Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2014) 38: 436 – 453
© TÜBİTAK
doi:10.3906/mat-1303-44

Horizontally submersions of contact CR-submanifolds

Fortuné MASSAMBA[1,*], **Tshikunguila TSHIKUNA-MATAMBA**[2]

[1]School of Mathematics, Statistics, and Computer Science, University of KwaZulu-Natal, Scottsville, South Africa
[2]Department of Mathematical Sciences, College of Science, Engineering, and Technology, University of South Africa, Pretoria, South Africa

Received: 22.03.2013 • **Accepted:** 09.08.2013 • **Published Online:** 14.03.2014 • **Printed:** 11.04.2014

Abstract: In this paper, we discuss some geometric properties of almost contact metric submersions involving symplectic manifolds. We show that the structures of quasi-K-cosymplectic and quasi-Kenmotsu manifolds are related to $(1,2)$-symplectic structures. For horizontally submersions of contact CR-submanifolds of quasi-K-cosymplectic and quasi-Kenmotsu manifolds, we study the principal characteristics and prove that their total spaces are CR-product. Curvature properties between curvatures of quasi-K-cosymplectic and quasi-Kenmotsu manifolds and the base spaces of such submersions are also established. We finally prove that, under a certain condition, the contact CR-submanifold of a quasi Kenmotsu manifold is locally a product of a totally geodesic leaf of an integrable horizontal distribution and a curve tangent to the normal distribution.

Key words: CR-submanifold, almost Hermitian manifold, almost contact metric submersion, symplectic manifold, horizontal submersion

1. Introduction

Riemannian submersions between Riemannian manifolds were initiated by O'Neill [14]. Almost contact metric submersions were developed by Chinea [8] and Watson [17]. The theory of almost contact metric submersions intertwines contact geometry with the almost Hermitian one. For instance, the base space of an almost contact metric submersion of type II, in the sense of Watson [17], is an almost Hermitian manifold. However, certain classes of almost Hermitian manifolds are closely related to symplectic manifolds. Specifically, almost Kähler manifolds are endowed with symplectic manifolds while quasi-Kählerians are related to $(1,2)$-symplectic ones. Symplectic and almost contact manifolds were treated in [4]. Recall that almost contact metric submersions were initiated by Chinea [7] and Watson [17].

On the other hand, the study of CR-submanifolds of an Hermitian manifold was initiated by Bejancu in [1]. He generalized both totally real and holomorphic immersions. Given an almost Hermitian manifold, (M,J,g), a submanifold M is called a CR-submanifold if there exists a differentiable distribution D on M such it is holomorphic, and its complementary orthogonal distribution $D^{\perp}$ is totally real $JD_x \subseteq D_x$ and $J(D_x^{\perp}) \subseteq T_xM^{\perp}$, for all $x \in M$. Since then, many authors have treated CR-submanifolds on different ambient manifolds and have amplified the definition to other decompositions of the tangent bundle (semislant and

*Correspondence: massamba@ukzn.ac.za

2010 *AMS Mathematics Subject Classification:* Primary 53C15, 53C20; Secondary 53D12, 53D15, 53D35.

almost semiinvariant submanifolds). In [16], Sahin considered horizontally conformal submersions and proved that every horizontally homothetic submersion is a Riemannian submersion.

The subject was considered later for Riemannian manifolds with an almost contact structure. In this sense, Benjacu and Papaghiuc studied semiinvariant submanifolds of a Sasakian manifold or a Sasakian space form (see [2, 3] and [15] and references therein).

In this paper, we study almost contact metric submersions of type II involving the classes of symplectic structures. We are interested in the following problem. Let $\pi : M^{2m+1} \longrightarrow M'^{2m'}$ be an almost contact metric submersion of type II. Under what conditions is the base space M' a $(1,2)$-symplectic manifold and, conversely, if the base space is a $(1,2)$-symplectic manifold, what is the structure of the total space M?

The paper is organized in the following way. In Section 2, devoted to the preliminaries on manifolds, we review the main classes of almost Hermitian manifolds that have some relation with almost symplectic structures. Almost contact metric manifolds that can be used as total space of fibration are also reviewed. Section 3 deals with almost contact metric submersions. Here, after recalling some fundamental properties, it is shown that quasi-K-cosymplectic and quasi-Kenmotsu manifolds, which have a common relation, are related to $(1,2)$-symplectic manifolds, that is, quasi-Kähler manifolds. Almost α-Kenmotsu manifolds are related to symplectic manifolds. In Section 4, we recall the definition of contact CR-submanifolds given by Yano and Kon in [18] and give the decomposition of their tangent and normal bundles. In Section 5, we consider Riemannian submersions of contact CR-submanifolds of quasi-K-cosymplectic, quasi-Kenmotsu manifolds. We study the integrability of all the distributions involved in the definition of a contact CR-submanifold. We prove that the base spaces of such submersions are quasi-K-cosymplectic and quasi-Kenmotsu, and under a certain condition, they are $(1,2)$-symplectic. By Theorem 5.15, we show that the total spaces of the submersions involved are CR-product. Finally, we give, in Section 6, some curvature properties by deriving expressions relating curvatures of the ambient manifolds and the base spaces of the submersions. Under a certain condition, we prove that the total space of the submersion of a contact CR-submanifold of a quasi-Kenmotsu manifold is locally a product $M^* \times C$, where M^* is a totally geodesic leaf of the horizontal distribution $D \oplus \{\xi\}$ in Definition 4.1 and C is a curve tangent to the distribution $D^\perp$ (Theorem 6.3).

2. Preliminaries

An almost Hermitian manifold is a Riemannian manifold, (M,g), endowed with a tensor field J of type $(1,1)$ satisfying the following 2 conditions:

(i) $J^2 X = -X$, and

(ii) $g(JX, JY) = g(X,Y)$, for any X, $Y \in \Gamma(TM)$.

It is known that any almost Hermitian manifold, (M,g,J), is of even dimension, say $2m$, and possesses a fundamental 2-form Ω defined by

$$\Omega(X,Y) = g(X, JY).$$

Following Gray and Hervella [13], an almost Hermitian manifold (M^{2m}, g, J) is said to be *quasi-Kählerian* if

$$(\nabla_X \Omega)(Y,Z) + (\nabla_{JX}\Omega)(JY,Z) = 0, \tag{2.1}$$

and *almost Kählerian* if

$$d\Omega(X,Y,Z) = 0. \tag{2.2}$$

Let M be a differentiable manifold of dimension $2m+1$. An almost contact structure on M is a triple (φ, ξ, η), where ξ is a characteristic vector field, η is a 1-form such that $\eta(\xi) = 1$, and φ is a tensor field of type $(1,1)$ satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0, \tag{2.3}$$

where I is the identity transformation. If M is equipped with a Riemannian metric g such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.4}$$

then (g, φ, ξ, η) is called an almost contact metric structure. So, the quintuple $(M^{2m+1}, g, \varphi, \xi, \eta)$ is an almost contact metric manifold. As in the case of almost Hermitian manifolds, any almost contact metric manifold admits a fundamental 2-form ϕ defined by

$$\phi(X, Y) = g(X, \varphi Y). \tag{2.5}$$

In this case, we will be interested in the following structures:

(1) quasi-K-cosymplectic if

$$(\nabla_X \varphi)Y + (\nabla_{\varphi X}\varphi)\varphi Y - \eta(Y)(\nabla_{\varphi X}\xi) = 0, \tag{2.6}$$

(2) quasi-Kenmotsu if $d\eta = 0$ and

$$(\nabla_X \phi)(Y, Z) + (\nabla_{\varphi X}\phi)(\varphi Y, Z) = \eta(Y)\phi(Z, X) + 2\eta(Z)\phi(X, Y), \tag{2.7}$$

(3) almost α-Kenmotsu if

$$d\phi(X, Y, Z) = \frac{\alpha}{3}\mathcal{G}(\eta(X)\phi(Y, Z)) \text{ and } d\eta = 0, \tag{2.8}$$

where α is real and $\mathcal{G}$ denotes the cyclic sum over X, Y and Z.

As examples of quasi-K-cosymplectic and quasi-Kenmotsu manifolds, we have the following. Let $l(t) = ce^t$ with $t \in \mathbb{R}$ and $c \in \mathbb{R}^*$. It is known that $S^2 \times \mathbb{R}^4$ is a quasi-Kähler manifold according to the almost complex structure defined by the Cayley numbers. Thus, using the warped product as treated by Kenmotsu, it can be shown that $M = \mathbb{R} \times_l (S^2 \times \mathbb{R}^4)$ is a quasi-Kenmotsu manifold.

In [10], the author showed that the product $S^2 \times \mathbb{R}^{2n+1}$ is a quasi-K-cosymplectic manifold.

3. Almost contact metric submersions

In [14], O'Neill defined a Riemannian submersion as a surjective mapping

$$\pi : M \longrightarrow B$$

between 2 Riemannian manifolds such that (i) π is of maximal rank and (ii) $\pi_*/(ker\pi_*)^\perp$ is a linear isometry.

