Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2014) 38: 896 – 904

doi:10.3906/mat-1402-12

A characterization of the projective transformation in Minkowski 3-space

Yasemin ALAGÖZ*
Department of Mathematics, Faculty of Arts and Sciences, Yıldız Technical University, İstanbul, Turkey

Received: 04.02.2014 • **Accepted:** 01.05.2014 • **Published Online:** 01.07.2014 • **Printed:** 31.07.2014

Abstract: We consider transformations preserving asymptotic directions of surfaces in Minkowski 3-space and show that a transformation preserves the asymptotic directions of a surface if only if it is the projective one. Therefore, we obtain a characterization of the projective transformation.

Key words: Minkowski space, asymptotic direction, projective transformation

1. Introduction

The projective transformation has been studied by many researchers in the Euclidean space. They characterize some properties of this transformation as follows. A transformation is the projective one if and only if it transforms a straight line to the other straight line [5]. In 3-dimensional Euclidean space, the projective transformation transforms an infinitesimally rigid surface to the other infinitesimally rigid surface, that is, it preserves the infinitesimal rigidity [8, 6, p.355]. The projective transformation also preserves the asymptotic lines of surfaces [3, p.202]. The transformations preserving asymptotic directions of hypersurfaces in the Euclidean space were considered by Alagöz and Soyuçok in [2]. Moreover, they gave a characterization of the projective transformation in [1].

In this study, we investigate the properties of transformation preserving asymptotic directions of surfaces in Minkowski 3-space. We also show that a transformation preserves the asymptotic directions of a Minkowski surface if and only if it is the projective one.

2. Preliminaries

Let E_1^3 be a Minkowski 3-space with the scalar product

$$\mathbf{A}.\mathbf{B} = a_1b_1 + a_2b_2 - a_3b_3 \tag{1}$$

for vectors $\mathbf{A} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 = (a_1, a_2, a_3)$ and $\mathbf{B} = (b_1, b_2, b_3)$. The Minkowski vector product of $\mathbf{A}$ and $\mathbf{B}$ is given as

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} e_1 & e_2 & -e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \tag{2}$$

*Correspondence: ygulluk@yildiz.edu.tr
2000 *AMS Mathematics Subject Classification:* 53A35.

[4, 9]. Therefore, the Minkowski triple scalar product is given by

$$(\mathbf{A}\times\mathbf{B}).\mathbf{C}=(\mathbf{ABC})=\det\begin{bmatrix} a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{bmatrix}=\begin{vmatrix} a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix} \tag{3}$$

where $\mathbf{C}=(c_1,c_2,c_3)$.

Let us consider a surface S in the Minkowski 3-space, which is given by the parametric representation

$$r(u^1,u^2)=(x^1(u^1,u^2),x^2(u^1,u^2),x^3(u^1,u^2)) \tag{4}$$

where x^1,x^2, and x^3 are cartesian coordinates. The Minkowski first fundamental form is defined by

$$I=d\mathbf{r}.d\mathbf{r}=E(du^1)^2+2Fdu^1du^2+G(du^2)^2 \tag{5}$$

with the coefficients

$$E=\mathbf{r}_{,1}.\mathbf{r}_{,1},\qquad F=\mathbf{r}_{,1}.\mathbf{r}_{,2},\qquad G=\mathbf{r}_{,2}.\mathbf{r}_{,2},\qquad (\mathbf{r}_{,i}=\frac{\partial r}{\partial u^i};\ i=1,2) \tag{6}$$

where

$$\det I=EG-F^2\neq 0 \tag{7}$$

When $\det I>0$, S is called a spacelike surface; when $\det I<0$, S is called a timelike surface [4, 7, 9].

