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On transformations of index 1

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Abstract: The *index* and the *period* of an element a of a finite semigroup are defined as the smallest values of $m \geq 1$ and $r \geq 1$ such that $a^{m+r} = a^m$, respectively. If $m = 1$ then a is called an element *of index* 1. The aim of this paper is to find some properties of the elements of index 1 in T_n, which we call *transformations of index* 1.

Key words: Transformations, orbit, index, period

1. Introduction

The full transformation semigroup $\mathcal{T}_X$ on a set X and the semigroup analogue of the symmetric group $\mathcal{S}_X$ have been much studied over the last 50 years, both in the finite and in the infinite cases. Here we are concerned solely with the case where $X = X_n = \{1, \dots, n\}$, and we write respectively T_n and S_n rather than $\mathcal{T}_X$ and $\mathcal{S}_X$. The *image*, *Defect set*, *defect*, *kernel*, and *Fix* of $\alpha \in T_n$ are defined by

$$\begin{aligned}
\operatorname{im}(\alpha) &= \{y \in X_n : \text{ there exists } x \in X_n \text{ such that } x\alpha = y\}, \\
\operatorname{Def}(\alpha) &= X_n \setminus \operatorname{im}(\alpha), \\
\operatorname{def}(\alpha) &= |\operatorname{Def}(\alpha)|, \\
\ker(\alpha) &= \{(x, y) \in X_n \times X_n : x\alpha = y\alpha\}, \\
\operatorname{Fix}(\alpha) &= \{x \in X_n : x\alpha = x\},
\end{aligned}$$

respectively. For any $\alpha, \beta \in T_n$, it is easy to show by using the definitions of Green's equivalences that

$$\begin{aligned}
(\alpha, \beta) \in \mathcal{D} \quad &\Leftrightarrow \quad |\operatorname{im}(\alpha)| = |\operatorname{im}(\beta)| \quad \Leftrightarrow \quad \operatorname{def}(\alpha) = \operatorname{def}(\beta), \\
(\alpha, \beta) \in \mathcal{H} \quad &\Leftrightarrow \quad \ker(\alpha) = \ker(\beta) \text{ and } \operatorname{im}(\alpha) = \operatorname{im}(\beta)
\end{aligned}$$

(see for definitions of Green's equivalences [4, pp. 45–47]). We denote Green's $\mathcal{D}$-class of all singular self maps of defect k by D_{n-k} for $1 \leq k \leq n-1$, and Green's $\mathcal{H}$-class containing $\alpha \in T_n$ by H_α. The equivalence relation generated by $R \subseteq Y \times Y$ on a set Y is defined by the smallest equivalence relation containing R and denoted by R^e. It is clear that $\alpha \in D_{n-1}$ if and only if there exists unique $(i, j) \in X_n \times X_n$ such that $i < j$ and $\ker(\alpha) = \{(i, j)\}^e$, or, equivalently, there exists unique $l \in X_n$ such that $\operatorname{Def}(\alpha) = \{l\}$. We denote the set of all idempotents in any subset U of any semigroup by $E(U)$. It is clear that $\alpha \in E(D_{n-1})$ if and only

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if there exist unique $(i,j) \in X_n \times X_n$ such that $i\alpha = j$ and $l\alpha = l$, for each $l \in X_n \setminus \{i\}$. We denote this idempotent by $\begin{pmatrix} i \\ j \end{pmatrix}$.

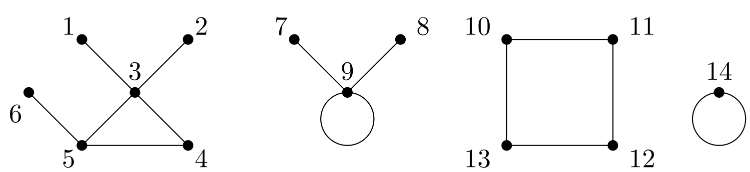
For $\alpha \in T_n$, the equivalence relation $\equiv$ on X_n, defined by

$$x \equiv y \text{ if and only if } (\exists r, s \geq 0)\, x\alpha^r = x\alpha^s,$$

parts X_n into *orbits* $\Omega_1, \Omega_2, \ldots, \Omega_t$ $(t \geq 1)$. The orbits are the connected components of the function graph and provide valuable information about the structure of the map α (for example, see [1], [3]). Typically, an orbit consists of a cycle with some trees attached. If there are no attached trees, we say that the orbit Ω_i is *cyclic*; in particular, if Ω_i consists of a single fixed point, we say that it is *trivial* or a *loop*. For example, let α be the map