The tangent bundle TM, of the total space M, admits an orthogonal decomposition

$$TM = \mathcal{V}(M) \oplus \mathcal{H}(M), \tag{3.1}$$

where $\mathcal{V}(M)$ and $\mathcal{H}(M)$ denote respectively the vertical and horizontal distributions. We denote by h and v the vertical and horizontal projections, respectively.

A vector field X of the horizontal distribution is called a basic vector field if it is π-related to a vector field X_* of the base space B. Such a vector field means that $\pi_* X = X_*$.

On the base space, tensors and other objects will be denoted by a prime $'$ while those tangent to the fibers will be specified by a carret $\hat{}$.

When the base space is an almost Hermitian manifold, $(B^{2m'}, g', J')$, the Riemannian submersion

$$\pi : M^{2m+1} \to B^{2m'}$$

is called an almost contact metric submersion of type II [17] if

$$\pi_* \varphi = J' \pi_*.$$

This type of submersion is called (φ, J)-holomorphic in [5].

Proposition 3.1 *Let $\pi : M^{2m+1} \longrightarrow B^{2m'}$ be an almost contact metric submersion of type II. Then*

(a) $\pi^* \Omega' = \phi$,

(b) $\eta(X) = 0$, $\forall X \in H(M)$.

Proof See Watson [17]. □

Proposition 3.2 *The fibers of an almost contact metric submersion of type II are almost contact metric manifolds.*

Proof Since the total space is of dimension $2m+1$ and the base space has $2m'$ as its dimension, the fibers have dimension $2(m-m')+1$. This shows that the dimension of the fibers is odd. Let $(\hat{g}, \hat{\varphi}, \hat{\xi}, \hat{\eta})$ be the restriction of the almost contact metric structure (g, φ, ξ, η) of the total space on the fibers. We have to show that $(\hat{g}, \hat{\varphi}, \hat{\xi}, \hat{\eta})$ is an almost contact metric structure. In fact, (i) $\hat{\varphi}^2 U = -U + \hat{\eta}(U)\hat{\xi}$, (ii) $(\hat{\eta}) = \hat{g}(\hat{\xi}, \hat{\xi}) = g(\xi, \xi) = 1$, (iii) $\hat{g}(\hat{\varphi} U, \hat{\varphi} V) = -\hat{g}(U, \hat{\varphi}^2 V) = \hat{g}(U, V) - \hat{g}(U, \hat{\eta}(V)\hat{\xi})$. But $\hat{g}(U, \hat{\eta}(V)\hat{\xi}) = \hat{g}(U, \hat{\xi})\hat{\eta}(V) = \hat{\eta}(U)\hat{\eta}(V)$. Thus, $\hat{g}(\hat{\varphi U}, \hat{\varphi V}) = \hat{g}(U, V) - \hat{\eta}(U)\hat{\eta}(V)$, which completes the proof. □

Definition 3.3 [5] An almost Hermitian manifold (M, g, J) is called a $(1,2)$-symplectic manifold if

$$(\nabla_X J)Y + (\nabla_{JX} J)JY = 0, \ \ \forall X, Y \in \Gamma(TM). \tag{3.2}$$

The $(1,2)$-symplectic manifold is also called a quasi-Kähler manifold [11].

Proposition 3.4 *Let $\pi : M^{2m+1} \longrightarrow M'^{2m'}$ be an almost contact metric submersion of type II. If the total space is quasi-K-cosymplectic or quasi-Kenmotsu, then the base space is a quasi-Kähler manifold.*

Proof Note that all these manifolds have in common the following relation:

$$(\nabla_D \phi)(E, G) + (\nabla_{\varphi D} \phi)(\varphi E, G) = \alpha \eta(D) C,$$

where C is a factor determined by the class of the manifold. For instance, if $\alpha = 1$ and $C = \eta(E)(\nabla_{\varphi D} \xi)$, we get the defining relation of a quasi-K-cosymplectic structure. If $\alpha = 1$ and $C = \eta(E)\phi(G, D) +$

$2\eta(G)\phi(D,E)$, we obtain the principal defining relation of a quasi-Kenmotsu structure. Let X, Y, and Z be 3 basic vector fields. Since η vanishes on horizontal vector fields, the common relation becomes $(\nabla_X\phi)(Y,Z)+(\nabla_{\varphi X}\phi)(\varphi Y,Z)=0$. As $\pi^*\Omega'=\phi$, we get $(\nabla'_{X_*}\Omega')(Y_*,Z_*)+(\nabla'_{J'X_*}\Omega')(J'Y_*,Z_*)=0$. This relation means that the base space has the quasi-Kählerian structure. Recalling that $(\nabla'_{X_*}\Omega')(Y_*,Z_*)=g'((\nabla'_{X_*}J')Y_*,Z_*)$, and $(\nabla'_{J'X_*}\Omega')(J'Y_*,Z_*)=g'((\nabla'_{J'X_*}J')J'Y_*,Z_*)$, we then obtain $g'((\nabla'_{X_*}J')Y_*,Z_*)+g'((\nabla'_{J'X_*}J')J'Y_*,Z_*)=0$, which is equivalent to $g'((\nabla'_{X_*}J')Y_*)+(\nabla'_{J'X_*}J')J'Y_*,Z_*)=0$, from which we have $(\nabla'_{X_*}J')Y_*+(\nabla'_{J'X_*}J')J'Y_*=0$ following. By Definition 3.3, the base space is a quasi-Kähler manifold, that is, a $(1,2)$-symplectic manifold as noted in [5]. □

Proposition 3.5 *Let $\pi: M^{2m+1}\longrightarrow M'^{2m'}$ be an almost contact metric submersion of type II. If the base space is a quasi-Kähler manifold, then the horizontal distribution of the total space looks like a quasi-K-cosymplectic or a quasi-Kenmotsu manifold.*

Proof Let X, Y and Z be basic vector fields. It is known that on the base space $\pi_*X=X_*$, $\pi_*Y=Y_*$ and $\pi_*Z=Z_*$. Consider that the base space is defined by $(\nabla'_{X_*}J')Y_*+(\nabla'_{J'X_*}J')J'Y_*=0$. This implies that $(\nabla'_{X_*}\Omega')(Y_*,Z_*)+(\nabla'_{J'X_*}\Omega')(J'Y_*,Z_*)=0$. Using $\pi^*\Omega'=\phi$, we obtain $\pi^*(\nabla'_{X_*}\Omega')(Y_*,Z_*)=(\nabla_X\phi)(Y,Z)$ and $\pi^*(\nabla'_{J'X_*}\Omega')J'Y_*,Z_*)=(\nabla_{\varphi X}\phi)(\varphi Y,Z)$. These relations lead to $(\nabla_X\phi)(Y,Z)+(\nabla_{\varphi X}\phi)(\varphi Y,Z)=0$. Taking into account that η vanishes on the horizontal distribution, the last relation means that this distribution is of kind $(\nabla_X\phi)(Y,Z)+(\nabla_{\varphi X}\phi)(\varphi Y,Z)=\eta(Z)C$, which completes the proof. □

Proposition 3.6 *Let $\pi: M^{2m+1}\longrightarrow M'^{2m'}$ be an almost contact metric submersion of type II. Assume that the base space admits a $(\mathbf{1},\mathbf{2})$ symplectic structure. Then the total space is an almost α-Kenmotsu manifold.*

Proof If $(M'^{2m'},g',J')$ admits a $(\mathbf{1},\mathbf{2})$ symplectic structure, we have $d\Omega'=0$ on horizontal vector fields. Referring to Proposition 3.1, $\pi^*\Omega'=\phi$, which implies that $d(\pi^*\Omega')=d\phi$. On the other hand, taking $d\Omega'=0$ implies that $d\phi=0$. To get $d\phi=0$ on horizontal vector fields, turn to Proposition 3.1 (b), where η vanishes on horizontal distribution. Thus, we claim that the total space is an almost α-Kenmotsu manifold. □

4. Contact CR-submanifolds

In this section, we introduce the notion of the contact CR-submanifold of a manifold (see [18] for details). Let M be an finite-dimensional isometrically immersed submanifold of a $(2m+1)$-dimensional manifold $\overline{M}$ and let g be the metric tensor on $\overline{M}$ as well as the induced metric on M.

Definition 4.1 [18] A Riemannian submanifold M of a quasi-K-cosymplectic (resp. quasi-Kenmotsu) manifold $\overline{M}$ is called a contact CR-submanifold if ξ is tangent to M and there exists on M a differential distribution $D: x\longmapsto D_x\subset T_xM$ such that

(i) D_x is invariant under φ (i.e. $\varphi D_x\subset D_x$), for each $x\in M$;

(ii) the orthogonal complementary distribution $D^\perp: x\longmapsto D_x^\perp\subset T_xM$ of the distribution D on M is totally real (i.e. $\varphi D^\perp\subset T_xM^\perp$);

(iii) $TM = D \oplus D^{\perp} \oplus \{\xi\}$, where T_xM and $T_xM^{\perp}$ are the tangent space and the normal space of M at x, respectively, and $\oplus$ denotes the orthogonal direct sum.

