Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2014) 38: 803 – 811
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doi:10.3906/mat-1309-63

Hausdorff dimension of the graph of the error-sum function of α-Lüroth series

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Received: 25.09.2013 • **Accepted:** 03.04.2014 • **Published Online:** 01.07.2014 • **Printed:** 31.07.2014

Abstract: Let α be a countable partition of the unit interval $[0,1]$. In this paper, we will introduce the error-sum function of α-Lüroth series and determine the Hausdorff dimension of its graph when the partition α is eventually decreasing. Some other properties of the error-sum function are also investigated.

Key words: Hausdorff dimension, the α-Lüroth series, error-sum function

1. Introduction

Recently, Kesseböhmer et al. introduced the concept of α-Lüroth series (see [7]), which is a generalization of the concept of the classical alternating Lüroth series (see [5, 6]). In [7], the authors studied some topological and ergodic theoretic properties of the α-Lüroth series and gave a complete description of its Lyapunov spectra in terms of the thermodynamical formalism. Meanwhile, in a related paper [8], Munday computed the Hausdorff dimension of some α-Good type sets. Soon after, Chen and Wen made a further contribution on the same topic in [1] by determining the Hausdorff dimension of sets of points whose digits are bounded below by a positive function ϕ satisfying $\phi(n) \to \infty$ as $n \to \infty$. In the present paper, we would like to give some other discoveries on the properties of α-Lüroth series by investigating its error-sum function.

Before the presentation of our results, we need to introduce some definitions and notations of the α-Lüroth series for reference. Let I be the unit interval $[0,1]$ and $\alpha = \{A_n, n \in \mathbb{N}\}$ a countable partition of I consisting of left-open, right-closed intervals. We always assume that the elements $\{A_n\}_{n\geq 1}$ are ordered from right to left, starting from A_1. Denote by $a_n = \mathcal{L}(A_n)$ the Lebesgue measure of the element A_n and $t_n = \sum_{i=n}^{\infty} a_n$ the Lebesgue measure of the nth *tail* of α. Moreover, a partition α is said to be *eventually decreasing* if $a_{n+1} \leq a_n$ for all $n \in \mathbb{N}$ sufficiently large.

For a given partition α, define the α*-Lüroth map* $L_\alpha \colon I \to I$ by

$$L_\alpha(x) := \begin{cases} (t_n - x)/a_n & \text{for } x \in A_n,\ n \in \mathbb{N}, \\ 0 & \text{if } x = 0. \end{cases} \tag{1.1}$$

Each $x \in I \setminus \{0\}$ can then be developed uniquely by the map L_α into an alternating series in the following

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2010 *AMS Mathematics Subject Classification:* 11K55, 28A80.

form:

$$x = t_{l_1(x)} + \sum_{j=2}^{\infty}(-1)^{j-1}\Big(\prod_{1\le i<j} a_{l_i(x)}\Big)t_{l_j(x)}, \tag{1.2}$$

where $l_n(x) = l_n \in \mathbb{N}$ if $L_\alpha^{n-1}(x) \in A_{l_n}$. Note that the sum is always supposed to be finite if the sequence $\{l_n(x)\}_{n\ge1}$ is finite. More specifically, the sequence $\{l_n(x)\}_{n\ge1}$ is terminated in k if and only if $L_\alpha^{k-1}(x) = t_n$ for some $n \ge 2$. Moreover, one can even find the fact $l_k(x) \ge 2$ in this situation. For simplicity, denote the finite α-Lüroth series of x by $[l_1(x), \ldots, l_k(x)]_\alpha$ and call it an *α-rational number*. Write I^*, the set of all the α-rational numbers. One can easily see that the set I^* is numerable and of Hausdorff dimension zero. On the other hand, if the sequence $\{l_n(x)\}_{n\ge1}$ is infinite, then we denote it by $[l_1(x), l_2(x), \ldots]_\alpha$ and call it an *α-irrational number*. Note that L_α acts as a shift map on the α-Lüroth series since