The Minkowski unit normal vector is

$$\mathbf{N}=\frac{\mathbf{r}_{,1}\times\mathbf{r}_{,2}}{k},\qquad k=\sqrt{|\mathbf{r}_{,1}\times\mathbf{r}_{,2}|}=\sqrt{\det I} \tag{8}$$

The Minkowski second fundamental form is given by

$$II=-d\mathbf{r}.d\mathbf{N}=\mathbf{N}.d^2\mathbf{r}=L_{11}(du^1)^2+2L_{12}du^1du^2+L_{22}(du^2)^2 \tag{9}$$

or

$$II=L_{ij}du^idu^j,\qquad (i,j=1,2) \tag{10}$$

where

$$kL_{ij}=k(\mathbf{N}.\mathbf{r}_{ij})=(\mathbf{r}_{,1},\mathbf{r}_{,2},\mathbf{r}_{ij}),\qquad (\mathbf{r}_{,ij}=\frac{\partial^2 r}{\partial u^i\partial u^j},\ i,j=1,2) \tag{11}$$

[9].

3. The equation of the asymptotic directions of a surface

The asymptotic directions of a surface S in the Minkowski 3-space are defined by the equation

$$II=0$$

[9]. Regarding (10), the above equation can be written as

$$L_{ij}du^idu^j=0,\qquad (i,j=1,2) \tag{12}$$

A spacelike surface S can be described by the Monge representation

$$r(x^1, x^2) = (x^1, x^2, x^3(x^1, x^2)) \tag{13}$$

A timelike surface S can be described by the Monge representation

$$r(x^1, x^3) = (x^1, x^2(x^1, x^3), x^3) \tag{14}$$

or

$$r(x^2, x^3) = (x^1(x^2, x^3), x^2, x^3) \tag{15}$$

[9]. Accordingly from (11) and (8):
For a spacelike surface, using (13) we have

$$kL_{ij} = \begin{vmatrix} 1 & 0 & x^3_{,1} \\ 0 & 1 & x^3_{,2} \\ 0 & 0 & x^3_{,ij} \end{vmatrix} = x^3_{,ij} \qquad (i,j = 1,2) \tag{16}$$

where

$$k = \sqrt{|\mathbf{r}_{,1} \times \mathbf{r}_{,2}|} = \sqrt{\left|1 - (x^3_{,1})^2 - (x^3_{,2})^2\right|}$$

For a timelike surface, using (14) we have

$$kL_{ij} = \begin{vmatrix} 0 & x^2_{,3} & 1 \\ 1 & x^2_{,1} & 0 \\ 0 & x^2_{,ij} & 0 \end{vmatrix} = x^2_{,ij} \qquad (i,j = 1,3) \tag{17}$$

where

$$k = \sqrt{|\mathbf{r}_{,1} \times \mathbf{r}_{,2}|} = \sqrt{\left|1 - (x^2_{,1})^2 - (x^2_{,3})^2\right|}$$

or using (15) we have

$$kL_{ij} = \begin{vmatrix} x^1_{,2} & 1 & 0 \\ x^1_{,3} & 0 & 1 \\ x^1_{,ij} & 0 & 0 \end{vmatrix} = x^1_{,ij} \qquad (i,j = 2,3)$$

where

$$k = \sqrt{|\mathbf{r}_{,1} \times \mathbf{r}_{,2}|} = \sqrt{\left|(x^1_{,3})^2 - 1 - (x^1_{,2})^2\right|}$$

Therefore, from (12), the equation of the asymptotics of a Minkowski surface can be written as follows:
For a spacelike surface,

$$x^3_{,ij} dx^i dx^j = 0, \qquad (i,j = 1,2) \tag{18}$$

For a timelike surface,

$$x^2_{,ij} dx^i dx^j = 0, \qquad (i,j = 1,3) \tag{19}$$

or

$$x^1_{,ij} dx^i dx^j = 0, \qquad (i,j = 2,3) \tag{20}$$

4. Conditions for a transformation preserving the asymptotic directions

In this section, we determine transformations that preserve the asymptotic directions in the Minkowski 3-space. Let

$$\mathbf{T} : y^a = y^a \left(x^1, x^2, x^3\right), \quad (a = 1, 2, 3) \tag{21}$$

be a coordinate transformation in E_1^3. We assume that $\mathbf{T}$ is differentiable of order 3 and

$$\Delta = \det \begin{bmatrix} \mathbf{T}_{,1} & \mathbf{T}_{,2} & \mathbf{T}_{,3} \end{bmatrix} = \begin{vmatrix} \mathbf{T}_{,1} & \mathbf{T}_{,2} & \mathbf{T}_{,3} \end{vmatrix} \neq 0 \tag{22}$$

where

$$\mathbf{T}_{,b} = \begin{bmatrix} y^1_{,b} \\ y^2_{,b} \\ y^3_{,b} \end{bmatrix}, \left(b = 1, 2, 3;\ y^a_{,b} = \frac{\partial y^a}{\partial x^b}\right). \tag{23}$$