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 3 & 3 & 4 & 5 & 3 & 5 & 9 & 9 & 9 & 11 & 12 & 13 & 10 & 14 \end{pmatrix} \in T_{14}.$$

The orbits of α (with the convention that arrows point towards the cycle or fixed point, and that arrows go counterclockwise within the cycles) can be depicted thus:



In the general case, it is clear that, for each $x \in X_n$, the sequence

$$x, x\alpha, x\alpha^2, \ldots$$

eventually arrives in a cycle (or a fixed point, which of course we may regard as a special case of a cycle) and remains there for all subsequent iterations. Denote the set of all elements contained in the cycle on Ω_i by $Z(\Omega_i)$ $(1 \leq i \leq t)$, and let

$$Z(\alpha) = \bigcup_{i=1}^{t} Z(\Omega_i).$$

In our example,

$$Z(\Omega_1) = \{3,4,5\}, \quad Z(\Omega_2) = \{9\}, \quad Z(\Omega_3) = \{10,11,12,13\}, \quad Z(\Omega_4) = \{14\},$$

$$Z(\alpha) = \{3,4,5,9,10,11,12,13,14\},$$

and notice that the orbits are either cyclic or a cycle with some trees attached.

The *index* and the *period* of an element a of a finite semigroup are defined as the smallest values of $m \geq 1$ and $r \geq 1$ such that the elements $a, a^2, \ldots, a^{m+r-1}$ are different and $a^{m+r} = a^m$, respectively. In particular, a is called an element *of index* 1 if $m = 1$ (see [2, 4] for other terms in semigroup theory that are not explained here). The aim of this paper is to find some properties of the elements of index 1 in T_n, which we call *transformations of index* 1. First we find the orbit structure of $\alpha \in T_n$ where $\mathrm{im}(\alpha^k) = \mathrm{im}(\alpha)$ for all $k \in \mathbb{Z}^+$. Then we prove that $\alpha \in T_n$ is a transformation of index 1 if and only if $\mathrm{im}(\alpha^k) = \mathrm{im}(\alpha)$ for all $k \in \mathbb{Z}^+$, and we give some related results.

2. Transformations of index 1

First we state and prove the following lemma, which will be useful throughout this paper.

Lemma 2.1 *Let $\Omega_1, \ldots, \Omega_t$ be the orbits of $\alpha \in T_n$. Then, for all $k \in \mathbb{Z}^+$, $\mathrm{im}(\alpha^k) = \mathrm{im}(\alpha)$ if and only if, for each $x \in X_n$, there exists unique $1 \leq i \leq t$ such that $x\alpha \in Z(\Omega_i)$.*

Proof ($\Rightarrow$) Let $\alpha \in T_n$ and $\mathrm{im}(\alpha^k) = \mathrm{im}(\alpha)$, for all $k \in \mathbb{Z}^+$. If the set $\mathrm{Def}(\alpha)$ is empty, then $\alpha \in S_n$, and so the condition is clearly satisfied since the orbits of a permutation are cyclic.

Suppose that $\mathrm{Def}(\alpha) \neq \emptyset$ and take any $x \in \mathrm{Def}(\alpha)$. Then there exists unique $1 \leq i \leq t$ such that $x \in \Omega_i \setminus Z(\Omega_i)$ since $Z(\Omega_i) \subseteq \mathrm{im}(\alpha)$. Moreover, there exists an integer $p \geq 1$ such that $x\alpha^p \in Z(\Omega_i)$ but $x\alpha^{p-1} \notin Z(\Omega_i)$. We also suppose that if there exist $y \in \Omega_i \setminus Z(\Omega_i)$ and $q \in \mathbb{Z}^+$ such that $y\alpha^q \in Z(\Omega_i)$ but $y\alpha^{q-1} \notin Z(\Omega_i)$, then $q \leq p$.