$$L_\alpha[l_1(x), l_2(x), \ldots]_\alpha = [l_2(x), l_3(x), \ldots]_\alpha$$

for each α-irrational number x. Given a number x in I, for $n \ge 2$, set

$$Q_n(x) = Q_n(l_1(x), \ldots, l_n(x)) = \frac{1}{\Big(\prod_{1\le i<n} a_{l_i(x)}\Big)t_{l_n(x)}}, \tag{1.3}$$

$$P_n(x) = Q_n(x)\Bigg(t_{l_1(x)} + \sum_{j=2}^{n}(-1)^{j-1}\Big(\prod_{1\le i<j} a_{l_i(x)}\Big)t_{l_j(x)}\Bigg); \tag{1.4}$$

for $n = 1$, set $Q_n(x) = 1/t_{l_1(x)}$, $P_n(x) = 1$. Then it follows that

$$x = \frac{P_n(x)}{Q_n(x)} + \frac{(-1)^n L_\alpha^n(x)}{Q_{n+1}(x)t_{l_{n+1}(x)}}. \tag{1.5}$$

Here, $P_n(x)/Q_n(x)$ is called the nth *convergent* of x in its α-Lüroth series. Accordingly, the *error-sum function* $\mathcal{E}_\alpha$ of the α-Lüroth series is defined by

$$\mathcal{E}_\alpha(x) = \begin{cases} \sum_{n=1}^{\infty}\Big(x - \frac{P_n(x)}{Q_n(x)}\Big), & x \in I \setminus \{0\}, \\ 0, & x = 0. \end{cases} \tag{1.6}$$

Further, denote by

$$G(\mathcal{E}_\alpha) = \{(x, y) \colon y = \mathcal{E}_\alpha(x), x \in I\} \tag{1.7}$$

the graph of the error-sum function $\mathcal{E}_\alpha$. To become acquainted with the features of the graphs of the error-sum functions, one can investigate the Figure in Section 3, which consists of 2 typical graphs of the functions $\mathcal{E}_{\alpha_D}$ and $\mathcal{E}_{\alpha_H}$.

The concept of the error-sum function and its graph were first introduced by Ridley and Petruska in [9] in terms of the regular continued fraction expansion. Including some elementary properties of the error-sum function, they studied the graph of that function by giving an upper bound of its Hausdorff dimension. Later, in [10], Shen and Wu considered the same questions in the Lüroth series and determined the exact Hausdorff dimension of the graph of the corresponding error-sum function. Recently, Dai and Tang in [2] also studied the error-sum functions described by the tent map base series. For some further and latest descriptions of the characters of the error-sum function, one can refer to [3] by Elsner and Stein and the references therein.

Inspired by the above works, in the present paper we would like to give the following main result on the consideration of the size of the graph of the error-sum function of α-Lüroth series.

Theorem 1.1 *For any eventually decreasing partition α of the unit interval, we have*

$$\dim_H G(\mathcal{E}_\alpha) = 1.$$

Here, $\dim_H$ denotes the Hausdorff dimension.

In the following section, we will present some elementary properties of the error-sum function $\mathcal{E}_\alpha$. Section 3 is then devoted to the proof of Theorem 1.1. The reader is assumed to be familiar with the definitions and basic properties of Hausdorff dimension and Hausdorff measure. For this subject, Falconer's book [4] is recommended.

2. Some properties of $\mathcal{E}_\alpha$

In this section, we give some elementary properties of the α-Lüroth series on account of the interest in the study of its characters and as preparation for the proof of Theorem 1.1. Without loss of generality, we would like to mention here that the partition α is always assumed to be decreasing in the sequel for better comprehension and expression. Thus,

$$\max_{n\geq 1}\{a_n\} = a_1 \in (0,1). \tag{2.1}$$

This property will be used throughout this paper if there are no other special statements.

Proposition 2.1 *The function $\mathcal{E}_\alpha$ is bounded. More precisely, we have the estimation*

$$-\frac{a_1}{1-a_1^2} \leq \mathcal{E}_\alpha(x) \leq \frac{a_1^2}{1-a_1^2}$$

for any $x \in I$.