If the transformation $\mathbf{T}$ is applied to a Minkowski surface S defined by one of the equations (13), (14), or (15), we have respectively

$$y^a = y^a \left(x^1, x^2, x^3 \left(x^1, x^2\right)\right), \quad a = (1, 2, 3) \tag{24}$$

$$y^a = y^a \left(x^1, x^2(x^1, x^3), x^3\right), \quad a = (1, 2, 3) \tag{25}$$

$$y^a = y^a \left(x^1(x^2, x^3), x^2, x^3\right), \quad a = (1, 2, 3) \tag{26}$$

Therefore, $\mathbf{T}$ transforms a spacelike surface S to a surface S^*, which is given by the equation

$$\mathbf{r}^* \left(x^1, x^2\right) = \left(y^1(x^1, x^2, x^3(x^1, x^2)), y^2(x^1, x^2, x^3(x^1, x^2)), y^3(x^1, x^2, x^3(x^1, x^2))\right) \tag{27}$$

and it transforms a timelike surface S to a surface S^*, which is given by the equation

$$\mathbf{r}^* \left(x^1, x^3\right) = \left(y^1(x^1, x^2(x^1, x^3), x^3), y^2(x^1, x^2(x^1, x^3), x^3), y^3(x^1, x^2(x^1, x^3), x^3)\right) \tag{28}$$

or

$$\mathbf{r}^* \left(x^2, x^3\right) = \left(y^1(x^1(x^2, x^3), x^2, x^3), y^2(x^1(x^2, x^3), x^2, x^3), y^3(x^1(x^2, x^3), x^2, x^3)\right) \tag{29}$$

From (12), the asymptotic directions of the surface S^* given by (27) or (28) or (29) can be written, respectively, as

$$L^*_{ij} dx^i dx^j = 0, \quad (i, j = 1, 2) \tag{27'}$$

or

$$L^*_{ij} dx^i dx^j = 0, \quad (i, j = 1, 3) \tag{28'}$$

or

$$L^*_{ij} dx^i dx^j = 0, \quad (i, j = 2, 3) \tag{29'}$$

where

$$k^* L^*_{ij} = (\mathbf{r}^*_{,1}, \mathbf{r}^*_{,2}, \mathbf{r}^*_{ij}), \quad (i, j = 1, 2), \quad k = \sqrt{\left|\mathbf{r}^*_{,1} \times \mathbf{r}^*_{,2}\right|} \tag{27''}$$

or

$$k^* L^*_{ij} = (\mathbf{r}^*_{,3}, \mathbf{r}^*_{,1}, \mathbf{r}^*_{ij}), \quad (i, j = 1, 3), \quad k = \sqrt{\left|\mathbf{r}^*_{,3} \times \mathbf{r}^*_{,1}\right|} \tag{28''}$$

or

$$k^*L^*_{ij} = (\mathbf{r}^*_{,2}, \mathbf{r}^*_{,3}, \mathbf{r}^*_{ij}), \qquad (i,j=2,3), \qquad k = \sqrt{\left|\mathbf{r}^*_{,2} \times \mathbf{r}^*_{,3}\right|} \tag{29''}$$

respectively.