We call D (resp. $D^{\perp}$) the horizontal (resp. vertical) distribution. We denote by g the metric tensor field of M as well as that induced on M. Let $\overline{\nabla}$ (resp. ∇) be the covariant differentiation with respect to the Levi-Civita connection on $\overline{M}$ (resp. M). The Gauss and Weingarten formulas for M are respectively given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{4.1}$$

$$\text{and } \overline{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \tag{4.2}$$

for any X, $Y \in \Gamma(TM)$, $V \in \Gamma(TM^{\perp})$, where $h : \Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TM^{\perp})$ is a normal bundle valued symmetric bilinear form on M, the linear operator A_V is the fundamental form tensor of Weingarten with respect to the normal section V, and the differential operator $\nabla^{\perp}$ defines a linear connection on the normal bundle $TM^{\perp}$, called the normal connection on M. Moreover, we have

$$g(h(X, Y), V) = g(A_V X, Y). \tag{4.3}$$

The submanifold M is said to be *totally geodesic* if h vanishes identically.

The projection of TM to D and $D^{\perp}$ are denoted by h and v, respectively, i.e. for any $X \in \Gamma(TM)$, we have

$$X = hX + vX + \eta(X)\xi. \tag{4.4}$$

Applying φ to X, we have,

$$\varphi X = FX + NX, \ \ \forall X \in \Gamma(TM), \tag{4.5}$$

where $FX = \varphi hX$ and $NX = \varphi vX$ are tangential and normal components of φX, respectively.

The normal bundle to M has the decomposition

$$TM^{\perp} = \varphi D^{\perp} \oplus \nu, \tag{4.6}$$

where ν denotes the orthogonal complementary distribution of $\varphi D^{\perp}$, and is an invariant normal subbundle of $TM^{\perp}$ under φ. For any $V \in TM^{\perp}$, we put

$$V = pV + qV, \tag{4.7}$$

where $pV \in \varphi D^{\perp}$, $qV \in \nu$. From the above equation, we have,

$$\varphi V = fV + nV, \ \ \forall V \in TM^{\perp}, \tag{4.8}$$

where $fV = \varphi pV \in D^{\perp}$ and $nV = \varphi qV \in \nu$.

5. Contact CR-submersions

Let M be a contact CR-submanifold of a quasi-K-cosymplectic (respectively, quasi-Kenmotsu) manifold $\overline{M}$ and M' be an almost contact metric manifold with the almost contact metric structure $(\varphi', \xi', \eta', g')$.

Next, we study the distributions involved and we characterize the horizontal one. Assume that there is a submersion $\pi : M \longrightarrow M'$ such that:

(i) $D^{\perp} = \ker(\pi_*)$, where $\pi_* : TM \longrightarrow TM'$ is the tangent mapping to π,

(ii) $\pi_* : D_x \oplus \{\xi\} \longrightarrow T_{\pi(x)}M'$ is an isometry for each x that satisfies: $\pi_* \circ \phi = \phi' \circ \pi_*$, $\eta = \eta' \circ \pi_*$, $\pi_*(\xi_x) = \xi'_{\pi(x)}$, where $T_{\pi(x)}M'$ denotes the tangent space of M' at $\pi(x)$.

Comparing tangential and normal components in (2.3) and (2.4), we obtain the next 2 Lemmas.

Lemma 5.1 *For a contact CR-submanifold M of a quasi-K-cosymplectic (resp. quasi-Kenmotsu) manifold $\overline{M}$, the following equalities hold:*

$$F^2 + fN = -I + \eta \otimes \xi, \tag{5.1}$$

$$NF + nN = 0, \tag{5.2}$$

$$Ff + fn = 0, \tag{5.3}$$

$$n^2 + Nf = -I. \tag{5.4}$$

Lemma 5.2 *Let M be a contact CR-submanifold M of an almost contact manifold $(\overline{M}, \varphi, \xi, \eta, g)$. Then,*

$$(\nabla_X F)Y - A_{NY}X - fh(X,Y) = F((\overline{\nabla}_X \varphi)Y), \tag{5.5}$$

$$(\nabla_X N)Y + h(X, FY) - nh(X,Y) = N((\overline{\nabla}_X \varphi)Y), \tag{5.6}$$

$$(\nabla_X f)V - A_{nV}X + FA_V X = f((\overline{\nabla}_X \varphi)V), \tag{5.7}$$

$$(\nabla^{\perp}_X n)V + h(X, fV) + NA_V X = n((\overline{\nabla}_X \varphi)V), \tag{5.8}$$

where $F((\overline{\nabla}_X \varphi)Y)$, $f((\overline{\nabla}_X \varphi)V)$, $n((\overline{\nabla}_X \varphi)V)$, and $N((\overline{\nabla}_X \varphi)Y)$ are, respectively, tangential and normal components of $(\overline{\nabla}_X \varphi)Y$ and $(\overline{\nabla}_X \varphi)V$, for any X, $Y \in \Gamma(TM)$ and $V \in \Gamma(TM^{\perp})$.

Proposition 5.3 *For a contact CR-submanifold M of a quasi-K-cosymplectic (resp. quasi-Kenmotsu) manifold $\overline{M}$, the following equalities hold:*

(i) $\ker(F) = D^{\perp} \oplus \{\xi\}$,

(ii) $\ker(N) = D \oplus \{\xi\}$,

(iii) $\ker(n) = ND^{\perp}$,

(iv) $\ker(f) = \nu$.

Proof (i) and (ii) are directly deduced from the definition of a contact CR-submanifold. For (iii), if $X \in D^{\perp}$, then, by (5.2), $nNX = -NFX = 0$, i.e. $nD^{\perp} \subset \ker(n)$. Conversely, let us consider $U \in \ker(n)$. From (5.3) and (5.4), it follows that $FfU = -fnU = 0$ and $U = -n^2U - NfU = -NfU$. From the first equality, $fU \in D^{\perp}$, and then the second one implies that $U \in ND^{\perp}$. Now let us prove (iv). For $V \in \ker(f)$, we have $fV = 0$ and, by (5.2) and (5.4), $0 = FfV + fnV = fnV$ and $n^2V + NfV = -V$, which implies $\varphi nV = n^2V = -V$, using (4.5). Applying φ and n to this equation, we have $n\varphi^2 nV = -n\varphi V$, i.e. $V = -n\varphi V \in \nu$. Thus, $\ker(f) \subset \nu$. For the other inclusion, notice that, for any V normal to M, $fV \in D^{\perp}$. Then, using (5.2) and

(5.4), $fnV = FfV = 0$ and $\varphi fV = NfV = -V - n^2V$, i.e. $n^2V = -V$. Therefore, $fV = 0$. □

For a quasi-Kenmostu manifold, (2.7) is equivalent to, for any X, $Y \in \Gamma(TM)$,

$$(\overline{\nabla}_X \varphi)Y - \varphi((\overline{\nabla}_{\varphi X}\varphi)Y) = g(\varphi X, Y)\xi - 2\eta(Y)\varphi X. \tag{5.9}$$

The covariant derivative of the structure vector field ξ is given, for a quasi-K-cosymplectic manifold, by,

$$\overline{\nabla}_X \xi = \varphi(\overline{\nabla}_{\varphi X}\xi), \;\; \forall X, \, Y \in \Gamma(TM), \tag{5.10}$$

and for a quasi-Kenmotsu manifold by

$$\overline{\nabla}_X \xi = -2\varphi^2 X + \varphi(\overline{\nabla}_{\varphi X}\xi), \;\; \forall X, \, Y \in \Gamma(TM). \tag{5.11}$$

Note that, for both ambient almost contact manifolds, the following identities hold:

$$\nabla_\xi \xi = 0 \;\text{ and }\; h(\xi, \xi) = 0. \tag{5.12}$$

Now we study the integrability of all the distributions involved in the definition of contact CR-submanifolds. First of all, we have:

Lemma 5.4 *For any $X \in \Gamma(D \oplus \{\xi\})$, $\varphi X = FX \in \Gamma(D \oplus \{\xi\})$.*

Proof For any $X \in \Gamma(D \oplus \{\xi\}$, $X = hX + \eta(X)\xi$. Applying φ to this equation, one has $\varphi X = \varphi hX + \eta(X)\varphi\xi$, i.e. $\varphi X = FX$. Therefore, for any $Y \in \Gamma(TM)$, $\overline{g}(\varphi X, vY) = -\overline{g}(X, \varphi vY)$. Since $\varphi vY \in \Gamma(\varphi D^\perp) \subset \Gamma(TM^\perp)$, we have $\overline{g}(X, \varphi vY) = 0$, that is, $\overline{g}(\varphi X, vY) = 0$, which completes the proof. □

The above lemma means that $D \oplus \{\xi\}$ is invariant under φ.