Proof Let $x = [l_1(x), l_2(x), \ldots]_\alpha$. Upon combining (1.5) with (1.6), the following is yielded:

$$\mathcal{E}_\alpha(x) = \sum_{n=1}^{\infty} \frac{(-1)^n L_\alpha^n(x)}{Q_{n+1}(x) t_{l_{n+1}(x)}} = \sum_{n=1}^{\infty} (-1)^n a_{l_1(x)} \ldots a_{l_n(x)} L_\alpha^n(x).$$

Thus, we have

$$\mathcal{E}_\alpha(x) \geq -\sum_{n=1}^{\infty} a_{l_1(x)} \ldots a_{l_{2n-1}(x)} L_\alpha^{2n-1}(x) \geq -\sum_{n=1}^{\infty} a_1^{2n-1} = -\frac{a_1}{1-a_1^2}$$

and

$$\mathcal{E}_\alpha(x) \leq \sum_{n=1}^{\infty} a_{l_1(x)} \ldots a_{l_{2n}(x)} L_\alpha^{2n}(x) \leq \sum_{n=1}^{\infty} a_1^{2n} = \frac{a_1^2}{1-a_1^2}.$$

Apparently, the estimation is also true for $x = 0$ or $x = [l_1(x), \ldots, l_k(x)]_\alpha$ for some $k \geq 1$. The proof is finished now. □

Proposition 2.2 *For any $x \in I$, we have*

$$\mathcal{E}_\alpha(x) = \sum_{i=1}^{n} \left(x - \frac{P_i(x)}{Q_i(x)} \right) + (-1)^n a_{l_1(x)} \dots a_{l_n(x)} \mathcal{E}_\alpha(L_\alpha^n(x)).$$

Proof If $i > n$, then we have, by the definition of the convergent $P_n(x)/Q_n(x)$,

$$\frac{P_i(x)}{Q_i(x)} - \frac{P_n(x)}{Q_n(x)} = (-1)^n a_{l_1(x)} \dots a_{l_n(x)} \frac{P_{i-n}(L_\alpha^n(x))}{Q_{i-n}(L_\alpha^n(x))}.$$

It follows that

$$\begin{aligned}
\mathcal{E}_\alpha(x) &= \sum_{i=1}^{n} \left(x - \frac{P_i(x)}{Q_i(x)} \right) + \sum_{i=n+1}^{\infty} \left(x - \frac{P_n(x)}{Q_n(x)} + \frac{P_n(x)}{Q_n(x)} - \frac{P_i(x)}{Q_i(x)} \right) \\
&= \sum_{i=1}^{n} \left(x - \frac{P_i(x)}{Q_i(x)} \right) + (-1)^n a_{l_1(x)} \dots a_{l_n(x)} \sum_{j=1}^{\infty} \left(L_\alpha^n(x) - \frac{P_j(L_\alpha^n(x))}{Q_j(L_\alpha^n(x))} \right) \\
&= \sum_{i=1}^{n} \left(x - \frac{P_i(x)}{Q_i(x)} \right) + (-1)^n a_{l_1(x)} \dots a_{l_n(x)} \mathcal{E}_\alpha(L_\alpha^n(x)).
\end{aligned}$$

This ends the proof. □

Proposition 2.3 *If the partition α is eventually decreasing, then*

$$\int_0^1 \mathcal{E}_\alpha(x)dx = \frac{\frac{1}{2} - \sum_{i=1}^{\infty} t_i a_i}{1 + \sum_{i=1}^{\infty} a_i^2}.$$

Proof By Proposition 2.2, we have

$$\begin{aligned}
\int_0^1 \mathcal{E}_\alpha(x)dx &= \sum_{i=1}^{\infty} \int_{t_{i+1}}^{t_i} \mathcal{E}_\alpha(x)dx = \sum_{i=1}^{\infty} \int_{t_{i+1}}^{t_i} (x - t_i - a_i \mathcal{E}_\alpha(L_\alpha(x)))dx \\
&= \frac{1}{2} \sum_{i=1}^{\infty} \left(t_i^2 - t_{i+1}^2 \right) - \sum_{i=1}^{\infty} t_i(t_i - t_{i+1}) + \sum_{i=1}^{\infty} \int_{t_{i+1}}^{t_i} a_i^2 \mathcal{E}_\alpha(L_\alpha(x)) d(L_\alpha(x)) \\
&= \frac{1}{2} - \sum_{i=1}^{\infty} t_i a_i - \left(\sum_{i=1}^{\infty} a_i^2 \right) \int_0^1 \mathcal{E}_\alpha(u)du.
\end{aligned}$$

