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An alternative approach to the Adem relations in the mod 2 Steenrod algebra

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Abstract: The Leibniz–Hopf algebra $\mathcal{F}$ is the free associative algebra over $\mathbf{Z}$ on one generator S^n in each degree $n > 0$, with coproduct given by $\Delta(S^n) = \sum_{i+j=n} S^i \otimes S^j$. We introduce a new perspective on the Adem relations in the mod 2 Steenrod algebra $\mathcal{A}_2$ by studying the map π^* dual to the Hopf algebra epimorphism $\pi\colon \mathcal{F} \otimes \mathbf{Z}/2 \to \mathcal{A}_2$. We also express Milnor's Hopf algebra conjugation formula in $\mathcal{A}_2^*$ in a different form and give a new approach for the conjugation invariant problem in $\mathcal{A}_2^*$.

Key words: Adem relations, Hopf algebra, Leibniz–Hopf algebra, antipode, Steenrod algebra, quasisymmetric functions

1. Introduction

The *Leibniz–Hopf algebra* is the free associative $\mathbf{Z}$-algebra $\mathcal{F}$ on one generator S^n in each positive degree with the graded, connected Hopf algebra structure determined by giving S^n degree n and $\Delta(S^n) = \sum_{i+j=n} S^i \otimes S^j$ (where S^0 denotes 1) [11]. This Hopf algebra is cocommutative and has been studied as the *ring of noncommutative symmetric functions* [4, 10, 12]. A topological model for this Hopf algebra is given by interpreting it as the homology of the loop space of the suspension of the infinite complex projective space, $H_*(\Omega\Sigma\mathbf{C}P^\infty)$ [2]. The graded dual of the Leibniz–Hopf algebra $\mathcal{F}^*$ is the *ring of quasisymmetric functions* with the outer coproduct [4, 14], which was the subject of the Ditters conjecture [3, 11, 12, 13] and isomorphic to the cohomology of $\Omega\Sigma\mathbf{C}P^\infty$ [2], making it relevant to a wide area of combinatorics, algebra, and topology. Note that in [11, Section 1] the graded dual of $\mathcal{F}$ over the integers is denoted by $\mathcal{M}$ and is called the *overlapping shuffle algebra*.

The mod 2 reduction $\mathcal{F} \otimes \mathbf{Z}/2$ also has a connection with topology, since it has the mod 2 Steenrod algebra $\mathcal{A}_2$ [4, Section 5] as a quotient. $\mathcal{A}_2$ is a vector space over $\mathbf{Z}/2$ with a basis made by admissible monomials [17]. Milnor [15] showed that the mod 2 dual Steenrod algebra $\mathcal{A}_2^*$ is a polynomial algebra on $\xi_1, \xi_2, \xi_3, \ldots$, where the grading of ξ_i is $2^i - 1$. In [15], Milnor also showed that $\mathcal{A}_2^*$ is also a Hopf algebra with a unique *antipode* or *conjugation*, here denoted by $\chi_{\mathcal{A}_2^*}$. This conjugation is an important tool in algebraic topology, since it is relevant for the commutativity of ring spectra [1, Lecture 3]. An element $x \in \mathcal{A}_2^*$ is an invariant under $\chi_{\mathcal{A}_2^*}$ if and only if $\chi_{\mathcal{A}_2^*}(x) = x$. In other words, $(\chi_{\mathcal{A}_2^*} - 1)(x) = 0$ (where 1 denotes the identity homomorphism). Thus, $\mathrm{Ker}(\chi_{\mathcal{A}_2^*} - 1)$ is a subvector space of $\mathcal{A}_2^*$, which is formed by the conjugation invariants in $\mathcal{A}_2^*$.

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In [7], some progress was made to calculate $\mathrm{Ker}(\chi_{\mathcal{A}_2^*}-1)$; however, a complete picture was not achieved. In [6], as another approach, motivated by the work of Crossley and Whitehouse [7, 8], a vector space basis was calculated for the conjugation invariants in the mod 2 dual Leibniz–Hopf algebra $\mathcal{F}_2^*$ (where $\mathcal{F}\otimes\mathbf{Z}/2$ is denoted by $\mathcal{F}_2$). The problem of finding conjugation invariants is interesting and was also studied in [5].

In this paper we introduce a different view of the Adem relations in terms of $\pi^*\colon \mathcal{A}_2^* \to \mathcal{F}^*\otimes\mathbf{Z}/2$. A description of π^* gives rise to a new perspective on the Adem relations in $\mathcal{A}_2$ (details are given in Section 3) and leads us to present $\chi_{A_2^*}$ in a different form. Our results also lead to express duals of admissible monomials in terms of $\xi_1, \xi_2, \xi_3, \ldots$. In the last section, we give a detailed description of the connection between π^* and the vector space $\mathrm{Ker}(\chi_{\mathcal{A}_2^*}-1)$. At the end of the paper we give tables to support the calculations throughout the paper.