For some other considerations, the submanifold M may be considered to be of odd or even codimension, but while either the dimension of M is odd or even, the distribution D is always of even dimension.

Lemma 5.5 *Let M be a contact CR-submanifold of an almost contact manifold $\overline{M}$. Then, if $\overline{M}$ is quasi-K-cosymplectic, we have the following identities:*

$$\nabla_X \xi = F(\nabla_{FX}\xi) + fh(FX, \xi), \tag{5.13}$$

$$h(X, \xi) = N(\nabla_{FX}\xi) + nh(FX, \xi), \;\; \forall \, X \in \Gamma(D). \tag{5.14}$$

Moreover, if $\overline{M}$ is quasi-Kenmotsu, we have

$$\begin{aligned} \nabla_X \xi &= 2\{X - \eta(X)\xi\} + F(\nabla_{FX}\xi) + fh(FX, \xi), \\ h(X, \xi) &= N(\nabla_{FX}\xi) + nh(FX, \xi), \;\; \forall \, X \in \Gamma(D). \end{aligned} \tag{5.15}$$

Proof If $\overline{M}$ is a quasi-K-cosymplectic, from (5.10), one has, for any $X \in \Gamma(D)$,

$$\begin{aligned} \nabla_X \xi + h(X, \xi) &= \overline{\nabla}_X \xi = \varphi(\overline{\nabla}_{FX}\xi) = \varphi(\nabla_{FX}\xi) + \varphi h(FX, \xi) \\ &= F(\nabla_{FX}\xi) + N(\nabla_{FX}\xi) + fh(FX, \xi) + nh(FX, \xi). \end{aligned}$$

On the other hand, if $\overline{M}$ is a quasi-Kenmotsu, from (5.11), we get

$$\begin{aligned} \nabla_X \xi + h(X, \xi) &= \overline{\nabla}_X \xi = -2\varphi^2 X + \varphi(\nabla_{FX}\xi) + \varphi h(FX, \xi) \\ &= -2\varphi^2 X + F(\nabla_{FX}\xi) + N(\nabla_{FX}\xi) + fh(FX, \xi) + nh(FX, \xi). \end{aligned}$$

Then, comparing tangential and normal components of both sides of these equations, we complete the proof. □

Proposition 5.6 *Let M be a contact CR-submanifold of an almost contact manifold $\overline{M}$. Then, the following assertions hold:*

(i) The distributions D, $D^\perp$ and $D \oplus D^\perp$ are ξ-parallel if and only if $h(\xi, FX) \in \Gamma(\nu)$, for any $X \in \Gamma(D)$.

(ii) If $\overline{M}$ is quasi-K-cosymplectic (or quasi-Kenmotsu), then, for any $X \in \Gamma(D)$, $[X,\xi] \in \Gamma(D)$ if and only if $h(\xi, FX) \in \Gamma(\nu)$.

(iii) If $\overline{M}$ is quasi-Kenmotsu (or quasi-K-cosymplectic), then $[X,\xi] \in \Gamma(D^\perp)$, for any $X \in \Gamma(D^\perp)$.

Proof (i) For $X \in \Gamma(D)$, $Y \in \Gamma(D^\perp)$, using (5.5), we get

$$\begin{aligned} g(\nabla_\xi X, \xi) &= \xi(g(X,\xi)) - g(X, \nabla_\xi \xi) = 0, \\ g(\nabla_\xi X, Y) &= \xi(g(X,Y)) - g(X, \nabla_\xi Y) = g(F^2X, \nabla_\xi Y) = -g(FX, F\nabla_\xi Y) \\ &= -g(FX, \nabla_\xi FY) + g(FX, A_{NY}\xi) + g(FX, fh(\xi, Y)) \\ &= g(h(\xi, FX), \varphi Y), \end{aligned}$$

$\nabla_\xi X \in \Gamma(D^\perp)$ if and only if $h(\xi, FX) \in \nu$. Similarly, we can proceed for $D^\perp$. Finally, if D and $D^\perp$ are ξ-parallel, then $D \oplus D^\perp$ also is. (ii) If M is a contact CR-submanifold of a quasi-K-cosymplectic manifold $\overline{M}$, then, by (5.13) and (5.5), and for any $X \in \Gamma(TM)$ and $Y \in \Gamma(D^\perp)$, we have

$$g(\nabla_X \xi, \xi) = 0, \tag{5.16}$$

$$\begin{aligned} g(\nabla_X \xi, Y) &= g(\overline{\nabla}_X \xi, Y) - g(h(X,\xi), Y) = g(\varphi(\overline{\nabla}_{\varphi X}\xi), Y) \\ &= -g(h(\varphi X, \xi), \varphi Y). \end{aligned} \tag{5.17}$$

Then, $\nabla_X \xi \in \Gamma(D)$ if and only if $g(h(\varphi X, \xi) \in \nu$. Consequently, $[X,\xi] = \nabla_X \xi - \nabla_\xi X \in \Gamma(D)$ if and only if $g(h(\varphi X, \xi) \in \nu$. On the other hand, if M is a contact CR-submanifold of a quasi-Kenmotsu manifold $\overline{M}$, for any $X \in \Gamma(TM)$ and $Y \in \Gamma(D^\perp)$ and since $\varphi D^\perp \subset TM^\perp$, we have

$$\begin{aligned} g(\nabla_X \xi, Y) &= g(\overline{\nabla}_X \xi, Y) - g(h(X,\xi), Y) = -2g(\varphi^2 X, Y) + g(\varphi(\overline{\nabla}_{\varphi X}\xi), Y) \\ &= 2g(X,Y) + g(\varphi(\nabla_{\varphi X}\xi), Y) + g(\varphi h(\varphi X, \xi), Y) \\ &= 2g(X,Y) - g(h(\varphi X, \xi), \varphi Y). \end{aligned} \tag{5.18}$$

The latter vanishes if and only if $g(h(\varphi X, \xi) \in \nu$, for any $X \in \Gamma(D)$. Thus, $[X,\xi] = \nabla_X \xi - \nabla_\xi X \in \Gamma(D)$ if and only if $g(h(\varphi X, \xi) \in \nu$, $\forall X \in \Gamma(D)$. The assertion (iii) is obvious, using the relations in (2.6) and (5.9), which completes the proof. □

The differential of the second fundamental form ϕ in (2.5) gives, for any X, Y, $Z \in \Gamma(TM)$,

$$\begin{aligned} 3d\phi(X,Y,Z) &= X(\phi(Y,Z)) + Y(\phi(Z,X)) + Z(\phi(X,Y)) \\ &\quad - \phi([X,Y], Z) - \phi([Z,X], Y) - \phi([Y,Z], X). \end{aligned} \tag{5.19}$$

Using this differential, we have, for any Y, $Z \in \Gamma(D^{\perp})$,

$$3d\phi(X,Y,Z) = -\phi([Y,Z],X) = -g([Y,Z],\varphi X) = g(\varphi[Y,Z],X). \tag{5.20}$$

So, $d\phi(X,Y,Z) = 0$ if and only if $[Y,Z] \in \ker(F) = D^{\perp} \oplus \{\xi\}$. This is equivalent to

$$[Y,Z] = v[Y,Z] + \eta([Y,Z])\xi.$$

But,

$$\begin{aligned}\eta([Y,Z]) &= g(\xi,\overline{\nabla}_Y Z) - g(\xi,\overline{\nabla}_Z Y)\\ &= g(\overline{\nabla}_Z\xi,Y) - g(\overline{\nabla}_Y\xi,Z).\end{aligned}$$

If $\overline{M}$ is a quasi-K-cosymplectic manifold, we have, $\overline{\nabla}_Z\xi = \varphi(\overline{\nabla}_{\varphi Z}\xi)$ and this implies that, for any Y, $Z \in \Gamma(D^{\perp})$,

$$\begin{aligned}g(\overline{\nabla}_Z\xi,Y) &= -g(\nabla_{\varphi Z}\xi,\varphi Y) - g(h(\varphi Z,\xi),\varphi Y)\\ &= -g(A_{\varphi Y}\xi,\varphi Z) = 0,\end{aligned} \tag{5.21}$$

since $A_{\varphi Y}\xi \in \Gamma(TM)$ and $\varphi Z \in \Gamma(\varphi D^{\perp})$. Consequently, $\eta([Y,Z]) = 0$ and $[Y,Z] \in D^{\perp}$. On the other hand, if $\overline{M}$ is a quasi-Kenmotsu manifold, then, by its definition, $0 = 2d\eta(Y,Z) = -\eta([Y,Z])$ and $[Y,Z] \in D^{\perp}$. Therefore, we have:

Lemma 5.7 *Let M be a contact CR-submanifold of a quasi-K-cosymplectic (or quasi-Kenmotsu) manifold $\overline{M}$. The distribution $D^{\perp}$ is integrable if and only if $d\phi(X,Y,Z) = 0$, for any X tangent to M and Y, $Z \in \Gamma(D^{\perp})$.*

From this Lemma, we deduce:

Theorem 5.8 *Let M be a contact CR-submanifold of a quasi-Kenmotsu manifold $\overline{M}$. Then, the distribution $D^{\perp}$ is always integrable .*

Proof For any X, Y, $Z \in \Gamma(D^{\perp})$, $3d\phi(X,Y,Z) = -g([Y,Z],\varphi X) = 0$, since $[Y,Z] \in \Gamma(TM)$ and $\varphi X \in \Gamma(\varphi D^{\perp}) \subset \Gamma(TM^{\perp})$. By Lemma 5.7, we complete the proof. □

Finally, we characterize the integrability of $D \oplus \{\xi\}$.