Since the estimations

$$\sum_{i=1}^{n} t_i a_i \le \sum_{i=1}^{\infty} a_i = 1 \quad \text{and} \quad \sum_{i=1}^{n} a_i^2 \le \sum_{i=1}^{\infty} a_i = 1$$

hold for all $n \ge 1$, the series of positive terms $\sum_{i=1}^{\infty} t_i a_i$ and $\sum_{i=1}^{\infty} a_i^2$ are both convergent. The result thus follows by solving the above equation. □

Let us take 2 special cases as examples. Denote by α_D the *doubly decreasing partition* and α_H the *harmonic partition*, which are given by

$$a_n = \frac{1}{2^n} \quad \text{and} \quad a_n = \frac{1}{n(n+1)},$$

respectively. It is well known that the alternating Lüroth series can be developed by the harmonic partition α_H. Then

$$\int_0^1 \mathcal{E}_{\alpha_D}(x)dx = -\frac{1}{8}, \quad \int_0^1 \mathcal{E}_{\alpha_H}(x)dx = \frac{\pi^2 - 9}{12 - 2\pi^2}.$$

We omit the verifications here since they are elementary.

The following lemma describes the continuity of the function $\mathcal{E}_\alpha$, which plays an important role in Section 3.

Proposition 2.4 *Let $x_0 \in I^*$. Then we have the following 2 distinguishable conclusions:*

(1) If $x_0 = [l_1(x_0), \dots, l_{2k+1}(x_0)]_\alpha$ for some k, then $\mathcal{E}_\alpha$ is left continuous at x_0, but not right continuous at x_0. More precisely, we have

$$\lim_{x_L \to x_0^-} \mathcal{E}_\alpha(x_L) = \mathcal{E}_\alpha(x_0), \quad \lim_{x_R \to x_0^+} \mathcal{E}_\alpha(x_R) = \mathcal{E}_\alpha(x_0) - a_{l_1(x_0)} \dots a_{l_{2k}(x_0)} a_{l_{2k+1}(x_0)-1}.$$

(2) If $x_0 = [l_1(x_0), \dots, l_{2k}(x_0)]_\alpha$ for some k, then $\mathcal{E}_\alpha$ is not left continuous at x_0, but right continuous at x_0. More precisely, we have

$$\lim_{x_L \to x_0^-} \mathcal{E}_\alpha(x_L) = \mathcal{E}_\alpha(x_0) + a_{l_1(x_0)} \dots a_{l_{2k-1}(x_0)} a_{l_{2k}(x_0)-1}, \quad \lim_{x_R \to x_0^+} \mathcal{E}_\alpha(x_R) = \mathcal{E}_\alpha(x_0).$$

Proof We only give the proof of conclusion (1) since conclusion (2) can be treated in an analogous way. For brevity's sake, write $x_0 = [l_1, \dots, l_{2k+1}]_\alpha$.

For the left continuity, write $x_L = [l_1, \dots, l_{2k+1}, K, \dots]_\alpha$. It is then easy to check that

$$x_L \to x_0^- \Longleftrightarrow K \to \infty \Longleftrightarrow a_K \to 0. \tag{2.2}$$

Thus, by Proposition 2.2,

$$\begin{aligned}
&\mathcal{E}_\alpha(x_L) - \mathcal{E}_\alpha(x_0) \\
&= \sum_{i=1}^{2k+1} \left(x_L - \frac{P_i(x_L)}{Q_i(x_L)} \right) + \left(x_L - \frac{P_{2k+2}(x_L)}{Q_{2k+2}(x_L)} \right) \\
&\quad + a_{l_1} \dots a_{l_{2k+1}} a_K \mathcal{E}_\alpha \left(L_\alpha^{2k+2}(x_L) \right) - \sum_{i=1}^{2k+1} \left(x_0 - \frac{P_i(x_0)}{Q_i(x_0)} \right) \\
&= (2k+1)(x_L - x_0) + a_{l_1} \dots a_{l_{2k+1}} a_K \left(L_\alpha^{2k+2}(x_L) + \mathcal{E}_\alpha \left(L_\alpha^{2k+2}(x_L) \right) \right).
\end{aligned}$$

Note that the function $\mathcal{E}_\alpha$ is bounded by Proposition 2.1, as well as L_α. This, together with (2.2), yields that $\lim_{x_L \to x_0^-} \mathcal{E}_\alpha(x_L) = \mathcal{E}_\alpha(x_0)$.