Since the transformation $\mathbf{T}$ transforms the asymptotic directions of a surface S to the asymptotic directions of the corresponding surface S^*, it must transform the equation (18) to the equation ($27'$), the equation (19) to the equation ($28'$), and the equation (20) to the equation ($29'$). Accordingly, our conditions are respectively

$$L^*_{ij} = tx^3_{,ij}, \qquad (i,j=1,2) \tag{30}$$

$$L^*_{ij} = tx^2_{,ij}, \qquad (i,j=1,3) \tag{31}$$

or

$$L^*_{ij} = tx^1_{,ij}, \qquad (i,j=2,3) \tag{32}$$

Now let us carry out the calculations for the corresponding surface S^* defined by (27). Thus the conditions for the transformations are given by (30). Since, for this case,

$$\mathbf{r}^*_{,i} = (y^1_{,i} + y^1_{,3}x^3_{,i}, y^2_{,i} + y^2_{,3}x^3_{,i}, y^3_{,i} + y^3_{,3}x^3_{,i})$$

$$\mathbf{r}^*_{,i} = \mathbf{T}_{,i} + \mathbf{T}_{,3}x^3_{,i}, \quad (i,j=1,2)$$

and

$$\mathbf{r}^*_{,ij} = \mathbf{T}_{,ij} + \mathbf{T}_{,i3}x^3_{,j} + \mathbf{T}_{,3j}x^3_{,i} + \mathbf{T}_{,33}x^3_{,i}x^3_{,j} + \mathbf{T}_{,3}x^3_{,ij}, \quad (i,j=1,2)$$

From ($27''$) we have

$$\begin{aligned} k^*L^*_{ij} = & \left| \mathbf{T}_{,1} \quad \mathbf{T}_{,2} \quad \mathbf{T}_{,ij} \right| + \left| \mathbf{T}_{,3} \quad \mathbf{T}_{,2} \quad \mathbf{T}_{,ij} \right| x^3_{,1} + \left| \mathbf{T}_{,1} \quad \mathbf{T}_{,3} \quad \mathbf{T}_{,ij} \right| x^3_{,2} \\ & + \left| \mathbf{T}_{,1} \quad \mathbf{T}_{,2} \quad \mathbf{T}_{,3j} \right| x^3_{,i} + \left| \mathbf{T}_{,1} \quad \mathbf{T}_{,2} \quad \mathbf{T}_{,i3} \right| x^3_{,j} \\ & + \left| \mathbf{T}_{,3} \quad \mathbf{T}_{,2} \quad \mathbf{T}_{,3j} \right| x^3_{,i}x^3_{,1} + \left| \mathbf{T}_{,1} \quad \mathbf{T}_{,3} \quad \mathbf{T}_{,3j} \right| x^3_{,i}x^3_{,2} \\ & + \left| \mathbf{T}_{,3} \quad \mathbf{T}_{,2} \quad \mathbf{T}_{,i3} \right| x^3_{,j}x^3_{,1} + \left| \mathbf{T}_{,1} \quad \mathbf{T}_{,3} \quad \mathbf{T}_{,i3} \right| x^3_{,j}x^3_{,2} \\ & + \left| \mathbf{T}_{,1} \quad \mathbf{T}_{,2} \quad \mathbf{T}_{,33} \right| x^3_{,i}x^3_{,j} + \left| \mathbf{T}_{,3} \quad \mathbf{T}_{,2} \quad \mathbf{T}_{,33} \right| x^3_{,i}x^3_{,j}x^3_{,1} \\ & + \left| \mathbf{T}_{,1} \quad \mathbf{T}_{,3} \quad \mathbf{T}_{,33} \right| x^3_{,i}x^3_{,j}x^3_{,2} + \Delta . x^3_{,ij}, \qquad (i,j=1,2) \end{aligned} \tag{33}$$

The equations (30) must be satisfied by any surface. Thus, from (33) we obtain the necessary conditions for the transformation preserving the asymptotic directions of a Minkowski surface.

For $i=j=1$, we have

$$\left| \mathbf{T}_{,1} \quad \mathbf{T}_{,2} \quad \mathbf{T}_{,11} \right| = 0, \left| \mathbf{T}_{,1} \quad \mathbf{T}_{,3} \quad \mathbf{T}_{,11} \right| = 0 \tag{34}$$

$$\left| \mathbf{T}_{,3} \quad \mathbf{T}_{,2} \quad \mathbf{T}_{,33} \right| = 0, \left| \mathbf{T}_{,1} \quad \mathbf{T}_{,3} \quad \mathbf{T}_{,33} \right| = 0 \tag{35}$$