Theorem 5.9 *Let M be a contact CR-submanifold of a quasi-K-cosymplectic manifold $\overline{M}$. If the horizontal distribution $D \oplus \{\xi\}$ is integrable, then,*

$$h(FX,Y) = h(X,FY), \;\; \forall X,\, Y \in \Gamma(D \oplus \{\xi\}). \tag{5.22}$$

Proof For any X, $Y \in \Gamma(D \oplus \{\xi\})$, we have,

$$\begin{aligned}\varphi[\varphi X,\varphi Y] &= \overline{\nabla}_{\varphi X}\varphi^2 Y - (\overline{\nabla}_{\varphi X}\varphi)\varphi Y - \overline{\nabla}_{\varphi Y}\varphi^2 X + (\overline{\nabla}_{\varphi Y}\varphi)\varphi X\\ &= -\overline{\nabla}_{\varphi X}Y + \varphi X(\eta(Y))\xi + \eta(Y)\overline{\nabla}_{\varphi X}\xi - (\overline{\nabla}_{\varphi X}\varphi)\varphi Y\\ &+ \overline{\nabla}_{\varphi Y}X - \varphi Y(\eta(X))\xi - \eta(X)\overline{\nabla}_{\varphi Y}\xi + (\overline{\nabla}_{\varphi Y}\varphi)\varphi X\\ &= \overline{\nabla}_{\varphi Y}X - \overline{\nabla}_{\varphi X}Y + \{\varphi X(\eta(Y)) - \varphi Y(\eta(X))\}\xi + \eta(Y)(\overline{\nabla}_{\varphi X}\xi)\\ &- (\overline{\nabla}_{\varphi X}\varphi)\varphi Y - \eta(X)(\overline{\nabla}_{\varphi Y}\xi) + (\overline{\nabla}_{\varphi Y}\varphi)\varphi X.\end{aligned} \tag{5.23}$$

Likewise, using the Gauss equation, we get, for any X, $Y \in \Gamma(D \oplus \{\xi\})$,

$$\begin{aligned}\varphi[X,Y] = \nabla_X \varphi Y - \nabla_Y \varphi X + (\overline{\nabla}_Y \varphi)X - (\overline{\nabla}_X \varphi)Y \\ + h(X, \varphi Y) - h(\varphi X, Y),\end{aligned} \tag{5.24}$$

since $\overline{M}$ is a quasi-K-cosymplectic manifold. Then, we have,

$$(\overline{\nabla}_X \varphi)Y + (\overline{\nabla}_{\varphi X} \varphi)\varphi Y = \eta(Y)(\overline{\nabla}_{\varphi X}\xi),$$

and the relation (5.23) becomes

$$\begin{aligned}\varphi[\varphi X, \varphi Y] = \nabla_{\varphi Y} X - \nabla_{\varphi X} Y + h(X, \varphi Y) - h(\varphi X, Y) \\ + \{\varphi X(\eta(Y)) - \varphi Y(\eta(X))\}\xi + (\overline{\nabla}_X \varphi)Y - (\overline{\nabla}_Y \varphi)X.\end{aligned} \tag{5.25}$$

Adding (5.25) and (5.24), one obtains

$$\begin{aligned}\varphi[\varphi X, \varphi Y] + \varphi[X,Y] + \{\varphi Y(\eta(X)) - \varphi X(\eta(Y))\}\xi \\ = \nabla_{\varphi Y} X - \nabla_{\varphi X} Y + \nabla_X \varphi Y - \nabla_Y \varphi X \\ + 2\{h(X, \varphi Y) - h(\varphi X, Y)\}.\end{aligned}$$

If $D \oplus \{\xi\}$ is integrable and since $\varphi X = FX$, for any $X \in \Gamma(D \oplus \{\xi\})$, the terms on the left-hand side are tangential to M. Then, equating normal components in the above equation, we obtain the desired relation. □

A vector field X on M is said to be basic if $X \in \Gamma(D_x \oplus \{\xi\})$ and X is π-related to a vector field on M', i.e. there exists a vector field $X_* \in TM'$ such that $\pi_*(X_x) = X_{*\pi(x)}$, for each $x \in M$. Note that, by condition (ii) above Lemma 5.1, it shows that the structural vector field ξ is a basic vector field.

Lemma 5.10 *[15] Let X and Y be basic vector fields on M. Then*

(i) $g(X,Y) = g'(X_*, Y_*) \circ \pi$*;*

(ii) the component $h([X,Y]) + \eta([X,Y])\xi$ *of* $[X,Y]$ *is a basic vector field and corresponds to* $[X_*, Y_*]$*, i.e.* $\pi_*(h([X,Y]) + \eta([X,Y])\xi) = [X_*, Y_*]$*;*

(iii) $[U, X] \in D^\perp$*, for any* $U \in D^\perp$*;*

(iv) $h([X,Y]) + \eta([X,Y])\xi$ *is a basic vector field corresponding to* $\nabla^*_{X_*} Y_*$*, where* ∇^* *denotes the Levi-Civita connection on* M'*.*

For basic vector fields on M, we define the operator $\widetilde{\nabla}^*$ corresponding to ∇^* by setting, for any X, $Y \in \Gamma(D \oplus \{\xi\})$,

$$\widetilde{\nabla}^*_X Y = h(\nabla_X Y) + \eta(\nabla_X Y)\xi. \tag{5.26}$$

By (iv) of Lemma 5.10, $\widetilde{\nabla}^*_X Y$ is a basic vector field, and we have

$$\pi_*(\widetilde{\nabla}^*_X Y) = \nabla^*_{X_*} Y_*. \tag{5.27}$$

Define the tensor field C by, for any X, $Y \in \Gamma(D \oplus \{\xi\})$,

$$\nabla_X Y = \widetilde{\nabla}^*_X Y + C(X, Y), \tag{5.28}$$

where $C(X, Y)$ is the vertical part of $\nabla_X Y$. It is known that C is skew-symmetric and satisfies

$$C(X, Y) = \frac{1}{2} v[X, Y], \quad X,\, Y \in \Gamma(D \oplus \{\xi\}). \tag{5.29}$$

Next, we want to examine the influence of a given structure defined on the ambient $\overline{M}$ on the determination of the corresponding structure on the contact CR-submanifold M and the base space M'.

The curvature tensors R, R^* of the connection ∇, ∇^* on M and M', respectively, are related by [15], for any X, Y, Z, $W \in \Gamma(D \oplus \{\xi\})$,

$$\begin{aligned} R(X, Y, Z, W) = R^*(X_*, Y_*, Z_*, W_*) - g(C(Y, Z), C(X, W)) \\ + g(C(X, Z), C(Y, W)) + 2g(C(X, Y), C(Z, W)), \end{aligned} \tag{5.30}$$

where $\pi_* X = X_*$, $\pi_* Y = Y_*$, $\pi_* Z = Z_*$, and $\pi_* W = W_*$.

We now pay attention to the different ambient manifolds involved, namely quasi-K-cosymplectic and quasi-Kenmotsu manifolds. First of all, we have, for any X, $Y \in \Gamma(D \oplus \{\xi\})$,

$$\begin{aligned} \overline{\nabla}_X Y &= \nabla_X Y + h(X, Y) = \nabla_X Y + ph(X, Y) + qh(X, Y) \\ &= \widetilde{\nabla}^*_X Y + C(X, Y) + ph(X, Y) + qh(X, Y). \end{aligned} \tag{5.31}$$

Using this, we have

$$\varphi(\overline{\nabla}_X Y) = \varphi \widetilde{\nabla}^*_X Y + \varphi C(X, Y) + \varphi ph(X, Y) + \varphi qh(X, Y). \tag{5.32}$$

Replacing Y with φY into the relation (5.31), we obtain

$$\overline{\nabla}_X \varphi Y = \widetilde{\nabla}^*_X \varphi Y + C(X, \varphi Y) + ph(X, \varphi Y) + qh(X, \varphi Y). \tag{5.33}$$

If $\overline{M}$ is a quasi-K-cosymplectic manifold, we find

$$(\overline{\nabla}_X \varphi) Y = \overline{\nabla}_X \varphi Y - \varphi(\overline{\nabla}_X Y) = -(\overline{\nabla}_X \varphi)\varphi Y + \eta(Y)(\overline{\nabla}_{\varphi X} \xi). \tag{5.34}$$