For the right continuity, write $x_R = [l_1, \dots, l_{2k}, l_{2k+1} - 1, 1, K, \dots]$. Similarly, it holds that

$$x_R \to x_0^+ \Longleftrightarrow K \to \infty \Longleftrightarrow a_K \to 0 \Longleftrightarrow [K, \dots]_\alpha \to 0. \tag{2.3}$$

Then, by Proposition 2.2 again, we have

$$\begin{aligned}
&\mathcal{E}_\alpha(x_R)-\mathcal{E}_\alpha(x_0)\\
&=\sum_{i=1}^{2k}\left(x_R-\frac{P_i(x_R)}{Q_i(x_R)}\right)+\left(x_R-\frac{P_{2k+1}(x_R)}{Q_{2k+1}(x_R)}\right)+\left(x_R-\frac{P_{2k+2}(x_R)}{Q_{2k+2}(x_R)}\right)\\
&\quad+\left(x_R-\frac{P_{2k+3}(x_R)}{Q_{2k+3}(x_R)}\right)-a_{l_1}\dots a_{l_{2k+1}-1}a_1a_K\mathcal{E}_\alpha\left(L_\alpha^{2k+3}(x_R)\right)\\
&\quad-\sum_{i=1}^{2k}\left(x_0-\frac{P_i(x_0)}{Q_i(x_0)}\right)-\left(x_0-\frac{P_{2k+1}(x_0)}{Q_{2k+1}(x_0)}\right)\\
&=(2k+1)(x_R-x_0)-a_{l_1}\dots a_{l_{2k}}a_{l_{2k+1}-1}+a_{l_1}\dots a_{l_{2k+1}-1}a_1[K,\dots]_\alpha\\
&\quad-a_{l_1}\dots a_{l_{2k+1}-1}a_1a_K\Big(L_\alpha^{2k+3}(x_R)+\mathcal{E}_\alpha\left(L_\alpha^{2k+3}(x_R)\right)\Big).
\end{aligned}$$

Once again, relation (2.3) and the boundedness of L_α and $\mathcal{E}_\alpha$ lead to the result:

$$\lim_{x_R\to x_0^+}\mathcal{E}_\alpha(x_R)=\mathcal{E}_\alpha(x_0)-a_{l_1}\dots a_{l_{2k}}a_{l_{2k+1}-1}.$$

The proof is completed now. □

3. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. To obtain a suitable cover of the graph of the error-sum function $\mathcal{E}_\alpha$, we need the following crucial Lemma 3.1. Before the presentation, first we introduce some notations for ready use.

Denote by $\Sigma=\bigcup_{n=1}^{\infty}\Sigma_n$, where

$$\Sigma_n=\{(l_1,\dots,l_n)\in\mathbb{N}^n\colon l_i\ge 1, 1\le i\le n\}$$

is the set of all blocks of length n. For any $\sigma_n=(l_1,\dots,l_n)\in\Sigma_n$, call

$$I_{\sigma_n}=I_n(l_1,\dots,l_n)=\{x\in I\colon l_1(x)=l_1,\dots,l_n(x)=l_n\}$$

a *basic interval* of order n. We also sometimes put

$$I_n(x)=I_n(l_1(x),\dots,l_n(x)).$$

In other words, $I_n(x)$ is the set of numbers whose first n digits coincide with those of x. Moreover, write

$$S_{\sigma_n}=t_{l_1}-a_{l_1}t_{l_2}+\dots+(-1)^{n-1}a_{l_1}\dots a_{l_{n-1}}t_{l_n}, \tag{3.1}$$

$$T_{\sigma_n}=t_{l_1}-a_{l_1}t_{l_2}+\dots+(-1)^{n-1}a_{l_1}\dots a_{l_{n-1}}t_{l_n+1}. \tag{3.2}$$