2. Preliminaries

In this section we introduce the main algebraic structures that are used in this work. Let O be an algebraic object. In the rest of the paper we denote the degree of O by $\deg(O)$, spanning set of O by $\mathrm{Span}(O)$, dimension of O by $\dim(O)$, and rank of O by rank (O).

$\mathcal{F}_2$ is the free $\mathbf{Z}/2$-algebra on generators $S^1, S^2, \ldots$, where S^i is of degree i. This algebra has a basis given by all words $S^{j_1}S^{j_2}\cdots S^{j_l}$. We denote the dual basis for $\mathcal{F}_2^*$ by $\{S_{j_1,j_2,\ldots,j_l}\}$. We now give a slightly revised version of the definition of an overlapping shuffle product [4, Section 2]:

The *overlapping shuffle product* of $S_{a_1,\ldots,a_k}$ and $S_{b_1,\ldots,b_m}$ is denoted by μ and defined by

$$\mu(S_{a_1,\ldots,a_k}\otimes S_{b_1,\ldots,b_m}) = \sum_h h(S_{a_1,\ldots,a_k,b_1,\ldots,b_m}),$$

where h first inserts a certain number ℓ of 0s into $a_1,\ldots,a_k$, and inserts a number of $\ell^{'}$ of 0s into $b_1,\ldots,b_m$, where

$$0\le \ell\le m, \quad 0\le \ell^{'}\le k, \quad k+\ell = m+\ell^{'},$$

and then it adds the first indices together, then the second, and so on. The sum is over all such h for which the result contains no 0. In $\mathcal{F}_2^*$, as an example,

$$\mu(S_{3,2}\otimes S_4) = S_{3,2,4}+S_{3,4,2}+S_{4,3,2}+S_{7,2}+S_{3,6}.$$
$$\mu(S_4\otimes S_4) = S_{4,4}+S_{4,4}+S_8 = S_8$$

(see [11, Section 2] for an alternative description of this product).

Ehrenborg [9, Proposition 3.4] gave a formula for the conjugation on $\mathcal{F}^*$. The mod 2 reduction of this formula is given by

$$\chi(S_{j_1,j_2,\ldots,j_l}) = \sum S_{i_1,\ldots,i_k}$$

summed over all coarsenings $i_1,\ldots,i_k$ of the *reversed* word $j_l,\ldots,j_2,j_1$, i.e. all words $i_1,\ldots,i_k$ that admit $j_l,\ldots,j_2,j_1$ as a refinement [6].

In order to make Section 5 of this paper more clear, we recall some of the terminology from [6]. A word $S_{j_1,j_2,\ldots,j_n}$ is a *palindrome* if $j_g = j_{n-(g-1)}$ for all $g\in\{1,\ldots,n\}$. A palindrome is referred to as an *even-length* *palindrome* or an *odd-length palindrome*, here denoted by ELP and OLP, respectively, according to whether its length is even or odd. Hence, for example, $S_{2,3,2}$ is an OLP and $S_{4,1,1,4}$ is an ELP. A non-palindrome $S_{j_1,\ldots,j_k}$ is referred to as a *higher non-palindrome*, here denoted by HNP, if $j_1,\ldots,j_k$ is bigger than its reverse $j_k,\ldots,j_1$, with respect to left lexicographic ordering. For instance, $S_{5,4,2,5}$ is an HNP, since $5,4,2,5$ is lexicographically bigger than $5,2,4,5$. Let $S_{j_1,\ldots,j_{2m+1}}$ be an OLP. The "λ"-image is defined as

$$\lambda(S_{j_1,\ldots,j_{2m+1}}) = \sum S_{i_1,\ldots,i_k,j_{m+2},\ldots,j_{2m+1}}$$

summed over all words $S_{i_1,\ldots,i_k,j_{m+2},\ldots,j_{2m+1}}$, where $j_1,\ldots,j_{m+1}$ is a refinement of $i_1,\ldots,i_k$. For example, $\lambda(S_{1,1,1,1,1}) = S_{1,1,1,1,1} + S_{2,1,1,1} + S_{1,2,1,1} + S_{3,1,1}$.

3. A different view of the Adem relations

We now turn our attention to $\mathcal{A}_2$. Let Sq^n denote the Steenrod square of degree n [17]. Then $\mathcal{A}_2$ is defined as a quotient of $\mathcal{F}_2$ by the Adem relations:

$$Sq^aSq^b = \sum_{j=0}^{[\frac{a}{2}]} \binom{b-1-j}{a-2j} Sq^{a+b-j}Sq^j, \qquad 0 < a < 2b, \tag{1}$$

and $Sq^0 = 1$, giving a graded algebra epimorphism $\pi\colon \mathcal{F}_2 \to \mathcal{A}_2$ (i.e. it preserves degrees), where $\pi(S^n) = Sq^n$. Furthermore, π is a graded Hopf algebra epimorphism, because the coproduct on the generators is defined in the same way for $\mathcal{F}_2$ as for $\mathcal{A}_2$. Note that $\mathcal{A}_2$ is also a connected algebra.