$$\left| \mathbf{T}_{,1} \quad \mathbf{T}_{,3} \quad \mathbf{T}_{,13} \right| = 0, \left| \mathbf{T}_{,3} \quad \mathbf{T}_{,2} \quad \mathbf{T}_{,11} \right| + 2\left| \mathbf{T}_{,1} \quad \mathbf{T}_{,2} \quad \mathbf{T}_{,13} \right| = 0 \tag{36}$$

$$\left| \mathbf{T}_{,1} \quad \mathbf{T}_{,2} \quad \mathbf{T}_{,33} \right| + 2\left| \mathbf{T}_{,3} \quad \mathbf{T}_{,2} \quad \mathbf{T}_{,13} \right| = 0 \tag{37}$$

For $i = j = 2$, we have

$$\left| \mathbf{T}_{,1} \quad \mathbf{T}_{,2} \quad \mathbf{T}_{,22} \right| = 0, \left| \mathbf{T}_{,3} \quad \mathbf{T}_{,2} \quad \mathbf{T}_{,22} \right| = 0 \tag{38}$$

$$\left| \mathbf{T}_{,3} \quad \mathbf{T}_{,2} \quad \mathbf{T}_{,23} \right| = 0, \left| \mathbf{T}_{,1} \quad \mathbf{T}_{,3} \quad \mathbf{T}_{,22} \right| + 2 \left| \mathbf{T}_{,1} \quad \mathbf{T}_{,2} \quad \mathbf{T}_{,23} \right| = 0 \tag{39}$$

$$\left| \mathbf{T}_{,1} \quad \mathbf{T}_{,2} \quad \mathbf{T}_{,33} \right| + 2 \left| \mathbf{T}_{,1} \quad \mathbf{T}_{,3} \quad \mathbf{T}_{,23} \right| = 0 \tag{40}$$

and also the equations (35).
Finally, for $i = 1,\ j = 2$ or $i = 2,\ j = 1$, apart from the above equations we have

$$\left| \mathbf{T}_{,1} \quad \mathbf{T}_{,2} \quad \mathbf{T}_{,12} \right| = 0, \left| \mathbf{T}_{,3} \quad \mathbf{T}_{,2} \quad \mathbf{T}_{,12} \right| + \left| \mathbf{T}_{,1} \quad \mathbf{T}_{,2} \quad \mathbf{T}_{,32} \right| = 0 \tag{41}$$

$$\left| \mathbf{T}_{,1} \quad \mathbf{T}_{,3} \quad \mathbf{T}_{,12} \right| + \left| \mathbf{T}_{,1} \quad \mathbf{T}_{,2} \quad \mathbf{T}_{,13} \right| = 0 \tag{42}$$

$$\left| \mathbf{T}_{,1} \quad \mathbf{T}_{,3} \quad \mathbf{T}_{,32} \right| + \left| \mathbf{T}_{,3} \quad \mathbf{T}_{,2} \quad \mathbf{T}_{,13} \right| + \left| \mathbf{T}_{,1} \quad \mathbf{T}_{,2} \quad \mathbf{T}_{,33} \right| = 0 \tag{43}$$

From (34), (35), and (38), we have

$$\mathbf{T}_{,aa} = 2A_a \mathbf{T}_{,a}, \qquad (a = 1, 2, 3) \tag{44}$$

where A_1, A_2, and A_3 are arbitrary functions.
From the remaining equations, using (44) we obtain

$$\mathbf{T}_{,ab} = A_a \mathbf{T}_{,b} + A_b \mathbf{T}_{,a}, \qquad (a, b = 1, 2, 3) \tag{45}$$

Equations (34) to (43) are all satisfied by (45).
Carrying out similar calculations for the corresponding surface S^* defined by (28) or (29) where the conditions for the transformation are respectively given by (31) or (32), we obtain the same equation (45).
Thus we have the following lemma.