It can then be checked that

$$I^*=\{S_{\sigma_n},T_{\sigma_n}\colon \sigma_n\in\Sigma_n, n\ge 1\}$$

and

$$I_{\sigma_n} = \begin{cases} (S_{\sigma_n}, T_{\sigma_n}], & \text{when } n \text{ is even,} \\ (T_{\sigma_n}, S_{\sigma_n}], & \text{when } n \text{ is odd.} \end{cases} \tag{3.3}$$

It follows that

$$\mathcal{L}(I_{\sigma_n}) = |S_{\sigma_n} - T_{\sigma_n}| = a_{l_1} \dots a_{l_n} \tag{3.4}$$

for any $\sigma_n = (l_1, \dots, l_n)$, $n \geq 1$.

Lemma 3.1 *For any $\sigma_n \in \Sigma_n$ with $n \geq 1$, we have*

$$\sup_{x,y \in I_{\sigma_n}} |\mathcal{E}_\alpha(x) - \mathcal{E}_\alpha(y)| = n\mathcal{L}(I_{\sigma_n}).$$

Proof In the case of $n = 2k+1$, write $\sigma_n = (l_1, \dots, l_n)$. By the discussion in Proposition 2.4, we have

$$\mathcal{E}_\alpha(T_{\sigma_n}) - a_{l_1} \dots a_{l_{n-1}} a_{(l_n+1)-1} \leq \mathcal{E}_\alpha(x) \leq \mathcal{E}_\alpha(S_{\sigma_n})$$

for any $x \in I_{\sigma_n}$. Thus,

$$\sup_{x,y \in I_{\sigma_n}} |\mathcal{E}_\alpha(x) - \mathcal{E}_\alpha(y)| = \mathcal{E}_\alpha(S_{\sigma_n}) - \mathcal{E}_\alpha(T_{\sigma_n}) + a_{l_1} \dots a_{l_n}. \tag{3.5}$$

In addition, by (3.4), we have that

$$\begin{aligned} \mathcal{E}_\alpha(S_{\sigma_n}) - \mathcal{E}_\alpha(T_{\sigma_n}) &= \sum_{i=1}^{n-1} \left(S_{\sigma_n} - \frac{P_i(S_{\sigma_n})}{Q_i(S_{\sigma_n})} \right) + \left(S_{\sigma_n} - \frac{P_n(S_{\sigma_n})}{Q_n(S_{\sigma_n})} \right) \\ &\quad - \sum_{i=1}^{n-1} \left(T_{\sigma_n} - \frac{P_i(T_{\sigma_n})}{Q_i(T_{\sigma_n})} \right) - \left(T_{\sigma_n} - \frac{P_n(T_{\sigma_n})}{Q_n(T_{\sigma_n})} \right) \\ &= n(S_{\sigma_n} - T_{\sigma_n}) - \left(\frac{P_n(S_{\sigma_n})}{Q_n(S_{\sigma_n})} - \frac{P_n(T_{\sigma_n})}{Q_n(T_{\sigma_n})} \right) \\ &= (n-1) a_{l_1} \dots a_{l_n} \\ &= (n-1)\mathcal{L}(I_{\sigma_n}). \end{aligned}$$

Substitute this result into (3.5), which finishes the proof of this case.

The case of $n = 2k$ can be verified in an analogous way, and we omit the details. □

Corollary 3.2 *$\mathcal{E}_\alpha$ is continuous on $I \setminus (I^* \cup \{0\})$.*

Proof Let $x_0 = [l_1(x_0), \dots, l_n(x_0), \dots]_\alpha \in I \setminus (I^* \cup \{0\})$. By Lemma 3.1, for any $x \in I_n(l_1(x_0), \dots, l_n(x_0))$, we then have that

$$|\mathcal{E}_\alpha(x) - \mathcal{E}_\alpha(x_0)| \leq n\mathcal{L}(I_n(l_1(x_0), \dots, l_n(x_0))) \leq n a_1^n \to 0$$

as $n \to \infty$. It implies our result. □

Proof [Proof of Theorem 1.1] On the one hand, we can easily see that $\{I_{\sigma_n} \times \mathcal{E}_\alpha(I_{\sigma_n})\}_{\sigma_n \in \Sigma_n}$ is a cover of the graph $G(\mathcal{E}_\alpha)$ for any $n \geq 1$, i.e.