Since $\mathrm{im}(\alpha^2) = \mathrm{im}(\alpha)$, there exists $z \in \Omega_i$ such that $z\alpha^2 = x\alpha$. It follows from the assumption of x that $z \in Z(\Omega_i)$ or $z\alpha \in Z(\Omega_i)$. Otherwise, that is, if $z \notin Z(\Omega_i)$ and $z\alpha \notin Z(\Omega_i)$, then $z\alpha^{p+1} \in Z(\Omega_i)$ but $z\alpha^p \notin Z(\Omega_i)$, which is a contradiction to the choice of x. Indeed,

$$z \to z\alpha \to z\alpha^2 = x\alpha \to \cdots \to z\alpha^p = x\alpha^{p-1} \notin Z(\Omega_i)$$
$$z \to z\alpha \to z\alpha^2 = x\alpha \to \cdots \to z\alpha^{p+1} = x\alpha^p \in Z(\Omega_i).$$

Since $x\alpha = z\alpha^2$ and $z \in Z(\Omega_i)$ or $z\alpha \in Z(\Omega_i)$, it follows that $x\alpha \in Z(\Omega_i)$; that is, $p = 1$. Moreover, for all $y \in \Omega_i$, it follows from the choice of x that $y\alpha \in Z(\Omega_i)$.

($\Leftarrow$) Suppose that, for each $x \in X_n$, there exists unique $1 \leq i \leq t$ such that $x\alpha \in Z(\Omega_i)$. For any $\alpha \in T_n$, since $\mathrm{im}(\alpha^k) \subseteq \mathrm{im}(\alpha)$ for all $k \in \mathbb{Z}^+$, it is enough to show that $\mathrm{im}(\alpha) \subseteq \mathrm{im}(\alpha^k)$.

For $y \in \mathrm{im}(\alpha)$ there exists $x \in \Omega_i$ $(1 \leq i \leq t)$ such that $x\alpha = y$, and so $y \in Z(\Omega_i)$. Since the restriction of α to $Z(\Omega_i)$, $\alpha_{|_{Z(\Omega_i)}}$, is a permutation of $Z(\Omega_i)$, it follows that $y \in \mathrm{im}(\alpha^k)$, and so $\mathrm{im}(\alpha) \subseteq \mathrm{im}(\alpha^k)$, for all $k \in \mathbb{Z}^+$. Therefore, $\mathrm{im}(\alpha^k) = \mathrm{im}(\alpha)$ for all $k \in \mathbb{Z}^+$, as required. □

Now we state an immediate result.

Corollary 2.2 *Let $\Omega_1, \ldots, \Omega_t$ be the orbits of $\alpha \in T_n$. Then $\mathrm{im}(\alpha^k) = \mathrm{im}(\alpha)$, for all $k \in \mathbb{Z}^+$, if and only if*

$$\mathrm{Def}(\alpha) = \bigcup_{1 \leq i \leq t} \big(\Omega_i \setminus Z(\Omega_i)\big) = X_n \setminus Z(\alpha). \qquad \square$$

Let $\alpha \in T_{14}$ be the transformation given above. It is easy to see that $\mathrm{im}(\alpha^k) = \mathrm{im}(\alpha)$, for all $k \in \mathbb{Z}^+$. Moreover,

$$\Omega_1 \setminus Z(\Omega_1) = \{1,2,6\}, \qquad \Omega_2 \setminus Z(\Omega_2) = \{7,8\},$$
$$\Omega_3 \setminus Z(\Omega_3) = \Omega_4 \setminus Z(\Omega_4) = \emptyset \quad \text{and} \quad \mathrm{Def}(\alpha) = \{1,2,6,7,8\},$$

as stated in Corollary 2.2.

Theorem 2.3 *Let $\Omega_1, \dots, \Omega_t$ be the orbits of $\alpha \in T_n$, and let r_i be the cardinality of $Z(\Omega_i)$ for each $1 \leq i \leq t$. Then α is a transformation of index 1 and period r if and only if, for all $k \in \mathbb{Z}^+$, $\mathrm{im}\,(\alpha^k) = \mathrm{im}\,(\alpha)$ and r is the lowest common multiple of $r_1, \dots, r_t$.*

Proof Let $\Omega_1, \dots, \Omega_t$ be the orbits of $\alpha \in T_n$, and let r_i be the cardinality of $Z(\Omega_i)$ for each $1 \leq i \leq t$.