Since π is a Hopf algebra epimorphism, its dual $\pi^*\colon \mathcal{A}_2^* \to \mathcal{F}_2^*$ is also a Hopf algebra morphism. In particular, π^* is multiplicative. It is also a Hopf algebra inclusion [6]. Note that by the dual we mean the graded dual of π.

We write $Sq^I = Sq^{i_1}\cdots Sq^{i_k}$, where $I = (i_1,\ldots,i_k)$ is a sequence of positive integers, where $\deg(I) = i_1 + i_2 + \cdots + i_k$, and say that I is admissible if $i_{r-1} \geq 2i_r$ for $2 \leq r \leq k$ and $i_r \geq 1$. $\mathcal{A}_2$ is a vector space over $\mathbf{Z}/2$ and its admissible monomials form a basis. We denote the corresponding dual basis element by Sq_I and define it by the duality as follows:

$$Sq_I(Sq^J) = \begin{cases} 1 & \text{if } I = J, \\ 0 & \text{otherwise}, \end{cases}$$

where J is a sequence of positive integers.

Up to degree 4, a vector space basis for $\mathcal{A}_2^*$ is

$$1, \quad Sq_1, \quad Sq_2, \quad Sq_3, \quad Sq_{2,1} \quad Sq_4, \quad Sq_{3,1}.$$

In this section we give some descriptions of π^* on Sq_Is. A description of π^* gives rise to a different view of the Adem relations. More precisely, this comes from looking at π^* rather than π. The Adem relations are the kernel of π, and π is defined directly from the Adem relations. Hence, π^* contains all information about the Adem relations. If we could give a formula for π^*, the Adem relations could be retrieved from it. It can be hard to give that formula in higher degrees, but in lower degrees we can see it. See Table 1 to observe what π^* does to each basis element in those degrees. We now give the following example:

Table 1. π^*-images of dual admissible basis elements up to degree 5.

Degree 1	$\pi^*(Sq_1) =$	S_1
Degree 2	$\pi^*(Sq_2) =$	S_2
Degree 3	$\pi^*(Sq_3) =$	$S_3 + S_{1,2}$
	$\pi^*(Sq_{2,1}) =$	$S_{2,1}$
Degree 4	$\pi^*(Sq_4) =$	S_4
	$\pi^*(Sq_{3,1}) =$	$S_{3,1} + S_{2,2} + S_{1,2,1}$
Degree 5	$\pi^*(Sq_5) =$	$S_5 + S_{2,3} + S_{2,1,2} + S_{1,4}$
	$\pi^*(Sq_{4,1}) =$	$S_{4,1} + S_{2,3} + S_{2,1,2}$

Example 3.1 Let us calculate $\pi^\star(Sq_3)$. Since $\pi^\star(Sq_3)$ is equal to $Sq_3 \circ \pi$ and $\deg(Sq_3) = 3$, it belongs to $\mathrm{Span}\left\{S^3, S^2S^1, S^1S^2, S^1S^1S^1\right\}$. The map π first gives:

$$\pi(S^3) = Sq^3, \quad \pi(S^2S^1) = Sq^2Sq^1, \quad \pi(S^1S^2) = Sq^1Sq^2, \quad \pi(S^1S^1S^1) = Sq^1Sq^1.$$

Since π is a quotient map, we get:

$$\pi(S^3) = Sq^3, \quad \pi(S^2S^1) = Sq^2Sq^1, \quad \pi(S^1S^2) = Sq^3, \quad \pi(S^1S^1S^1) = 0.$$

Hence, $\pi^\star(Sq_3)$ has S_3 and $S_{1,2}$ as a summand, i.e. $\pi^\star(Sq_3) = S_3 + S_{1,2}$.

Let C be an arbitrary length admissible sequence of degree m. It is natural to ask: what are the summands of $\pi^\star(Sq_C)$? By definition of $\pi^\star(Sq_C)$, we write:

$$\pi^\star(Sq_C) = \sum S_{i_1,i_2,\ldots,i_k},$$

summed over all sequences $i_1, i_2, \ldots, i_k$ of degree m for which $Sq^{i_1,i_2,\ldots,i_k}$ has Sq^C as a summand when expressed as a sum of elements in the admissible basis elements. More precisely, we write

$$\pi^\star(Sq_C) = S_C + \sum S_{j_1,j_2,\ldots,j_r}, \tag{2}$$

summed over all (non-admissible) sequences $j_1, j_2, \ldots, j_r$ for which $Sq^{j_1,j_2,\ldots,j_r}$ has Sq^C as a summand when expressed as a sum of elements in the admissible basis elements.

Problem 3.2 Can we find an explicit formula for $\pi^\star(Sq_C)$?