Lemma 1 *A transformation* $\mathbf{T}$ *preserving the asymptotic directions of a Minkowski surface must satisfy the equations*

$$\mathbf{T}_{,ab} = A_a \mathbf{T}_{,b} + A_b \mathbf{T}_{,a}, \qquad (a, b = 1, 2, 3) \tag{46}$$

where A_1, A_2*, and* A_3 *are arbitrary functions of variables* x^1, x^2*, and* x^3*.*

5. A characterization of the projective transformation

Firstly, let us consider the projective transformation

$$\mathbf{T} : y^m = \frac{C_0^m + C_1^m x^1 + C_2^m x^2 + C_3^m x^3}{C_0 + C_1 x^1 + C_2 x^2 + C_3 x^3} = \frac{C_p^m x^p}{C_p x^p}, \tag{47}$$

where $(m = 1, 2, 3)$, which can be expressed as

$$\mathbf{T} = \frac{\mathbf{C}_p x^p}{C_p x^p}, \qquad \left(\mathbf{C}_p = \left(C_p^1, C_p^2, C_p^3\right)\right) \tag{48}$$

where C_p^m and C_p are constants. For this transformation

$$\mathbf{T}_{,a} = \frac{\left(C_p \mathbf{C}_a - C_a \mathbf{C}_p\right) x^p}{\left(C_p x^p\right)^2}, \quad \mathbf{T}_{,b} = \frac{\left(C_p \mathbf{C}_b - C_b \mathbf{C}_p\right) x^p}{\left(C_p x^p\right)^2} \tag{49}$$

and

$$\mathbf{T}_{,ab} = \frac{-C_b(C_p \mathbf{C}_a - C_a \mathbf{C}_p) x^p - C_a \left(C_p \mathbf{C}_b - C_b \mathbf{C}_p\right) x^p}{\left(C_p x^p\right)^3}. \tag{50}$$

Therefore, we have

$$\mathbf{T}_{,ab} = \frac{-C_b}{C_p x^p} \mathbf{T}_{,a} + \frac{-C_a}{C_p x^p} \mathbf{T}_{,b}. \tag{51}$$

Accordingly, the projective transformation satisfies (46). Hence, according to Lemma 1, the projective transformation preserves the asymptotic directions of a Minkowski surface.

In the following, we show that a transformation satisfying the conditions of Lemma 1 is the projective one. Now let us consider the compatibility equations of the equations (46). If we use (46) in $\mathbf{T}_{,abc} = \mathbf{T}_{,acb}$ then we obtain

$$(A_{b,c} - A_{c,b})\mathbf{T}_{,a} + (A_{a,c} - A_a A_c)\mathbf{T}_{,b} + (A_a A_b - A_{a,b})\mathbf{T}_{,c} = 0 \tag{52}$$

where $A_{a,b} = \frac{\partial A_a}{\partial x^b}, \quad (a, b = 1, 2, 3)$.
From (52) we have,

$$A_{a,b} = A_a A_b$$

and so

$$A_{a,a} = A_a^2 \tag{53}$$

Thus we find

$$A_a = -\frac{C_a}{C_a x^a + B^a} \tag{54}$$

where $C_a =$const.$\neq 0$ and

$$B^1 = B^1\left(x^2, x^3\right), \quad B^2 = B^2\left(x^1, x^3\right), \quad B^3 = B^3\left(x^1, x^2\right) \tag{55}$$

Using (54) and (55), from (53) we first have

$$B^a_{,b} = C_b \frac{C_a x^a + B^a}{C_b x^b + B^b}, \quad (a \neq b)$$

and then

$$B^a_{,bc} = 0$$

and finally

$$B^1 = C_0 + C_2 x^2 + C_3 x^3, \quad B^2 = C_0 + C_1 x^1 + C_3 x^3, \quad B^3 = C_0 + C_1 x^1 + C_2 x^2$$

($C_0 =$const.). Therefore, (54) becomes

$$A_a = -\frac{C_a}{g} \tag{56}$$

where

$$g = C_p x^p = C_0 + C_1 x^1 + C_2 x^2 + C_3 x^3 \tag{57}$$

By this value of A_a, from (46), which is written for $a = b$, we find

$$\mathbf{T}_{,a} = \frac{\mathbf{f}_a}{g^2} \tag{58}$$

where

$$\mathbf{f}_1 = \mathbf{f}_1\left(x^2, x^3\right), \quad \mathbf{f}_2 = \mathbf{f}_2\left(x^1, x^3\right), \quad \mathbf{f}_3 = \mathbf{f}_3\left(x^1, x^2\right) \tag{59}$$