($\Leftarrow$) Suppose that $\mathrm{im}\,(\alpha^k) = \mathrm{im}\,(\alpha)$ for all $k \in \mathbb{Z}^+$, and that r is the lowest common multiple of $r_1, \dots, r_t$. For any $x \in X_n$, there exists $1 \leq i \leq t$ such that $x \in \Omega_i$. If $x \in Z(\Omega_i)$ it is clear that $x\alpha^{r_i} = x$, and so $x\alpha^{1+r_i} = x\alpha$. If $x \notin Z(\Omega_i)$, then it follows from Lemma 2.1 that $x\alpha \in Z(\Omega_i)$, and so $x\alpha^{1+r_i} = x\alpha$. Moreover, since there exists a $q_i \in \mathbb{Z}^+$ such that $r = q_i r_i$, it follows that

$$x\alpha^{1+r} = x\alpha^{1+q_i r_i} = (x\alpha^{1+r_i})\alpha^{(q_i-1)r_i} = (x\alpha)\alpha^{(q_i-1)r_i} \tag{1}$$
$$= \cdots = (x\alpha)\alpha^{r_i} = x\alpha^{1+r_i} = x\alpha.$$

Thus, $\alpha^{1+r} = \alpha$ and so the index of α is 1.

Now we show that the period of α is r. Suppose that there exists $p \in \mathbb{Z}^+$ such that $\alpha^{1+p} = \alpha$. For any $1 \leq i \leq t$, take any $x \in \Omega_i$. From the division algorithm, there exist $u_i, v_i \in \mathbb{Z}$ such that $p = u_i r_i + v_i$ and $0 \leq v_i \leq r_i - 1$. Notice that $p \geq r_i$, since the restriction of α to $Z(\Omega_i)$ is a permutation (even a cycle) and $|Z(\Omega_i)| = r_i$, and so $u_i \geq 1$. Assume that $v_i \neq 0$. Since $x\alpha^{1+u_i r_i} = x\alpha$ (as in Eq. (1)), it follows that

$$x\alpha = x\alpha^{1+p} = x\alpha^{1+u_i r_i + v_i} = (x\alpha^{1+u_i r_i})\alpha^{v_i} = (x\alpha)\alpha^{v_i} = x\alpha^{1+v_i},$$

which is in contradiction with the assumption of r_i. Thus, v_i must be zero; that is, r_i divides p. Therefore, r divides p, and so the period of α is r.

($\Rightarrow$) Let α be a transformation of index 1 and period r. If $1 \leq k \leq r$ then, since

$$\mathrm{im}\,(\alpha^k) \subseteq \mathrm{im}\,(\alpha) = \mathrm{im}\,(\alpha^{1+r}) = \mathrm{im}\,(\alpha^{1+r-k}\alpha^k) \subseteq \mathrm{im}\,(\alpha^k),$$

we have $\mathrm{im}\,(\alpha^k) = \mathrm{im}\,(\alpha)$. If $k > r$, then, from the division algorithm, there exist $u, v \in \mathbb{Z}$ such that $k = ur + v$ and $0 \leq v \leq r - 1$. Notice that $u \geq 1$. If $1 \leq v \leq r - 1$ then

$$\begin{aligned} \alpha^k &= \alpha^{ur+v} = \alpha^{1+r}\alpha^{(u-1)r+(v-1)} = \alpha\alpha^{(u-1)r+(v-1)} \\ &= \alpha^{(u-1)r+v} = \cdots = \alpha^{r+v} = \alpha^{1+r}\alpha^{v-1} = \alpha^v. \end{aligned}$$

Thus, since $0 \leq v < r$, it follows that

$$\mathrm{im}\,(\alpha^k) = \mathrm{im}\,(\alpha^v) = \mathrm{im}\,(\alpha).$$

If $v = 0$, then, since $k = ur$ and $u \geq 2$, it follows that

$$\begin{aligned} \alpha^k &= \alpha^{ur} = \alpha^{1+r}\alpha^{(u-1)r-1} = \alpha\alpha^{(u-1)r-1} \\ &= \alpha^{(u-1)r} = \cdots = \alpha^r. \end{aligned}$$