Substituting (5.32) and (5.33) in (5.34), one obtains

$$\begin{aligned} &\widetilde{\nabla}^*_X \varphi Y + C(X, \varphi Y) + ph(X, \varphi Y) + qh(X, \varphi Y) - \varphi \widetilde{\nabla}^*_X Y \\ &- \varphi C(X, Y) - \varphi ph(X, Y) - \varphi qh(X, Y) \\ &= -(\overline{\nabla}_{\varphi X} \varphi)\varphi Y + \eta(Y)(\overline{\nabla}_{\varphi X} \xi). \end{aligned} \tag{5.35}$$

On the other hand, if $\overline{M}$ is a quasi-Kenmotsu manifold, we get

$$(\overline{\nabla}_X \varphi) Y = \varphi((\overline{\nabla}_{\varphi X} \varphi) Y) + g(\varphi X, Y)\xi - 2\eta(Y)\varphi X. \tag{5.36}$$

Putting (5.32) and (5.33) in (5.36), with $(\overline{\nabla}_X\varphi)Y = \overline{\nabla}_X\varphi Y - \varphi(\overline{\nabla}_X Y)$, one has

$$\begin{aligned}
&\widetilde{\nabla}^*_X\varphi Y + C(X,\varphi Y) + ph(X,\varphi Y) + qh(X,\varphi Y) - \varphi\widetilde{\nabla}^*_X Y \\
&- \varphi C(X,Y) - \varphi ph(X,Y) - \varphi qh(X,Y) \\
&= \varphi((\overline{\nabla}_{\varphi X}\varphi)Y) + g(\varphi X,Y)\xi - 2\eta(Y)\varphi X.
\end{aligned} \tag{5.37}$$

We have the following results.

Theorem 5.11 *Let $\pi : M \longrightarrow M'$ be a submersion of a contact CR-submanifold of a manifold $\overline{M}$ onto an almost contact metric manifold M'. Then:*

(i) If $\overline{M}$ is quasi-K-cosymplectic, for any X, $Y \in \Gamma(D \oplus \{\xi\})$,

$$(\widetilde{\nabla}^*_X\varphi)Y + (\widetilde{\nabla}^*_{\varphi X}\varphi)\varphi Y = \eta(Y)\widetilde{\nabla}^*_{\varphi X}\xi, \tag{5.38}$$

$$C(X,\varphi Y) - C(\varphi X,Y) = f\{h(X,Y) + h(\varphi X,\varphi Y), \tag{5.39}$$

$$q\{h(X,\varphi Y) - h(\varphi X,Y)\} = n\{h(X,Y) + h(\varphi X,\varphi Y)\}, \tag{5.40}$$

$$p\{h(X,\varphi Y) - h(\varphi X,Y)\} = \varphi\{C(X,Y) + C(\varphi X,\varphi Y). \tag{5.41}$$

(ii) If $\overline{M}$ is quasi-Kenmotsu, for any X, $Y \in \Gamma(D \oplus \{\xi\})$,

$$(\widetilde{\nabla}^*_X\varphi)Y - \varphi((\widetilde{\nabla}^*_{\varphi X}\varphi)Y) = g(\varphi X,Y)\xi - 2\eta(Y)\varphi X, \tag{5.42}$$

$$C(X,\varphi Y) - C(\varphi X,Y) = fh(X,Y), \tag{5.43}$$

$$C(X,Y) = -C(\varphi X,\varphi Y), \tag{5.44}$$

$$ph(X,\varphi Y) = \varphi qh(X,Y). \tag{5.45}$$

Proof (i) If $\overline{M}$ is a quasi-K-cosymplectic manifold, we have,

$$\overline{\nabla}_{\varphi X}\xi = \nabla_{\varphi X}\xi + h(\varphi X,\xi) = \widetilde{\nabla}^*_{\varphi X}\xi + C(\varphi X,\xi) + h(\varphi X,\xi), \tag{5.46}$$

and

$$\begin{aligned}
(\overline{\nabla}_{\varphi X}\varphi)\varphi Y &= \overline{\nabla}_{\varphi X}\varphi^2 Y - \varphi(\overline{\nabla}_{\varphi X}\varphi Y) \\
&= \nabla_{\varphi X}\varphi^2 Y + h(\varphi X,\varphi^2 Y) - \varphi(\nabla_{\varphi X}\varphi Y + h(\varphi X,\varphi Y)) \\
&= \widetilde{\nabla}^*_{\varphi X}\varphi^2 Y + C(\varphi X,\varphi^2 Y) + h(\varphi X,\varphi^2 Y) - \varphi(\widetilde{\nabla}^*_{\varphi X}\varphi Y) \\
&- \varphi C(\varphi X,\varphi Y) - \varphi h(\varphi X,\varphi Y) \\
&= (\widetilde{\nabla}^*_{\varphi X}\varphi)\varphi Y - C(\varphi X,Y) + \eta(Y)C(\varphi X,\xi) - h(\varphi X,Y) \\
&+ \eta(Y)h(\varphi X,\xi) - \varphi C(\varphi X,\varphi Y) - \varphi h(\varphi X,\varphi Y)
\end{aligned} \tag{5.47}$$

for any X, $Y \in \Gamma(D \oplus \{\xi\})$. Putting the pieces (5.46) and (5.47) into (5.35), we have

$$\begin{aligned}
&(\widetilde{\nabla}^*_X\varphi)Y + C(X,\varphi Y) + ph(X,\varphi Y) + qh(X,\varphi Y) - \varphi C(X,Y) \\
&- \varphi nh(X,Y) - \varphi qh(X,Y) \\
&= -(\widetilde{\nabla}^*_{\varphi X}\varphi)\varphi Y + \eta(Y)\widetilde{\nabla}^*_{\varphi X}\xi + C(\varphi X,Y) + h(\varphi X,Y) \\
&+ \varphi C(\varphi X,\varphi Y) + \varphi h(\varphi X,\varphi Y). \qquad (5.48)
\end{aligned}$$

Comparing the components of $D \oplus \{\xi\}$, $D^\perp$, $\varphi D^\perp$, and ν, respectively, on both sides of (5.48), we find

$$\begin{aligned}
(\widetilde{\nabla}^*_X\varphi)Y + (\widetilde{\nabla}^*_{\varphi X}\varphi)\varphi Y &= \eta(Y)\widetilde{\nabla}^*_{\varphi X}\xi, \\
C(X,\varphi Y) - C(\varphi X,Y) &= \varphi p\{h(X,Y) + h(\varphi X,\varphi Y), \\
q\{h(X,\varphi Y) - h(\varphi X,Y)\} &= \varphi q\{h(X,Y) + h(\varphi X,\varphi Y)\}, \\
p\{h(X,\varphi Y) - h(\varphi X,Y)\} &= \varphi\{C(X,Y) + C(\varphi X,\varphi Y)\}.
\end{aligned}$$

(ii) Suppose that $\overline{M}$ is a quasi-Kenmotsu manifold. Using the fact that C is vertical, for any X, $Y \in \Gamma(D\oplus\{\xi\})$,

$$\begin{aligned}
\varphi(\overline{\nabla}_{\varphi X}\varphi)Y &= \varphi(\overline{\nabla}_{\varphi X}\varphi Y) - \varphi^2(\overline{\nabla}_{\varphi X}Y) \\
&= \varphi(\widetilde{\nabla}^*_{\varphi X}\varphi Y) - \varphi^2(\widetilde{\nabla}^*_{\varphi X}Y) + \varphi C(\varphi X,\varphi Y) - \varphi^2(C(\varphi X,Y)) \\
&= \varphi((\widetilde{\nabla}^*_{\varphi X}\varphi)Y) + \varphi C(\varphi X,\varphi Y) + C(\varphi X,Y). \qquad (5.49)
\end{aligned}$$

Putting (5.49) in (5.37), we get,

$$\begin{aligned}
&(\widetilde{\nabla}^*_X\varphi)Y + C(X,\varphi Y) + ph(X,\varphi Y) + qh(X,\varphi Y) - \varphi C(X,Y) \\
&- \varphi ph(X,Y) - \varphi qh(X,Y) \\
&= \varphi((\widetilde{\nabla}^*_{\varphi X}\varphi)Y) + g(\varphi X,Y)\xi - 2\eta(Y)\varphi X + \varphi C(\varphi X,\varphi Y) \\
&+ C(\varphi X,Y). \qquad (5.50)
\end{aligned}$$

Also, comparing the components of $D \oplus \{\xi\}$, $D^\perp$, $\varphi D^\perp$ and ν, respectively, on both sides of (5.50), we have $(\widetilde{\nabla}^*_X\varphi)Y - \varphi((\widetilde{\nabla}^*_{\varphi X}\varphi)Y) = g(\varphi X,Y)\xi - 2\eta(Y)\varphi X$, $C(X,\varphi Y) - C(\varphi X,Y) = \varphi ph(X,Y)$, $C(X,Y) = -C(\varphi X,\varphi Y)$ and $ph(X,\varphi Y) = \varphi qh(X,Y)$, which completes the proof. □

