeq G$, *where* $G' \in dg\widetilde{\mathcal{GF}}$ *with* $G'_i = 0$ *for* $i > n$.

(3) $\mathrm{Ext}_R^i(G, C) = 0$ *for any module* $C \in \mathcal{GC}$ *and* $i > n$.

(4) $\sup \mathrm{H}(G) \leq n$ *and* $\mathrm{C}_n(G')$ *is a Gorenstein flat module for each dg-Gorenstein flat resolution* $G' \longrightarrow G$.

(5) $\sup \mathrm{H}(G) \leq n$ *and* $\mathrm{C}_n(F)$ *is a Gorenstein flat module for each dg-flat resolution* $F \longrightarrow G$.

(6) $\sup \mathrm{H}(G) \leq n$ *and* $\mathrm{C}_n(P)$ *is a Gorenstein flat module for each dg-projective resolution* $P \longrightarrow G$.
(6)′ $\sup \mathrm{H}(G) \leq n$ *and* $\mathrm{C}_j(P)$ *is a Gorenstein flat module for every* $j \geq n$, *for each dg-projective resolution* $P \longrightarrow G$.
(7) *There exists a dg-projective resolution* $P' \longrightarrow G$ *such that* $\mathrm{H}_j(P') = 0$ *for every* $j \geq n+1$, *and* $\mathrm{C}_n(P')$ *is a Gorenstein flat module.*
(7)′ *There exists a dg-projective resolution* $P' \longrightarrow G$ *such that* $\mathrm{H}_j(P') = 0$ *for every* $j \geq n+1$, *and* $\mathrm{C}_j(P')$ *is a Gorenstein flat module for every* $j > n$.

Proof (1) ⇔ (2). By Theorem 2.9.
(1) ⇔ (3) ⇔ (4) ⇔ (6) ⇔ (7) are obvious by Theorem 2.8.
(5) ⇔ (6). By [[20], Theorem 1].
(6) ⇒ (6′) is obvious, since $(\mathcal{GF}, \mathcal{GC})$ is complete and hereditary.
(7) ⇒ (7′) is obvious, since $(\mathcal{GF}, \mathcal{GC})$ is complete and hereditary. □

For unbounded complexes, Iacob defined the *Gorenstein flat dimension* (Gfd) of complexes over left GF-closed rings [[20], Definition 15].

Remark 3.6 *Let* R *be a left GF-closed ring and* G *an* R*-complex. Then* $\mathcal{GF}\text{-}\dim_R(G) = \mathrm{Gfd}G$.

Proof By Definition 2.5 and [[20], Theorem 1] the conclusion is obvious. □

By [[20], Remark 3], for a homologically bounded below complex, [[20], Definition 15] agreed with [[6], Definition 2.7]. Thus, our definition agrees with [[6], Definition 2.7] by the preceding remark.

The dual statements of Proposition 3.5 hold, which is the characterization of Gorenstein cotorsion dimensions of complexes, and we omit it.

3.3. FP-projective and FP-injective dimensions of R-complexes

Recall that an R-module M is FP-injective [26] if $\mathrm{Ext}^1_R(F, M) = 0$ for every finitely presented R-module F. An R-module N is FP-projective [21] if $\mathrm{Ext}^1_R(N, M) = 0$ for every FP-injective R-module M. Let $\mathcal{FP}, \mathcal{FI}$ denote the class of FP-projective R-modules and FP-injective R-modules, respectively. If R is coherent, then the cotorsion pair $(\mathcal{FP}, \mathcal{FI})$ is complete and hereditary [[23], Proposition 3.6]. Hence, the cotorsion pair $(\mathcal{FP}, \mathcal{FI})$ induces FP-projective and FP-injective dimensions for any complex Y, denoted by $\mathcal{FP}\text{-}\dim(Y)$ and $\mathcal{FI}\text{-}\dim(Y)$, respectively. Applying Theorem 2.8 and Theorem 2.9 to $(\mathcal{FP}, \mathcal{FI})$ will yield the following results of the FP-projective dimension of a complex.