Therefore,

$$\mathrm{im}\,(\alpha^k) = \mathrm{im}\,(\alpha^r) = \mathrm{im}\,(\alpha),$$

as required. It is easy to show as in the first part of the proof that r is the lowest common multiple of $r_1, \dots, r_t$. □

Corollary 2.4 *$\alpha \in T_n$ is a transformation of index 1 if and only if the restriction of α to $\mathrm{im}(\alpha)$ is a permutation. In particular, all permutations and all idempotents in T_n are transformations of index 1.*

Proof ($\Rightarrow$) Let $\alpha \in T_n$ be a transformation of index 1. It follows from Theorem 2.3 that $(\mathrm{im}(\alpha))\alpha = \mathrm{im}(\alpha^2) = \mathrm{im}(\alpha)$. That is, the restriction of α to $\mathrm{im}(\alpha)$ is onto, and so a permutation.

($\Leftarrow$) Let the restriction of α to $\mathrm{im}(\alpha)$ be a permutation. Then $\mathrm{im}(\alpha^2) = (\mathrm{im}(\alpha))\alpha = \mathrm{im}(\alpha)$, and so $\mathrm{im}(\alpha^k) = \mathrm{im}(\alpha)$ for all $k \in \mathbb{Z}^+$. From Theorem 2.3 $\alpha \in T_n$ is a transformation of index 1. □

Corollary 2.5 *Let H_α be Green's $\mathcal{H}$-class containing $\alpha \in T_n$. Then α is a transformation of index 1 if and only if H_α is a group.*

Proof ($\Rightarrow$) Suppose that α is a transformation of index 1 and the period of α is r. Since $\alpha^{1+r} = \alpha = \alpha^{r+1}$ and $\alpha\alpha^{r-1} = \alpha^r = \alpha^{r-1}\alpha$, we have $\alpha\mathcal{H}\alpha^r$, and so $\alpha^r \in H_\alpha$. Moreover, it is easy to see that α^r is an idempotent, and, from [4, Corollary 2.2.6], we have the fact that H_α is a group.

($\Leftarrow$) Suppose that H_α is a group. Then $\alpha^k \in H_\alpha$, and so $\mathrm{im}(\alpha^k) = \mathrm{im}(\alpha)$ for all $k \in \mathbb{Z}^+$. Thus, the result follows from Theorem 2.3. □

Consider Green's $\mathcal{D}$-class D_r for each $1 \leq r \leq n$. Since there exists $\binom{n}{r} r^{n-r}$ many idempotents in D_r (see, for example, [2]), exactly $\binom{n}{r} r^{n-r}$ many Green's $\mathcal{H}$-classes are groups $(1 \leq r \leq n)$. Since each Green's $\mathcal{H}$-class in D_r contains exactly $r!$ elements, we have the following corollary:

Corollary 2.6 *There exist*

$$\sum_{r=1}^{n} \binom{n}{r} r^{n-r} r! = \sum_{r=1}^{n} \frac{n!}{(n-r)!} r^{n-r}$$

transformations of index 1 in T_n. □

Theorem 2.7 *Let $\alpha \in T_n$ with defect $k \geq 1$. Then α is a transformation of index 1 if and only if there exist a permutation $\beta \in S_n$ and $\gamma \in E(D_{n-k})$ such that $\alpha = \beta\gamma$ and $\mathrm{Def}(\alpha) = \mathrm{Def}(\gamma) \subseteq \mathrm{Fix}(\beta)$.*

Proof ($\Rightarrow$) Suppose that α is a transformation of index 1. Then we define the map $\beta : X_n \to X_n$ by

$$x\beta = \begin{cases} x\alpha & x \in \mathrm{im}(\alpha) \\ x & x \in \mathrm{Def}(\alpha) \end{cases}$$

and the map $\gamma : X_n \to X_n$ by

$$x\gamma = \begin{cases} x & x \in \mathrm{im}(\alpha) \\ x\alpha & x \in \mathrm{Def}(\alpha) \end{cases}$$

for $x \in X_n$. Since α is a transformation of index 1, it follows from Corollary 2.4 that the restriction of α to $\mathrm{im}(\alpha)$ is a permutation, and so β is a permutation. Moreover, it is clear that $\mathrm{Def}(\gamma) = \mathrm{Def}(\alpha) \subseteq \mathrm{Fix}\,\beta$, $\gamma \in E(D_{n-k})$, and $\alpha = \beta\gamma$.