$$G(\mathcal{E}_\alpha) \subset \bigcup_{\sigma_n \in \Sigma_n} I_{\sigma_n} \times \mathcal{E}_\alpha(I_{\sigma_n}).$$

Moreover, by Lemma 3.1, $I_{\sigma_n} \times \mathcal{E}_\alpha(I_{\sigma_n})$ can be covered by n squares with the same side length $\mathcal{L}(I_{\sigma_n})$. Thus, for any $t > 1$, we have

$$\begin{aligned}
\mathcal{H}^t(G(\mathcal{E}_\alpha)) &\leq \liminf_{n\to\infty} \sum_{\sigma_n \in \Sigma_n} n(\sqrt{2})^t(\mathcal{L}(I_{\sigma_n}))^t \\
&\leq \liminf_{n\to\infty} n(\sqrt{2})^t(a_1)^{(t-1)n} \sum_{\sigma_n \in \Sigma_n} \mathcal{L}(I_{\sigma_n}) \\
&\leq (\sqrt{2})^t \liminf_{n\to\infty} n(a_1)^{(t-1)n} \\
&= 0.
\end{aligned}$$

The second inequality is followed by the fact $\mathcal{L}(I_{\sigma_n}) \leq (a_1)^n$. It follows that $\dim_H G(\mathcal{E}_\alpha) \leq t$ for any $t > 1$. Hence, $\dim_H G(\mathcal{E}_\alpha) \leq 1$.

On the other hand, since I is the orthogonal projection of $G(\mathcal{E}_\alpha)$ onto the real line and the projection is a Lipschitz mapping, we have, by Corollary 2.4(a) in [4],

$$\dim_H G(\mathcal{E}_\alpha) \geq \dim_H \left(\mathrm{Proj}(G(\mathcal{E}_\alpha)\right) = \dim_H I = 1.$$

Therefore, combining the above 2 conclusions, we conclude the proof. □

Denote by $G(\mathcal{E}_{\alpha_D})$ and $G(\mathcal{E}_{\alpha_H})$ the graphs of the error-sum functions $\mathcal{E}_{\alpha_D}$ and $\mathcal{E}_{\alpha_H}$, which are plotted in the Figure, respectively.

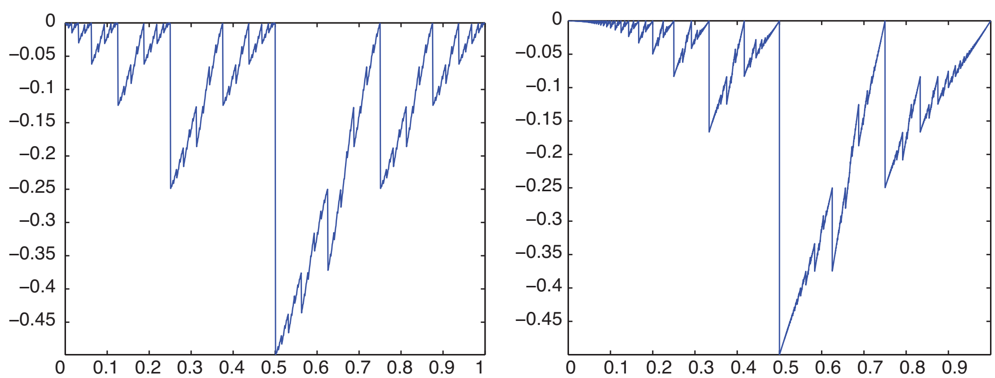

Figure. The graphs of $\mathcal{E}_{\alpha_D}$ (left) and $\mathcal{E}_{\alpha_H}$ (right).

Then we have the following result, which can be regarded as 2 special cases of Theorem 1.1.

Corollary 3.3 *For any partition α that is eventually decreasing, we have*

$$\dim_H G(\mathcal{E}_{\alpha_D}) = \dim_H G(\mathcal{E}_{\alpha_H}) = 1.$$

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