We give particular answers to Problem 3.2 in the following:

Proposition 3.3 *Let $a > 0$ and $b > 0$ be integers with $a + b = n$. Then*

$$\pi^\star(Sq_n) \text{ has } S_{a,b} \text{ as a summand} \iff \binom{b-1}{a} \equiv 1 \pmod 2.$$

Proof Let $\deg(S_{a,b}) = n$. $\pi^\star(Sq_n)$ has $S_{a,b}$ as a summand $\Leftrightarrow$ $Sq^{a,b}$ has Sq^n as a summand when written as sum of elements in the admissible basis. This is only possible for $j = 0$ and $n = a + b$ in the Adem relations in Eq. (1). □

Proposition 3.4 $\pi^\star(Sq_{2^n}) = S_{2^n}$ *for all* $n \geq 0$.

Proof $\pi^\star(Sq_{2^n})$ has $S_{i_1,\ldots,i_k}$ as a summand $\Leftrightarrow$ $Sq^{i_1}\cdots Sq^{i_k}$ has Sq^{2^n} as a summand when expressed as the sum of elements in the admissible basis. The rest of the proof follows from the fact that Sq^{2^n} is indecomposable (see Lemma 4.2 of [17, Chapter 1]). $\square$

Proposition 3.5 $\pi^\star(Sq_{2^{n-1},2^{n-2},\ldots,2^k}) = S_{2^{n-1},2^{n-2},\ldots,2^k}$ *for all* $n > k \geq 0$.

Proof The proof is inspired by proof of Proposition 1.2.3 in [16]. We first see $\deg(Sq_{2^{n-1},2^{n-2},\ldots,2^k}) = 2^n - 2^k$. Let $J = j_1, j_2, \ldots, j_v$ be any nonadmissible sequence of degree $2^n - 2^k$. $\pi^\star(Sq_{2^{n-1},2^{n-2},\ldots,2^k})$ has $S_{j_1,i_2,\ldots,j_v}$ as a summand $\Leftrightarrow$ $Sq^{j_1,j_2,\ldots,j_v}$ has $Sq^{2^{n-1},2^{n-2},\ldots,2^k}$ as a summand when expressed as sum of elements in the admissible basis. If J is nonadmissible, then $v > n-k$ and there exists r, $1 \leq r \leq v-1$ such that $j_r < 2j_{r+1}$. By the Adem relations, we get

$$Sq^J = \sum_{0}^{\left[\frac{j_r}{2}\right]} \lambda_y Sq^{J'} Sq^{j_r+j_{r+1}-y} Sq^y Sq^{J''},$$

where $\lambda_y \in \mathbf{Z}/2$, $J' = j_1, \ldots, j_{r-1}$, $J'' = j_{r+2}, \ldots, j_v$, $0 \leq y \leq \left[\frac{j_r}{2}\right]$. However, for Sq^J having $Sq^{2^{n-1},2^{n-2},\ldots,2^k}$ as a summand, $Sq^{j_r+j_{r+1}-y}$ must be equal to $Sq^{2^{n-r}}$. The rest of the proof can be seen by adapting the proof of Proposition 3.4. $\square$

One can wonder if Proposition 3.5 still holds for the $\pi^\star$-image of any admissible sequence. This first fails in degree 4, since $\pi^\star(Sq_{3,1}) = S_{3,1} + S_{2,2}$.

4. $\pi^\star$ via $\xi_1, \xi_2, \ldots$

We first recall the definition of ξ_n [17]:

$$< \xi_n, Sq^T >= \begin{cases} 1 & \text{if } T = T^n, \\ 0 & \text{otherwise,} \end{cases}$$

where $T^n = (2^{n-1}, 2^{n-2}, \ldots, 2, 1)$ for $n \geq 1$.

As $\mathcal{A}_2^*$ is a polynomial algebra, $\mathrm{Im}(\pi^\star)$ is generated by $\pi^\star(\xi_i)$, but we do not have a good description for $\pi^\star(\xi_1^{i_1}\xi_2^{i_2}\ldots\xi_n^{i_n})$.

Problem 4.1 It would be nice to have an algorithm to establish if a fixed element in $\mathcal{F}_2^*$, i.e. a linear combination of the independent monomials $\{S_{j_1,j_2,\ldots,j_l}\}$, belongs to $\mathrm{Im}(\pi^*)$, and, if this is the case, to identify the counter-image.

Although a proof is not given, a particular answer is given by Crossley [4, Section 5] in the following:

$$\pi^\star(\xi_n) = S_{2^{n-1},2^{n-2},\ldots,2,1}. \tag{3}$$

$$\pi^\star(\xi_n^{2^r}) = S_{2^{r+n-1},2^{r+n-2},\ldots,2^{r+1},2^r}. \tag{4}$$

By Eq. (3), we can easily see the following:

$$\pi^\star(Sq_{2^n,2^{n-1},\ldots,2,1}) = S_{2^n,2^{n-1},\ldots,2,1} \quad \text{for all} \quad n \geq 0. \tag{5}$$

Note that Eq. (5) can also be seen by Proposition 3.5.

We now consider some calculations of π^* in Propositions 4.2 and 4.3. It is worth mentioning that we will do our calculations in $\mathcal{F}_2^*$. By Section 3, we know π^* is a Hopf algebra morphism. In particular, π^* is an algebra morphism on the target overlapping shuffle product [11, Section 6].