Following the nature of ambient manifolds, that is, if $\overline{M}$ is a quasi-K-cosymplectic manifold, for any $X \in \Gamma(D \oplus \{\xi\})$,

$$C(X,\varphi X) = \frac{1}{2}\varphi p\{h(X,X) + h(\varphi X,\varphi X)\}, \qquad (5.51)$$

and if $\overline{M}$ is a quasi-Kenmotsu manifold, for any $X \in \Gamma(D \oplus \{\xi\})$,

$$C(X,\varphi X) = \frac{1}{2}\varphi ph(X,X). \qquad (5.52)$$

Lemma 5.12 *Let $\pi : M \longrightarrow M'$ be a submersion of a contact CR-submanifold of a quasi-K-cosymplectic manifold $\overline{M}$ onto an almost contact metric manifold M'. Then, $C(\xi,\xi) = 0$, $h(\xi,\xi) = 0$, and $C(X,\xi) = \varphi ph(\varphi X,\xi)$, $\forall\, X \in \Gamma(D \oplus \{\xi\})$.*

Proof Putting $Y = \xi$ in the relation (iv) of Theorem 5.11 and using the fact that $\varphi\xi = 0$, we get $ph(\varphi X, \xi) = -\varphi C(X, \xi)$. Applying φ to this equation and using (2.3), we have, for any $X \in \Gamma(D \oplus \{\xi\})$, $\varphi ph(\varphi X, \xi) = -\varphi^2 C(X, \xi) = C(X, \xi) - \eta(C(X, \xi))\xi = C(X, \xi)$, since $\eta(C(X, \xi)) = 0$ because of the fact that C is vertical and ξ is a basic vector, and this proves the last relation. The first 2 relations are obvious. □

Theorem 5.13 *Let $\pi : M \longrightarrow M'$ be a submersion of a contact CR-submanifold of a manifold $\overline{M}$ onto an almost contact metric manifold M'. Then:*

(1) If $\overline{M}$ is quasi-K-cosymplectic, then M' is also a quasi-K-cosymplectic manifold.

(2) If $\overline{M}$ is quasi-Kenmotsu, then M is $D \oplus \{\xi\}$-totally geodesic and M' is also a quasi-Kenmotsu manifold.

Proof (1) Using (i) of Theorem 5.11, we have, for any X, $Y \in \Gamma(D \oplus \{\xi\})$,

$$(\widetilde{\nabla}^*_X \varphi)Y + (\widetilde{\nabla}^*_{\varphi X} \varphi)\varphi Y = \eta(Y)\widetilde{\nabla}^*_{\varphi X}\xi.$$

Applying π_* to the above equation and using Lemma 5.10, equation (5.27), we derive

$$\pi_*((\widetilde{\nabla}^*_X \varphi)Y) + \pi_*((\widetilde{\nabla}^*_{\varphi X} \varphi)\varphi Y) = \pi_*(\eta(Y)\widetilde{\nabla}^*_{\varphi X}\xi).$$

That is,

$$(\nabla^*_{X_*}\varphi')Y_* + (\nabla^*_{\varphi' X_*}\varphi')\varphi' Y_* = \eta'(Y_*)\nabla^*_{\varphi' X_*}\xi',$$

which proves that M' is a quasi-K-cosymplectic manifold. (2) From (5.45), we have $ph(X, Y) = 0$ and $qh(X, Y) = 0$, and, therefore, $h(X, Y) = 0$, $\forall X$, $Y \in \Gamma(D \oplus \{\xi\})$. This proves that M is $D \oplus \{\xi\}$-totally geodesic. The last assertion follows from (5.42), mimicking the techniques used in (1). □
By Proposition 3.4, we deduce the following results.

Theorem 5.14 *Let $\pi : M \longrightarrow M'$ be a submersion of type II of contact CR-submanifold of a quasi-K-cosymplectic (or quasi-Kenmotsu) manifold $\overline{M}$ onto an almost contact metric manifold M' with $\dim D^{\perp} = 2k$. Then, the base space M' is a quasi-Kähler manifold.*

It is known that by a result of Chen [6] that the antiinvariant distribution $D^{\perp}$ of a CR-submanifold of a Kähler manifold is always integrable. This is still true for a CR-submanifold of a locally conformal Kähler manifold [12]. Now, we have:

Theorem 5.15 *Let $\pi : M \longrightarrow M'$ be a submersion of a contact CR-submanifold of a quasi-K-cosymplectic (or quasi-Kenmotsu) manifold $\overline{M}$ onto an almost contact metric manifold M'. If the horizontal distribution $D \oplus \{\xi\}$ is integrable and the vertical distribution $D^{\perp}$ is parallel, then M is CR-product.*

Proof Since the horizontal distribution $D \oplus \{\xi\}$ is integrable, then, for any X, $Y \in \Gamma(D \oplus \{\xi\})$, we have $[X, Y] \in \Gamma(D \oplus \{\xi\})$. Therefore, $v[X, Y] = 0$. Now, using the equation (5.29), we have $C(X, Y) = 0$, $\forall X$, $Y \in \Gamma(D \oplus \{\xi\})$. Putting the value of $C(X, Y)$ in (5.28), we have $\nabla_X Y = \widetilde{\nabla}^*_X Y \in \Gamma(D \oplus \{\xi\})$, which shows that $D \oplus \{\xi\}$ is parallel since the horizontal distribution $D \oplus \{\xi\}$ and vertical distribution $D^{\perp}$ are both parallel. Thus, using de Rham's theorem, it follows that M is the product $M_1 \times M_2$, where M_1 is the invariant submanifold of $\overline{M}$ and M_2 is the totally real submanifold of $\overline{M}$. Hence, M is a CR-product. □

6. Curvature properties

Next, we discuss the holomorphic sectional curvature of quasi- K-cosymplectic, quasi-Kenmotsu manifold $\overline{M}$ and M', respectively.

Let $\pi : M \longrightarrow M'$ be a submersion of a contact CR-submanifold of a manifold $\overline{M}$.

For any manifold $\overline{M}$ and putting $Y = \varphi X$, $Z = \varphi Y$, $W = Y$ in the Gauss equation,

$$\overline{R}(X,Y,Z,W) = R(X,Y,Z,W) - g(h(X,W),h(Y,Z)) + g(h(X,Z),h(Y,W)), \tag{6.1}$$

to obtain the following equation, for any X, $Y \in \Gamma(D \oplus \{\xi\})$,

$$\overline{R}(X,\varphi X,\varphi Y,Y) = R(X,\varphi X,\varphi Y,Y) - g(h(X,Y),h(\varphi X,\varphi Y)) + g(h(X,\varphi Y),h(\varphi X,Y)). \tag{6.2}$$

Substituting $h = ph + qh$, in the above equation and using (5.30), we derive

$$\begin{aligned}\overline{R}(X,\varphi X,\varphi Y,Y) &= R(X,\varphi X,\varphi Y,Y) - g(h(X,Y),h(\varphi X,\varphi Y))\\ &+ g(h(X,\varphi Y),h(\varphi X,Y))\\ &= R^*(X_*,\varphi' X_*,\varphi' Y_*,Y_*) - g(C(X,Y),C(\varphi X,\varphi Y))\\ &+ g(C(X,\varphi Y),C(\varphi X,Y)) + 2g(C(X,\varphi X),C(\varphi Y,Y))\\ &- g(ph(X,Y),ph(\varphi X,\varphi Y)) - g(qh(X,Y),qh(\varphi X,\varphi Y))\\ &+ g(ph(X,\varphi Y),ph(\varphi X,Y)) + g(qh(X,\varphi Y),qh(\varphi X,Y)).\end{aligned} \tag{6.3}$$

Suppose that the distribution $D \oplus \{\xi\}$ is integrable. Then, we have

$$C(X,Y) = \frac{1}{2}v[X,Y] = 0, \tag{6.4}$$

for any X, $Y \in \Gamma(D \oplus \{\xi\})$. Thus, from the definition of C, we have $\nabla_X Y = \widetilde{\nabla}^*_X Y \in \Gamma(D \oplus \{\xi\})$, i.e. $D \oplus \{\xi\}$ is parallel. By relation (5.22) and since $\varphi X = FX$, $h(\varphi X, \xi) = 0$, which implies that $h(X,\xi) = 0$, since $h(\xi,\xi) = 0$. Taking $Y = \varphi Y$ in (5.22), one obtains

$$h(\varphi X,\varphi Y) = -h(X,Y), \;\; \forall X,\; Y \in \Gamma(D \oplus \{\xi\}). \tag{6.5}$$

