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The property of real hypersurfaces in 2-dimensional complex space form with Ricci operator

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Abstract: Let M be a real hypersurface in a complex space form $M_2(c)$, $c \neq 0$. In this paper, we prove that $S\phi = \phi S$ on M if and only if M is pseudo-Einstein.

Key words: Real hypersurface, Ricci operator, Hopf hypersurface, Pseudo-Einstein hypersurface

1. Introduction

A complex n-dimensional Kaehlerian manifold of constant holomorphic sectional curvature c is called a *complex space form*, which is denoted by $M_n(c)$. As is well known, a complete and simply connected complex space form is complex analytically isometric to a complex projective space $P_n\mathbf{C}$, a complex Euclidean space $\mathbf{C}^n$ or a complex hyperbolic space $H_n\mathbf{C}$, according to $c > 0$, $c = 0$, or $c < 0$. In this paper, we consider a real hypersurface M in a complex space form $M_n(c)$, $c \neq 0$. Then M has an almost contact metric structure (ϕ, g, ξ, η) induced from the Kaehler metric and complex structure J on $M_n(c)$. The structure vector field ξ is said to be *principal* if $A\xi = \alpha\xi$ is satisfied, where A is the shape operator of M and $\alpha = \eta(A\xi)$. In this case, it is known that α is locally constant and that M is called a *Hopf hypersurface*.

Takagi [6] completely classified homogeneous real hypersurfaces in such hypersurfaces as 6 model spaces, A_1, A_2, B, C, D, and E. Berndt [1] classified all homogeneous Hopf hypersurfaces in $H_n\mathbf{C}$ as 4 model spaces, which are said to be A_0, A_1, A_2, and B.

The Ricci operator of M will be denoted by S. One of the most interesting problems in the study of real hypersurfaces M in $M_n(c)$ is to investigate a geometric characterization of these model spaces. M satisfying $\phi S = S\phi$ have been classified for $n \geq 3$. Refer to Theorems 6.18–6.19 in the Niebergall–Ryan survey[4].

The holomorphic distribution T_0 of a real hypersurface M in $M_n(c)$ is defined by

$$T_0(p) = \{X \in T_p(M) \mid g(X, \xi)_p = 0\},$$

where $T_p(M)$ is the tangent space of M at $p \in M$.

The Ricci operator S is said to be *η-parallel* if

$$g((\nabla_X S)Y, Z) = 0 \tag{1}$$

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for any vector field X, Y, and Z in T_0.

As for Ricci operator and structure tensor ϕ, one of the present authors proved the following.

Theorem 1 *([5]) Let M be a real hypersurface with η-parallel Ricci operator in a complex space form $M_n(c)$, $c \neq 0$, $n \geq 3$. If M satisfies*

$$g((S\phi - \phi S)X, Y) = 0 \tag{2}$$

for any X and Y in T_0, then M is locally congruent to one of the model spaces of type A or type B.

For the Ricci operator S on a real hypersurface M, we define pseudo-Einstein if there exist constants ρ and σ such that for any tangent vector X,

$$SX = \rho X + \sigma\eta(X)\xi$$

, where S and $\eta(X)$ denote the Ricci operator and the dual 1-form of the unit vector field ξ. Additionally, with respect to the Ricci operator and η-parallel, Song and 2 of the present authors [3] proved the following.

Theorem 2 *([3]) A real hypersurface in a complex space form $M_2(c)$, $c \neq 0$, satisfies (1) and (2) if and only if it is pseudo-Einstein.*

The purpose of this paper is to investigate the structure of space in tangent bundle TM by the Ricci operator. Concretely, we shall prove the following.

Main theorem *A real hypersurface in a complex space form $M_2(c)$, $c \neq 0$, satisfies $S\phi = \phi S$ if and only if it is pseudo-Einstein.*

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2. Preliminaries

Let M be a real hypersurface immersed in a complex space form $M_2(c)$, and let N be a unit normal vector field of M. By $\widetilde{\nabla}$ we denote the Levi-Civita connection with respect to the Fubini–Study metric tensor $\widetilde{g}$ of $M_2(c)$. Then the Gauss and Weingarten formulas are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \qquad \widetilde{\nabla}_X N = -AX$$

for any vector fields X and Y tangent to M, where g denotes the Riemannian metric tensor of M induced from $\widetilde{g}$ and A is the shape operator of M in $M_2(c)$. For any vector field X on M we put

$$JX = \phi X + \eta(X)N, \qquad JN = -\xi,$$

where J is the almost complex structure of $M_2(c)$. Then we see that M induces an almost contact metric structure (ϕ, g, ξ, η); that is,

$$\begin{gathered} \phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\xi) = 1, \\ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi) \end{gathered} \tag{3}$$