Proposition 4.2 *$\pi^\star(\xi_1^{2^n}) = S_{2^n}$ for all $n \geq 0$.*

Proof Proof is by induction on r. ξ_1 and S_1 are the only degree one basis elements in $\mathcal{A}_2^*$ and $\mathcal{F}_2^*$, respectively. On the other hand, $\pi^\star$ is a degree-preserving morphism, so $\pi^\star(\xi_1) = S_1$. By the inductive hypothesis, $\pi^\star(\xi_1^{2^{n-1}}) = S_{2^{n-1}}$. Since $\pi^\star$ is an algebra morphism, we write $\pi^\star(\xi_1^{2^n})$ as a product of 2 copies of $\pi^\star(\xi_1^{2^{n-1}})$, i.e. $\pi^\star(\xi_1^{2^n}) = \pi^\star(\xi_1^{2^{n-1}})\pi^\star(\xi_1^{2^{n-1}})$. Note that by product we mean the overlapping shuffle product. Hence, $\pi^\star(\xi_1^{2^{n-1}})\pi^\star(\xi_1^{2^{n-1}}) = S_{2^{n-1}}S_{2^{n-1}} = S_{2^n}$. $\square$

Proposition 4.3 *$\pi^\star(\xi_2^{2^n}) = S_{2^{n+1},2^n}$ for all $n \geq 0$.*

Proof Proof is by induction on n. When $n = 0$, Eq. (3) satisfies the first step of the induction. By the inductive hypothesis, $\pi^\star(\xi_2^{2^{n-1}}) = S_{2^n,2^{n-1}}$. Similar to the proof of Proposition 4.2, we arrive at:

$$\pi^\star(\xi_2^{2^n}) = S_{2^n,2^{n-1}}S_{2^n,2^{n-1}}.$$

By the overlapping shuffle product, we have:

$$\begin{aligned} S_{2^n,2^{n-1}}S_{2^n,2^{n-1}} &= S_{2^{n+1},2^n} + 2S_{2^n,2^n,2^n} + 2S_{2^n,2^n+2^{n-1},2^{n-1}} + 2S_{2^{n+1},2^{n-1},2^{n-1}} \\ &\quad + 2S_{2^n,2^{n-1},2^n,2^{n-1}} + 4S_{2^n,2^n,2^{n-1},2^{n-1}}. \end{aligned}$$

This completes the proof, since we work on mod 2. $\square$

Corollary 4.4 *$\xi_1^{2^n} = Sq_{2^n}$ for all $n \geq 0$.*

Theorem 4.5 *$\chi_{\mathcal{A}_2^*}(\xi_n) = Sq_{2^n-1}$ for all $n \geq 1$.*

Before proving Theorem 4.5, we first define the linear transformation $r\colon \mathcal{F}_2^* \to \mathcal{A}_2^*$ by:

$$r\colon \mathcal{F}_2^* \to \mathcal{A}_2^*, \quad r(S_I) = \begin{cases} Sq_I & \text{if I is admissible,} \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Lemma 4.6 *$r \circ \pi^*$ is the identity function on $\mathcal{A}_2^*$.*

Proof For an admissible sequence C, we first calculate $\pi^*(Sq_C)$. By Eq. (2), we write

$$\pi^\star(Sq_C) = S_C + \sum S_{j_1,j_2,\ldots,j_r}, \tag{7}$$

summed over all (nonadmissible) sequences $j_1, j_2, \ldots, j_r$ for which $Sq^{j_1,j_2,\ldots,j_r}$ has Sq^C as a summand when expressed as a sum of elements in the admissible basis elements. Applying r to both sides of Eq. (7), we get:

$$r(\pi^*(Sq_C)) = r\left(S_C + \sum S_{j_1,j_2,\ldots,j_r}\right) = Sq_C.$$

For all basis elements Sq_C, we have $r(\pi^*(Sq_C)) = Sq_C$. This completes the proof. □

Example 4.7 $(r \circ \pi^*)(Sq_5) = r(S_5 + S_{2,3} + S_{2,1,2} + S_{1,4}) = Sq_5$.

Proof [Proof of Theorem 4.5] Since π^* is a Hopf algebra morphism, the following diagram commutes.

$$\begin{tikzcd} \mathcal{A}_2^* \arrow[r, hook, "\pi^*"] \arrow[d, "\chi_{\mathcal{A}_2^*}"] & \mathcal{F}_2^* \arrow[d, "\chi"] \\ \mathcal{A}_2^* \arrow[r, hook, "\pi^*"] & \mathcal{F}_2^* \end{tikzcd} \tag{8}$$

By Lemma 4.6 and the commutativity of the diagram (8), we have the following commutative diagram.

$$\begin{tikzcd} \mathcal{A}_2^* \arrow[r, hook, "\pi^*"] \arrow[d, "\chi_{\mathcal{A}_2^*}"] & \mathcal{F}_2^* \arrow[d, "\chi"] \\ \mathcal{A}_2^* \arrow[r, hook, "\pi^*"] \arrow[d, equal] & \mathcal{F}_2^* \arrow[dl, "r"] \\ \mathcal{A}_2^* & \end{tikzcd}$$

Hence,

$$\chi_{\mathcal{A}_2^*} = r \circ \chi \circ \pi^*.$$

By definition of χ and admissible monomial sequence, it follows that:

$$\chi_{\mathcal{A}_2^*}(\xi_n) = Sq_{2^n-1}.$$

□

Theorem 4.5 gives a different form of $\chi_{\mathcal{A}_2^*}$ but does not say if $\chi_{\mathcal{A}_2^*}$ is multiplicative. However, this theorem leads to the following results.