Using this, the relation (6.3) becomes, for any X, $Y \in \Gamma(D \oplus \{\xi\})$,

$$\begin{aligned}\overline{R}(X,\varphi X,\varphi Y,Y) &= R^*(X_*,\varphi' X_*,\varphi' Y_*,Y_*) - g(ph(X,Y),ph(\varphi X,\varphi Y))\\ &- g(qh(X,Y),qh(\varphi X,\varphi Y)) + g(ph(X,\varphi Y),ph(\varphi X,Y))\\ &+ g(qh(X,\varphi Y),qh(\varphi X,Y))\\ &= R^*(X_*,\varphi' X_*,\varphi' Y_*,Y_*) + ||ph(X,Y)||^2 + ||qh(X,Y)||^2\\ &+ ||ph(X,\varphi Y)||^2 + ||qh(X,\varphi Y)||^2.\end{aligned} \tag{6.6}$$

It is easy to check that, for any X, $Y \in \Gamma(D \oplus \{\xi\})$, $||h(X,Y)||^2 = ||ph(X,Y)||^2 + ||qh(X,Y)||^2$. Therefore, we have,

$$\overline{R}(X, \varphi X, \varphi Y, Y) = R^*(X_*, \varphi' X_*, \varphi' Y_*, Y_*) + ||h(X,Y)||^2 + ||h(X, \varphi Y)||^2, \tag{6.7}$$

which implies that

$$\overline{H}(X) = H'(X_*) + ||h(X,X)||^2 + ||h(X, \varphi X)||^2, \tag{6.8}$$

where $\overline{H}(X) = \overline{R}(X, \varphi X, \varphi X, X)$ and $H'(X_*) = R^*(X_*, \varphi' X_*, \varphi' X_*, X_*)$ are the holomorphic sectional curvatures of $\overline{M}$ and M', respectively.

Theorem 6.1 *Let $\pi : M \longrightarrow M'$ be a submersion of a contact CR-submanifold of a quasi-K-cosymplectic manifold $\overline{M}$ onto an almost contact metric manifold M' with integrable $D \oplus \{\xi\}$. Then, the holomorphic sectional curvatures $\overline{H}$ and H^* of $\overline{M}$ and M', respectively, satisfy*

$$\overline{H}(X) \geq H'(X_*), \;\; \forall X \in \Gamma(D \oplus \{\xi\}), \; ||X|| = 1, \;\; \pi_* X = X_*, \tag{6.9}$$

and the equality holds if and only if M is $D \oplus \{\xi\}$-totally geodesic.

Proof The first assertion holds from (6.8). The equality holds if and only if $h(X,X) = 0$ and $h(X, \varphi X) = 0$, for any $X \in \Gamma(D \oplus \{\xi\})$, $||X|| = 1$. From $h(X,X) = 0$, $X \in \Gamma(D \oplus \{\xi\})$, $||X|| = 1$, and linearity of h it follows immediately that $h(X,Y) = 0$, for any X, $Y \in \Gamma(D \oplus \{\xi\})$, and proves that M is $D \oplus \{\xi\}$-totally geodesic. □

This result is similar to the one found in [11] for CR-submanifolds of a quasi-Kähler manifold onto an almost Hermitian manifold.

When the ambient manifold $\overline{M}$ is quasi-Kenmotsu, then, using (5.44), (6.3), (5.52), and (2) in Theorem 5.13, the curvature tensors $\overline{R}$ and R^* are related as

$$\begin{aligned}
\overline{R}(X, \varphi X, \varphi Y, Y) &= R^*(X_*, \varphi' X_*, \varphi' Y_*, Y_*) - g(C(X,Y), C(\varphi X, \varphi Y)) \\
&+ g(C(X, \varphi Y), C(\varphi X, Y)) + 2g(C(X, \varphi X), C(\varphi Y, Y)) \\
&= R^*(X_*, \varphi' X_*, \varphi' Y_*, Y_*) + ||C(X,Y)||^2 + ||C(X, \varphi Y)||^2 \\
&+ 2g(C(X, \varphi X), C(\varphi Y, Y)) \\
&= R^*(X_*, \varphi' X_*, \varphi' Y_*, Y_*) + ||C(X,Y)||^2 + ||C(X, \varphi Y)||^2,
\end{aligned} \tag{6.10}$$

for any X, $Y \in \Gamma(D \oplus \{\xi\})$, since $C(X, \varphi X) = \frac{1}{2} \varphi ph(X,X) = 0$. The relation (6.10) reduces to

$$\overline{H}(X) = H'(X_*) + ||C(X,X)||^2 + ||C(X, \varphi X)||^2. \tag{6.11}$$

Theorem 6.2 *Let $\pi : M \longrightarrow M'$ be a submersion of a contact CR-submanifold of a quasi-Kenmotsu manifold $\overline{M}$ onto an almost contact metric manifold M'. Then, the holomorphic sectional curvatures $\overline{H}$ and H^* of $\overline{M}$ and M', respectively, satisfy*

$$\overline{H}(X) \geq H'(X_*), \;\; \forall X \in \Gamma(D \oplus \{\xi\}), \; ||X|| = 1, \;\; \pi_* X = X_*, \tag{6.12}$$

and the equality holds if and only if the distribution $D \oplus \{\xi\}$ is integrable.

Proof The inequality follows the relation (6.11). If $\overline{H}(X) = H'(X_*)$ if and only the skew-symmetric tensor C vanishes, the distribution $D \oplus \{\xi\}$ is parallel and this completes the proof. □

Also, we have:

Theorem 6.3 *Let $\pi : M \longrightarrow M'$ be a submersion of a contact CR-submanifold of a quasi-Kenmotsu manifold $\overline{M}$ onto an almost contact metric manifold M' such that the holomorphic sectional curvatures $\overline{H}$ and H^* of $\overline{M}$ and M', respectively, coincide on $D \oplus \{\xi\}$. Then, M is locally a product $M^* \times C$, where M^* is a totally geodesic leaf of $D \oplus \{\xi\}$ and C is a curve tangent to the distribution $D^{\perp}$.*

Proof By Theorem 6.2, we have that the distribution $D \oplus \{\xi\}$ is integrable. We deduce that $D \oplus \{\xi\}$ determines a foliation and if M^* is a leaf of $D \oplus \{\xi\}$, it is totally geodesic. By Theorem 5.8, the distribution $D^{\perp}$ is integrable, and then it defines a foliation. So with $TM = D \oplus D^{\perp} \oplus \{\xi\}$, we complete the proof. □

Acknowledgments

This work was done within the framework of the Associateship Scheme of the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy. FM acknowledges with thanks the financial support from ICTP. The authors also thank the referee for helping them to improve the paper.

References

[1] Bejancu A. CR-submanifolds of a Kaehler manifold. I. Proc Amer Math Soc 1978; 69: 135–142.

[2] Bejancu A, Papaghiuc N. Semi-invariant submanifolds of a Sasakian manifold. An Ştinţ Univ Al I Cuza Iaşi Mat (NS) 1981; 27: 163–170.

[3] Bejancu A., Papaghiuc N. Semi-invariant submanifolds of a Sasakian space form. Colloq Math 1984; 48: 77–88.

[4] Blair DE. Riemannian Geometry of Contact and Symplectic Manifolds. Progress in Mathematics. New York, NY, USA: Birkhäuser, 2002.

[5] Burel JM. Almost contact structures and harmonic maps with minimal fibres. Houston J Math 2004; 30: 393–411.

[6] Chen BY. CR-submanifolds of a Kaehler manifold I, II. J Differ Geom 1981; 16: 305–322, 493–509.

[7] Chinea D. Quasi-K-cosymplectic submersions. Rend Circ Mat Palermo 1984; 33: 319–330.

[8] Chinea D. Almost contact metric submersions. Rend Circ Mat Palermo 1985; 34: 89–104.

[9] Chinea D. Harmonicity on maps between almost contact metric manifolds. Acta Math Hungar 2010; 126: 352–365.

[10] Chinea D, Gonzalez C. A classification of almost contact metric manifolds. Ann Mat Pura Appl 1990; 156: 15–36.

[11] Deshmukh S, Ghazal T, Hashem H. Submersions of CR-submanifolds on an almost Hermitian manifold. Yokohama Math J 1992; 40: 45–57.

[12] Dragomir S, Ornea L. Locally Conformal Kaehler Geometry. Progress in Mathematics. Boston, MA, USA: Birkhäuser, 1998.

[13] Gray A, Hervella LM. The sixteen classes of almost Hermitian manifolds and their linear invariants. Ann Mat Pura Appl 1980; 123: 35–58.

[14] O'Neill O. The fundamental equations of a submersion. Michigan Math J 1966; 13: 459–469.

[15] Papaghiuc N. Submersions of semi-invariant submanifolds of a Sasakian manifold. An Stint Univ Al I Cuza Iasi Sect I-a Mat 1989; 35: 281–288.

[16] Sahin B. Horizontally conformal submersions of CR-submanifolds. Kodai Math J 2008; 31: 46–53.

[17] Watson B. The differential geometry of two types of almost contact metric submersions. In: Rassias GM, editor. The. Mathematical Heritage of C. F. Gauss. Singapore: World Scientific Publishing, 1991, pp. 827–861.

[18] Yano K., Kon M. Structures on Manifolds. Series in Pure Mathematics. Singapore: World Scientific Publishing, 1984.