for any vector fields X and Y on M. Since the almost complex structure J is parallel, we can verify from the Gauss formula that

$$\nabla_X \xi = \phi AX. \tag{4}$$

Since the ambient manifold is of constant holomorphic sectional curvature c, we have the Gauss equation

$$\begin{aligned} R(X,Y)Z = & \tfrac{c}{4}\{g(Y,Z)X - g(X,Z)Y + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y \\ & -2g(\phi X,Y)\phi Z\} + g(AY,Z)AX - g(AX,Z)AY \end{aligned} \tag{5}$$

for any vector fields X, Y, and Z on M, where R denotes the Riemannian curvature tensor of M. From (3) the Ricci operator S of M is expressed by

$$SX = \frac{c}{4}\{(2n+1)X - 3\eta(X)\xi\} + mAX - A^2X, \tag{6}$$

where $m = traceA$ is the mean curvature of M.

3. Proof of the main theorem

Let W be a unit vector field on M with the same direction of the vector field $-\phi\nabla_\xi\xi$, and let μ be the length of the vector field $-\phi\nabla_\xi\xi$ if it does not vanish. It is not possible to define W without specifying that $\mu(p) \neq 0$. Then it is easily seen from (4) that

$$A\xi = \alpha\xi + \mu W, \tag{7}$$

where $\alpha = \eta(A\xi)$. We notice that W is orthogonal to ξ.

In this section, we assume that M is not Hopf. Then there are scalar fields γ, ε, and δ and a unit vector field W and ϕW orthogonal to ξ such that

$$AW = \mu\xi + \gamma W + \varepsilon\phi W, \qquad A\phi W = \varepsilon W + \delta\phi W \tag{8}$$

and

$$m = traceA = \alpha + \gamma + \delta \tag{9}$$

in $M_2(c)$. We first prove the following lemma.

Lemma 3 *Let M be a real hypersurface with $\mu \neq 0$ satisfying $S\phi = \phi S$ in a complex space form $M_2(c)$, $c \neq 0$. Then we have $AW = \mu\xi + \gamma W$, $A\phi W = 0$, and $\mu^2 = \alpha\gamma$.*

Proof By making the substitutions $X = \xi$, $X = W$, $X = \phi W$ in (6) and using (7)–(9), we have the following equations:

$$S\xi = (\frac{c}{2} + \alpha\gamma + \alpha\delta - \mu^2)\xi + \mu\delta W - \mu\varepsilon\phi W,$$

$$SW = \mu\delta\xi + (\frac{5c}{4} + \alpha\gamma + \gamma\delta - \mu^2 - \varepsilon^2)W + \alpha\varepsilon\phi W,$$

$$S\phi W = -\mu\varepsilon\xi + \alpha\varepsilon W + (\frac{5c}{4} + \alpha\delta + \gamma\delta - \varepsilon^2)\phi W.$$

If we apply ϕ to the above third equation, then we have

$$(S\phi - \phi S)W = -\mu\varepsilon\xi + 2\alpha\varepsilon W + (\alpha\delta - \alpha\gamma + \mu^2)\phi W. \tag{10}$$

The condition $S\phi = \phi S$ together with (6) implies that

$$(\phi A^2 - A^2\phi)X = m(\phi A - A\phi)X. \tag{11}$$

If we put $X = \xi$ into (11) and use (7)–(9), we have $\varepsilon = 0$ and $\delta = 0$. Therefore, it follows that AW is expressed in terms of ξ and W only and $A\phi W = 0$. Putting $X = W$ into (11) and using the results of the above, we obtain $\mu^2 = \alpha\gamma$. □

We shall prove the main theorem.

Proof of main theorem. Assume that M is not Hopf, and work in a small set where $A\xi \neq \alpha\xi$ and therefore W, μ, etc. can be defined. From Lemma 3.1, the Ricci operator S expressed that

$$S\xi = \frac{c}{2}\xi, \quad SW = \frac{5c}{4}W, \quad S\phi W = \frac{5c}{4}\phi W.$$

That is, M is pseudo-Einstein with

$$SX = \frac{5c}{4}X - \frac{3c}{4}g(X,\xi)\xi.$$

This contradicts a result of Kim and Ryan [2]. Having shown that M must be Hopf, one can choose W to be any unit vector field orthogonal to ξ and then the condition $S\phi = \phi S$ yields $\alpha(\gamma - \delta) = 0$ and the criteria are satisfied (see [2]). Thus, M is pseudo-Einstein. Conversely, if M is pseudo-Einstein, observe that $S\phi = \phi S$ must be satisfied. □

Remark. In this paper, we proved that $S\phi = \phi S$ on M if and only if S is η-parallel and $g((S\phi - \phi S)X, Y) = 0$ for all X and Y in T_0.

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