Corollary 4.8

$$Sq_{2^n-1} = \sum_{\alpha \in Part(n)} \prod_{i=1}^{l(\alpha)} \xi_{\alpha(i)}^{2^{\sigma(i)}} \quad \textit{for all} \quad n \geq 1,$$

where $Part(n)$ denotes the set of all ordered partitions of n, and for a given ordered partition $\alpha = (\alpha(1)|\alpha(2)|\cdots|\alpha(l)) \in Part(n)$, $\sigma(i) = \sigma(1) + \cdots + \sigma(i-1)$.

Proof It can be seen by Theorem 4.5 along with Lemma 1.1 of [7]. □

Corollary 4.9 $\chi_{\mathcal{A}_2^*}(Sq_{2^n-1}) = \xi_n$, *for* $n \geq 1$.

Proof Since $\mathcal{A}_2^*$ is a commutative Hopf algebra, $\chi^2_{\mathcal{A}_2^*} = 1$. Using this, the proof can be seen by Theorem 4.5. □

Theorem 4.10 $\pi^*(Sq_{2^n-1}) = \chi(S_{2^{n-1},2^{n-2},\ldots,2,1})$, *for* $n \geq 1$.

Proof By commutativity of diagram (8), Corollary 4.9, and Eq. (3), we arrive at

$$\chi(\pi^*(Sq_{2^n-1})) = S_{2^{n-1},\ldots,2,1}. \tag{9}$$

Applying χ to both sides of Eq. (9) completes the proof. □

5. A strategy for computing conjugation invariants in $\mathcal{A}_2^*$

In [7], although a complete description is not given for $\mathrm{Ker}(\chi_{\mathcal{A}_2^*} - 1)$, Crossley and Whitehouse established bounds on its dimension in each degree. In this section we introduce a method for determining $\mathrm{Ker}(\chi_{\mathcal{A}_2^*} - 1)$ and give examples. Our method proposes an understanding for a basis of $\mathrm{Ker}(\chi_{\mathcal{A}_2^*} - 1)$.

Theorem 5.1 *The space of conjugation invariants,* $\mathrm{Ker}(\chi - 1)$, *has a basis consisting of: (i)* $(\chi - 1)$*-images of all ELPs; (ii)* $(\chi - 1)$*-images of all HNPs; and (iii) the* λ*-images of all odd degree OLPs.*

Proof See [6, Theorem 2.7]. □

In each fixed degree, conjugation invariants in $\mathcal{A}_2^*$ have a link with π^* and conjugation invariants in $\mathcal{F}_2^*$ as follows:

Theorem 5.2

$$\pi^*(\mathrm{Ker}(\chi_{\mathcal{A}_2^*} - 1)) = \mathrm{Ker}(\chi - 1) \cap \pi^*(\mathcal{A}_2^*).$$

Proof Injectivity of π^* and commutativity of diagram (8) complete the proof. □

Theorem 5.3 *Let* $S_{2^a,2^b}$ *be an HNP or an ELP. Then*

$$\pi^\star(\xi_1^{2^a}\xi_1^{2^b}) = (\chi - 1)(S_{2^a,2^b}).$$

Proof Let $S_{2^a,2^b}$ be an HNP; then, by definition, $(\chi - 1)(S_{2^a,2^b}) = S_{2^a,2^b} + S_{2^b,2^a} + S_{2^b+2^a}$. On the other hand, since π^* is an algebra morphism, by Proposition 4.2, $\pi^\star(\xi_1^{2^a}\xi_1^{2^b}) = (\chi - 1)(S_{2^a,2^b})$. The same argument also works for the ELP case. □

Corollary 5.4 *Let* $S_{2^a,2^b}$ *be an HNP or an ELP. Then in* $2^a + 2^b$ *degrees*

$$(\chi - 1)(S_{2^a,2^b}) \in \mathrm{Ker}(\chi - 1) \cap \pi^*(\mathcal{A}_2^*).$$

To illustrate Theorem 5.2 we give the following examples.