Proposition 3.7 *Let* R *be left coherent. Then the following are equivalent:*
(1) $\mathcal{FP}\text{-}\dim_R(Y) \leq n$.
(2) *There exists a quasi-isomorphism* $Y' \simeq Y$, *where* $Y' \in dg\widetilde{\mathcal{FP}}$ *with* $Y'_i = 0$ *for* $i > n$.
(3) $\mathrm{Ext}^i_R(Y, X) = 0$ *for any module* $X \in \mathcal{FI}$ *and* $i > n$.
(4) $\sup \mathrm{H}(Y) \leq n$ *and* $\mathrm{C}_n(Y')$ *is a FP-projective module for each dg-FP-projective resolution* $Y' \longrightarrow Y$.
(5) $\sup \mathrm{H}(Y) \leq n$ *and* $\mathrm{C}_n(P)$ *is a FP-projective module for each dg-projective resolution* $P \longrightarrow Y$.
(5)′ $\sup \mathrm{H}(Y) \leq n$ *and* $\mathrm{C}_j(P)$ *is a FP-projective module for every* $j \geq n$, *for each dg-projective resolution* $P \longrightarrow Y$.
(6) *There exists a dg-projective resolution* $P' \longrightarrow Y$ *such that* $\mathrm{H}_j(P') = 0$ *for every* $j \geq n+1$, *and* $\mathrm{C}_n(P')$ *is a FP-projective module.*

(6)′ *There exists a dg-projective resolution* $P' \longrightarrow Y$ *such that* $\mathrm{H}_j(P') = 0$ *for every* $j \geq n+1$, *and* $\mathrm{C}_j(P')$ *is a FP-projective module for every* $j \geq n$.

Take Y be an R-module and the FP-projective dimension of R-module Y is denoted by $\mathrm{fpd}(Y)$; then we have the following:

Corollary 3.8 ([21], Proposition 7.2.1) *Let* R *be a left coherent ring. For a left* R*-module* Y *and an integer* n, *the following are equivalent:*
(1) $\mathrm{fpd}(Y) \leq n$.
(2) $\mathrm{Ext}_R^{n+1}(Y, X) = 0$ *for any FP-injective left* R*-module* X.
(3) $\mathrm{Ext}_R^{n+j}(Y, X) = 0$ *for any FP-injective left* R*-module* X *and* $j > 1$.
(4) *If the sequence* $0 \longrightarrow Y_n \longrightarrow Y_{n-1} \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Y_0 \longrightarrow Y \longrightarrow 0$ *is exact with* $Y_0, Y_1, \cdots, Y_{n-1}$ *FP-projective, then* Y_n *is also FP-projective.*

We omit the characterization for the FP-injective dimension of complexes and modules, which is dual to Theorem 3.7 and Corollary 3.8.

Now we give some characterizations of von Neumann regular and left Noetherian rings.

Corollary 3.9 *Let* R *be a left coherent ring. The following are equivalent:*
(1) R *is* von Neumann *regular.*
(2) $\mathcal{FI}\text{-}\dim_R X = -\inf \mathrm{H}(X)$ *for every complex of* R*-modules* X.
(3) $\mathrm{Ext}_R^i(Y, X) = 0$ *for every complex of* R*-modules* X, *any FP-projective module* Y, *and* $i > -\inf \mathrm{H}(X)$.

Proof By [[26], Proposition 3.6] and Theorem 2.19 the conclusion is obvious. □

Corollary 3.10 *Let* R *be a left coherent ring. The following are equivalent:*
(1) R *is* Noetherian.
(2) $\mathcal{FP}\text{-}\dim_R Y = \sup \mathrm{H}(Y)$ *for every complex of* R*-modules* Y.
(3) $\mathrm{Ext}_R^i(Y, X) = 0$ *for every complex of* R*-modules* Y, *any FP-injective module* X, *and* $i > \sup \mathrm{H}(Y)$.

Proof By [[21], Proposition 2.6], Proposition 2.14, and Theorem 2.18, the conclusion is obvious. □

3.4. Ding projective and Ding injective dimensions of R-complexes

Ding and Chen extended FC rings to n-FC rings [7, 8], which are seen to have many properties similar to those of n-Gorenstein rings. Just as a ring is called Gorenstein when it is n-Gorenstein for some nonnegative integer n (a ring R is called n-Gorenstein if it is a left and right Noetherian ring with self-injective dimension of at most n on both sides for some nonnegative integer n), Gillespie first called a ring Ding–Chen when it is n-FC for some n [[19], Definition 4.1]. An R-module M is called Ding projective if there exists an exact sequence of projective R-modules $\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P_{-1} \longrightarrow P_{-2} \longrightarrow \cdots$ with $M = \mathrm{Ker}(P_0 \longrightarrow P_{-1})$ and that remains exact after applying $\mathrm{Hom}(-, F)$ for any flat R-module F [10]. The class of Ding projective R-modules is denoted by $\mathcal{DP}$. An R-module N is called Ding injective if there exists an exact sequence of injective R-modules $\cdots \longrightarrow I_1 \longrightarrow I_0 \longrightarrow I_{-1} \longrightarrow I_{-2} \longrightarrow \cdots$ with $N = \mathrm{Ker}(I_0 \longrightarrow I_{-1})$ and that remains exact after applying $\mathrm{Hom}(E,-)$ for any FP-injective R-module E [24]. The class of Ding injective R-modules is denoted by $\mathcal{DI}$. Note that every Ding injective (respectively, Ding projective) R-module N is Gorenstein injective (respectively, Gorenstein projective), and if R is Gorenstein, then every Gorenstein injective R-module is Ding injective (respectively, Gorenstein projective) [19].