($\Leftarrow$) Suppose that there exist a permutation $\beta \in S_n$ and $\gamma \in E(D_{n-k})$ such that $\alpha = \beta\gamma$ and $\mathrm{Def}(\gamma) = \mathrm{Def}(\alpha) = \{x_1, \ldots, x_k\} \subseteq \mathrm{Fix}(\beta)$.

Take any $z \in \mathrm{im}(\alpha)$. Then there exist $x \in X_n$ such that $x\alpha = z$. Since $z \in \mathrm{im}(\gamma) = \mathrm{im}(\alpha)$ and γ is an idempotent, it follows that $z\gamma = z$. Moreover, since $\beta \in S_n$, there exist $y \in X_n$ such that $y\beta = z$. If $z \in \mathrm{Fix}(\beta)$, then we have

$$x\alpha^2 = (x\alpha)\alpha = z\alpha = (z\beta)\gamma = z\gamma = z.$$

If $z \notin \mathrm{Fix}(\beta)$, then $y \notin \mathrm{Fix}(\beta)$ since β is a permutation, and so $y \in \mathrm{im}(\alpha)$ since $\mathrm{Def}(\alpha) \subseteq \mathrm{Fix}(\beta)$. Thus there exists $w \in X_n$ such that $w\alpha = y$, and so we have

$$w\alpha^2 = (w\alpha)\alpha = y\alpha = (y\beta)\gamma = z\gamma = z.$$

In both cases, we have $z \in \mathrm{im}(\alpha^2)$, and so $\mathrm{im}(\alpha) \subseteq \mathrm{im}(\alpha^2)$. Since $\mathrm{im}(\alpha^2) \subseteq \mathrm{im}(\alpha)$ it follows that $(\mathrm{im}(\alpha))\alpha = \mathrm{im}(\alpha^2) = \mathrm{im}(\alpha)$. Therefore, the restriction of α to $\mathrm{im}(\alpha)$ is onto, and so a permutation. It follows from Corollary 2.4 that α is a transformation of index 1. □

3. Kernel structure

Theorem 3.1 *Let $\alpha \in T_n$ be a transformation of index 1, and let $\mathrm{Def}(\alpha) = \{x_1, \ldots, x_k\}$ for $k \geq 1$. Then there exist $m_1, \ldots, m_k \in \mathbb{Z}^+$ (not necessarily different) such that*

$$\ker(\alpha) = \{(x_1, x_1\alpha^{m_1}), \ldots, (x_k, x_k\alpha^{m_k})\}^e.$$

Proof Let α be a transformation of index 1, $\mathrm{Def}(\alpha) = \{x_1, \ldots, x_k\}$ for $k \geq 1$, and let $\Omega_1, \ldots, \Omega_t$ be the orbits of α. Then, for each $1 \leq i \leq k$, it follows from Lemma 2.1 that $x_i \in \Omega_j \setminus Z(\Omega_j)$ and $x_i\alpha \in Z(\Omega_j)$ for unique $1 \leq j \leq t$. Thus there exist some $m_i \in \mathbb{Z}^+$, which can be chosen as the cardinality of $Z(\Omega_j)$, such that $x_i\alpha^{m_i+1} = x_i\alpha$.

Let $R = \{(x_1, x_1\alpha^{m_1}), \ldots, (x_k, x_k\alpha^{m_k})\}$. It is clear that $(x_i, x_i\alpha^{m_i}) \in \ker(\alpha)$ for all $1 \leq i \leq k$, and so $R^e \subseteq \ker(\alpha)$.

Now, let $(x, y) \in \ker(\alpha)$ with $x \neq y$. Since $x\alpha = y\alpha$, it follows that both x and y are in the same orbit of α, say Ω_j $(1 \leq j \leq t)$. Since at most 1 of x and y is in $Z(\Omega_j)$, there are 2 cases:

1. neither of them is in $Z(\Omega_j)$;
2. exactly 1 of them is in $Z(\Omega_j)$.

First of all, suppose that $| Z(\Omega_j) |= m$.