Example 5.5 In degree 2, $\mathcal{F}_2^*$ has a basis: $\{S_2, S_{1,1}\}$. By Theorem 5.1, $(\chi-1)$-images of HNPs and ELPs form a basis for $\mathrm{Ker}(\chi-1)$. In our case, that is $(\chi-1)(S_{1,1}) = S_2$, and then we have:

$$\mathrm{Ker}(\chi-1) = \{0, S_2\}.$$

On the other hand, in the same degree $\mathcal{A}_2^*$ has a basis $\{\xi_1^2\}$, and by Proposition 4.2, $\pi^*(\xi_1^2) = S_2$. Hence, we have: $\pi^*(\mathcal{A}_2^*) = \{0, S_2\}$. Finally, by Theorem 5.2, we arrive at:

$$\pi^*(\mathrm{Ker}(\chi_{\mathcal{A}_2^*} - 1)) = \{0, S_2\}.$$

Recalling that π^* is a monomorphism, we conclude that $\mathrm{Ker}(\chi_{\mathcal{A}_2^*} - 1)$ has a basis $\{\xi_1^2\}$ in degree 2.

Example 5.6 In degree 3, $\mathcal{F}_2^*$ has a basis: $\{S_3, S_{2,1}, S_{1,2}, S_{1,1,1}\}$. By Theorem 5.1, the basis elements of $\mathrm{Ker}(\chi-1)$ are: $(\chi-1)(S_{2,1}) = S_{2,1} + S_{1,2} + S_3$, $\lambda(S_{1,1,1}) = S_{1,1,1} + S_{2,1}$, and $\lambda(S_3) = S_3$. Hence, we have:

$$\begin{aligned}\mathrm{Ker}(\chi-1) =&\{0, S_3, S_3 + S_{2,1} + S_{1,2}, S_{2,1} + S_{1,2}, S_{2,1} + S_{1,1,1}, S_{1,2} + S_{1,1,1},\\ &S_3 + S_{2,1} + S_{1,1,1}, S_3 + S_{1,2} + S_{1,1,1}\}\end{aligned}.$$

$\mathcal{A}_2^*$ has a basis $\{\xi_1^3, \xi_2\}$, and $\pi^*(\xi_1^3) = \pi^*(\xi_1^2)\pi^*(\xi_1) = S_3 + S_{2,1} + S_{1,2}$, $\pi^*(\xi_2) = S_{2,1}$. Hence, we have:

$$\pi^*(\mathcal{A}_2^*) = \{0, S_3 + S_{2,1} + S_{1,2}, S_{2,1}, S_3 + S_{1,2}\}.$$

Finally, we arrive at:

$$\pi^*(\mathrm{Ker}(\chi_{\mathcal{A}_2^*} - 1)) = \{0, S_3 + S_{2,1} + S_{1,2}\},$$

from which we conclude that $\mathrm{Ker}(\chi_{\mathcal{A}_2^*} - 1)$ has a basis $\{\xi_1^3\}$ in degree 3.

Example 5.7 In this example we introduce an efficient method for calculations in higher degrees. In degree 4, we first give an order to the monomial basis of $\mathcal{F}_2^*$ with respect to lexicographic order. We denote this ordered basis by Y, which is given in the following:

$$Y = \{S_4, S_{3,1}, S_{2,2}, S_{2,1,1}, S_{1,3}, S_{1,2,1}, S_{1,1,2}, S_{1,1,1,1}\}.$$

For instance, this basis tells us that $S_{2,1,1}$ is lexicographically bigger than $S_{1,3}$. We now recall linear algebra from pages 199–200 of [18]: if V is the column space of a matrix A, and W is the column space of a matrix B, then $V + W$ is the column space of the matrix $D = [A \;\; B]$ and $\dim(V + W) = \mathrm{rank}\ (D)$ and $\dim(V \cap W) =$ nullity of D, which leads to the following formula:

$$\dim(V + W) + \dim(V \cap W) = \dim(V) + \dim(W). \tag{10}$$

To use the method above, by Tables 2 and 3, we write the basis matrix of $\pi^*(\mathcal{A}_2^*)$, which is denoted by $[M]_Y$, and of $\mathrm{Ker}(\chi-1)$, which is denoted by $[N]_Y$, relative to the basis Y as follows:

$$[M]_Y = \begin{bmatrix}1 & 0\\0 & 1\\0 & 1\\0 & 0\\0 & 0\\0 & 1\\0 & 0\\0 & 0\end{bmatrix}, \quad [N]_Y = \begin{bmatrix}1 & 1 & 1 & 1\\1 & 0 & 0 & 1\\0 & 0 & 1 & 1\\0 & 0 & 1 & 1\\1 & 0 & 1 & 1\\0 & 0 & 0 & 1\\0 & 0 & 1 & 1\\0 & 0 & 0 & 0\end{bmatrix}.$$

Table 2. Basis elements of $\pi^*(\mathcal{A}_2^*)$ in degrees 4 and 5.