Using (58) and (59), from (46) we have

$$\mathbf{f}_{a,b} = \frac{C_b \mathbf{f}_a - C_a \mathbf{f}_b}{g} \tag{60}$$

and so

$$\mathbf{f}_{a,b} = -\mathbf{f}_{b,a} \tag{61}$$

Differentiating both sides of (60) we first obtain

$$\mathbf{f}_{a,bc} = 0 \tag{62}$$

Then we have

$$\mathbf{f}_a = \mathbf{E}_a + \mathbf{E}_{ab} x^b, \qquad (a, b = 1, 2, 3) \tag{63}$$

where

$$\mathbf{E}_{ab} = -\mathbf{E}_{ba} \tag{64}$$

$$C_0 \mathbf{E}_{ab} = C_b \mathbf{E}_a - C_a \mathbf{E}_b, \quad (C_1 \mathbf{E}_{23} = C_2 \mathbf{E}_{13} - C_3 \mathbf{E}_{12}) \tag{65}$$

Here $\mathbf{E}_a$ and $\mathbf{E}_{ab}$ are constant vectors. Thus (58) transforms to

$$\mathbf{T}_{,a} = \frac{\mathbf{E}_a + \mathbf{E}_{ab} x^b}{g^2} \tag{66}$$

By integration of the last equation we obtain

$$\mathbf{T} = -\frac{\mathbf{E}_a + \mathbf{E}_{ab} x^b}{C_a g} + \mathbf{h}_a, \quad (C_a \neq 0) \tag{67}$$

where

$$\mathbf{h}_1 = \mathbf{h}_1\left(x^2, x^3\right), \quad \mathbf{h}_2 = \mathbf{h}_2\left(x^1, x^3\right), \quad \mathbf{h}_3 = \mathbf{h}_3\left(x^1, x^2\right) \tag{68}$$

Using conditions (64) and (65), from (66) and (67) we find that the vectors $\mathbf{h}_1$, $\mathbf{h}_2$, and $\mathbf{h}_3$ are constant vectors. Therefore, we have

$$\mathbf{T} = \frac{\mathbf{e}_0 + \mathbf{e}_1 x^1 + \mathbf{e}_2 x^2 + \mathbf{e}_3 x^3}{C_0 + C_1 x^1 + C_2 x^2 + C_3 x^3} \tag{69}$$

where $\mathbf{e}_p$ vectors are constants. Thus, this transformation is the projective transformation. Therefore, we have the following theorem that gives a characterization of the projective transformation.

Theorem 2 *In Minkowski 3-space, a transformation preserves the asymptotic directions of a surface if and only if it is a projective transformation.*

Acknowledgment

The author thanks the referee for his/her valuable comments and suggestions.

References

[1] Alagöz Y, Soyuçok Z. A characterization of the projective transformation. Int J Phys Sci 2010; 5: 001–005.

[2] Alagöz Y, Soyuçok Z. On the transformation preserving asymptotic directions of hypersurfaces in the Euclidean space. Ejpam 2009; 2: 564–573.

[3] Eisenhart LP. A Treatise on the Differential Geometry of Curves and Surfaces. Mineola, NY, USA: Dover Publications, 1960.

[4] Milnor TK. Harmonic maps and classical surface theory in Minkowski 3-space. T Am Math Soc 1983; 280: 161–185.

[5] Lovett EO. Note general projective transformation. Ann Math 1895-1896; 10: 5–16.

[6] Efimov NV. Qualitative problems of the theory of deformation of surfaces. Am Math Soc; Translation no. 37: 355.

[7] O'Neill B. Semi Riemannian Geometry with Applications to Relativity. London, UK: Academic Press, 1983.

[8] Özkan MA, Soyuçok Z. Transformations preserving the asymptotic and Darboux directions of a surface and their effects on the infinitesimal bending. Bull Tech Univ 1987; 40: 497–512.

[9] Weinstein T. An Introduction to Lorentz Surfaces. Berlin, Germany: De Gruyter, 1996.