Case 1. Let $x, y \in \Omega_j \setminus Z(\Omega_j)$. From Corollary 2.2 we have $x, y \in \mathrm{Def}(\alpha)$. We also have $x\alpha^m = y\alpha^m$, since $x\alpha = y\alpha$. Thus, since $(x, x\alpha^m), (y, y\alpha^m) \in R$, it follows from the definition of R^e that $(x, x\alpha^m), (y\alpha^m, y) \in R^e$, and so $(x, y) \in R^e$, as required.

Case 2. Without loss of generality suppose that $x \in Z(\Omega_j)$, and that $y \in \Omega_j \setminus Z(\Omega_j) \subseteq \mathrm{Def}(\alpha)$. Since $y\alpha \in Z(\Omega_j)$ and $x \in Z(\Omega_j)$, we have $y\alpha^{m+1} = y\alpha$ and $x = x\alpha^m$. Moreover, since $x\alpha^m = y\alpha^m$, as in Case 1, and since $(y, y\alpha^m) \in R$, it follows that

$$(y, y\alpha^m) = (y, x\alpha^m) = (y, x) \in R,$$

and so $(x, y) \in R^e$, as required.

Therefore, in both cases, we have $\ker(\alpha) \subseteq R^e$. □

Let $\alpha \in T_n$ be a transformation of index 1 and period r. Now we prove that, if $\operatorname{def}(\alpha) = k$ for $1 \le k \le n-1$, then $\alpha^r \in E(D_{n-k})$ can be written as a product of k idempotents of defect 1, in the following corollary.

Corollary 3.2 *Let $\alpha \in T_n$ be a transformation of index 1 and period r, and let $\operatorname{Def}(\alpha) = \{x_1, \ldots, x_k\}$ for $1 \le k \le n-1$. Then there exist $m_1, \ldots, m_k \in \mathbb{Z}^+$ (not necessarily different) such that*

$$\alpha^r = \begin{pmatrix} x_1 \\ x_1\alpha^{m_1} \end{pmatrix} \cdots \begin{pmatrix} x_k \\ x_k\alpha^{m_k} \end{pmatrix} \in E(D_{n-k}).$$

Proof Let $\operatorname{Def}(\alpha) = \{x_1, \ldots, x_k\}$ for $1 \le k \le n-1$. From Theorem 2.3, we have the fact that $\operatorname{im}(\alpha^r) = \operatorname{im}(\alpha)$, and so $\ker(\alpha^r) = \ker(\alpha)$. It follows from Theorem 3.1 that there exist $m_1, \ldots, m_k \in \mathbb{Z}^+$ such that

$$\ker(\alpha^r) = \{(x_1, x_1\alpha^{m_1}), \ldots, (x_k, x_k\alpha^{m_k})\}^e.$$

Since $\alpha^r \in E(D_{n-k})$ and $x_i\alpha^{m_i} \in \operatorname{im}(\alpha^r) = \operatorname{im}(\alpha)$ for each $1 \le i \le k$, it follows that

$$\alpha^r = \begin{pmatrix} x_1 \\ x_1\alpha^{m_1} \end{pmatrix} \cdots \begin{pmatrix} x_k \\ x_k\alpha^{m_k} \end{pmatrix} \in E(D_{n-k}),$$

as required. □

Let us consider our example given above. With the notation given in Theorem 2.3, $r_1 = 3$, $r_2 = 1$, $r_3 = 4$, and $r_4 = 1$, and so $r = \operatorname{lcm}\{3, 4, 1\} = 12$. Notice that

$$\alpha^{12} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 5 & 5 & 3 & 4 & 5 & 4 & 9 & 9 & 9 & 10 & 11 & 12 & 13 & 14 \end{pmatrix},$$

and that

$$\alpha^{13} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 3 & 3 & 4 & 5 & 3 & 5 & 9 & 9 & 9 & 11 & 12 & 13 & 10 & 14 \end{pmatrix} = \alpha.$$

Moreover, $\ker(\alpha) = \{(1,5), (2,5), (6,4), (7,9), (8,9)\}^e$ and

$$\alpha^{12} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}\begin{pmatrix} 2 \\ 5 \end{pmatrix}\begin{pmatrix} 6 \\ 4 \end{pmatrix}\begin{pmatrix} 7 \\ 9 \end{pmatrix}\begin{pmatrix} 8 \\ 9 \end{pmatrix}.$$

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