From [[19], Theorem 4.2], we know that for a Ding–Chen ring R, the class of all modules with finite flat dimension and the class of all modules with finite FP-injective dimension are the same, and we use $\mathcal{W}$ to denote this class throughout this section. Ding and Mao proved that $({}^{\perp}\mathcal{W},\mathcal{W})$ forms a complete cotorsion pair when R is a Ding–Chen ring [[9], Theorem 3.8]. Also, $(\mathcal{W},\mathcal{W}^{\perp})$ forms a complete cotorsion pair when R is a Ding–Chen ring [[23], Theorem 3.4]. Moreover, Gillespie proved that an R-module M is Ding projective if and only if $M\in^{\perp}\mathcal{W}$, and an R-module N is Ding injective if and only if $N\in\mathcal{W}^{\perp}$ [[19], Corollaries 4.5 and 4.6]. So $(\mathcal{DP},\mathcal{W})$ and $(\mathcal{W},\mathcal{DI})$ are complete hereditary cotorsion pairs (each cogenerated by a set). Hence, they induce Ding projective and Ding injective dimensions for complex O, denoted by $\mathcal{DP}\text{-}\dim_R(O)$ and $\mathcal{DI}\text{-}\dim_R(O)$, respectively. Applying Theorem 2.8 and Theorem 2.9 to $(\mathcal{DP},\mathcal{W})$ will yield the following results of the Ding projective dimension of a complex.

Proposition 3.11 *Let R be a Ding–Chen ring. Then the following assertions are equivalent for an R-complex O:*
(1) $\mathcal{DP}\text{-}\dim_R(O)\leq n$.
(2) *There exists a quasi-isomorphism $O'\simeq O$, where $O'\in dg\widetilde{\mathcal{DP}}$ with $O'_i=0$ for $i>n$.*
(3) $\mathrm{Ext}^i_R(O,N')=0$ *for any module $N'\in\mathcal{W}$ and $i>n$.*
(4) $\sup\mathrm{H}(O)\leq n$ *and $\mathrm{C}_n(O')$ is a Ding projective R-module for each dg-Ding projective resolution $O'\longrightarrow O$.*
(5) $\sup\mathrm{H}(O)\leq n$ *and $\mathrm{C}_n(P)$ is a Ding projective R-module for each dg-projective resolution $P\longrightarrow O$.*
(5)′ $\sup\mathrm{H}(O)\leq n$ *and $\mathrm{C}_j(P)$ is a Ding projective R-module for every $j\geq n$, for each dg-projective resolution $P\longrightarrow O$.*
(6) *There exists a dg-projective resolution $P'\longrightarrow O$ such that $\mathrm{H}_j(P')=0$ for every $j\geq n+1$, and $\mathrm{C}_n(P')$ is a Ding projective R-module.*
(6)′ *There exists a dg-projective resolution $P'\longrightarrow O$ such that $\mathrm{H}_j(P')=0$ for every $j\geq n+1$, and $\mathrm{C}_j(P')$ is a Ding projective R-module for every $j\geq n$.*

The case of Ding injective dimension of complex is dual.

Now we give some characterizations of QF rings.

Corollary 3.12 *Let R be a Ding–Chen ring. The following are equivalent:*
(1) *R is a QF ring (that is, R is a 0-Gorenstein ring).*
(2) $\mathcal{DP}\text{-}\dim_R O=\sup\mathrm{H}(O)$ *for every complex of R-modules O.*
(3) $\mathrm{Ext}^i_R(O,N)=0$ *for every complex of R-modules O, any Ding injective R-module N, and $i>\sup\mathrm{H}(O)$.*
(4) $\mathcal{DI}\text{-}\dim O=-\inf\mathrm{H}(O)$ *for every complex of R-modules O.*
(5) $\mathrm{Ext}^i_R(M,O)=0$ *for every complex of R-modules O, any Ding projective R-module M, and $i>-\inf\mathrm{H}(O)$.*

Proof By [[10], Proposition 2.16], [[24], Proposition 4.5], Theorem 2.18, and Theorem 2.19, the conclusion is obvious. □

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