Degree 4	$m_1 = \pi^*(\xi_1^4) =$	S_4
	$m_2 = \pi^*(\xi_2\xi_1) =$	$S_{3,1} + S_{2,2} + S_{1,2,1}$
Degree 5	$m_1^{'} = \pi^*(\xi_1^5) =$	$S_5 + S_{4,1} + S_{1,4}$
	$m_2^{'} = \pi^*(\xi_2\xi_1^2) =$	$S_{4,1} + S_{2,3} + S_{2,1,2}$

Note that $\mathcal{A}_2^*$ has a basis $\{\xi_1^4, \xi_2\xi_1\}$ in degree 4. Hence, $\pi^*(\mathcal{A}_2^*)$ has $\{\pi^*(\xi_1^4), \pi^*(\xi_2\xi_1)\}$ as a basis in the same degree, since π^* is a monomorphism.

Let us be more precise. The first column of $[M]_Y$ represents the coordinate vector of basis element m_1 in Table 2, relative to the basis Y. On the other hand, the first column of $[N]_Y$ represents the coordinate vector of basis element n_1 in Table 3, relative to the basis Y, while the second column of $[N]_Y$ represents the coordinate vector of basis element n_2 in Table 3, relative to the basis Y and so on.

It is now easy to see that the rank of $D = [\ [M]_Y \ \ [N]_Y]$ is 5. Hence, by Eq. (10), we have $5 + \dim([M]_Y \cap [N]_Y) = 6$ which gives $\dim([M]_Y \cap [N]_Y) = 1$. By Tables 2 and 3, $\pi^*(\mathcal{A}_2^*)$ and $\mathrm{Ker}(\chi - 1)$ have $\pi^*(\xi_1^4)$ as a common basis element. Therefore, by dimension reason, $\{\pi^*(\xi_1^4)\}$ has to be a basis for $\mathrm{Ker}(\chi - 1) \cap \pi^*(\mathcal{A}_2^*)$, and, hence, $\mathrm{Ker}(\chi_{\mathcal{A}_2^*} - 1)$ has a basis $\{\xi_1^4\}$ in degree 4.

Example 5.8 In degree 5 we will use the same argument used in Example 5.7 and will not explain the full details of the calculations. We again first give an order to the monomial basis of $\mathcal{F}_2^*$ with respect to lexicographic order. We denote this ordered basis by $Y^{'}$, which is given in the following:

$$Y^{'} = \{S_5, S_{4,1}, S_{3,2}, S_{3,1,1}, S_{2,3}, S_{2,2,1}, S_{2,1,2}, S_{2,1,1,1}, S_{1,4}, S_{1,3,1}, S_{1,2,2}, S_{1,2,1,1}, S_{1,1,3},$$
$$S_{1,1,2,1}, S_{1,1,1,2}, S_{1,1,1,1,1}\}.$$

Table 3. Basis elements of $\mathrm{Ker}(\chi - 1)$ in degrees 4 and 5.

Degree 4	$n_1 = (\chi - 1)(S_{3,1}) =$	$S_4 + S_{3,1} + S_{1,3}$
	$n_2 = (\chi - 1)(S_{2,2}) =$	S_4
	$n_3 = (\chi - 1)(S_{2,1,1}) =$	$S_4 + S_{2,2} + S_{2,1,1} + S_{1,3} + S_{1,1,2}$
	$n_4 = (\chi - 1)(S_{1,1,1,1}) =$	$S_4 + S_{3,1} + S_{2,2} + S_{2,1,1} + S_{1,3} + S_{1,2,1} + S_{1,1,2}$
Degree 5	$n_1^{'} = (\chi - 1)(S_{4,1}) =$	$S_5 + S_{4,1} + S_{1,4}$
	$n_2^{'} = (\chi - 1)(S_{3,2}) =$	$S_5 + S_{3,2} + S_{2,3}$
	$n_3^{'} = (\chi - 1)(S_{3,1,1}) =$	$S_5 + S_{3,1,1} + S_{2,3} + S_{1,4} + S_{1,1,3}$
	$n_4^{'} = (\chi - 1)(S_{2,2,1}) =$	$S_5 + S_{3,2} + S_{2,2,1} + S_{1,4} + S_{1,2,2}$
	$n_4^{'} = (\chi - 1)(S_{2,2,1})=$	$S_5 + S_{3,2} + S_{2,2,1} + S_{1,4} + S_{1,2,2}$
	$n_5^{'} = (\chi - 1)(S_{2,1,1,1})=$	$S_5 + S_{3,2} + S_{2,3} + S_{2,1,2} + S_{2,1,1,1} + S_{1,4} + S_{1,2,2}+$ $S_{1,1,3} + S_{1,1,1,2}$
	$n_6^{'} = (\chi - 1)(S_{1,2,1,1})=$	$S_5 + S_{4,1} + S_{2,3} + S_{2,2,1} + S_{1,4} + S_{1,3,1} + S_{1,2,1,1}+$ $S_{1,1,3} + S_{1,1,2,1}$
	$n_7^{'} = \lambda(S_5) =$	S_5
	$n_8^{'} = \lambda(S_{1,3,1}) =$	$S_{4,1} + S_{1,3,1}$
	$n_9^{'} = \lambda(S_{2,1,2}) =$	$S_{3,2} + S_{2,1,2}$
	$n_{10}^{'} = \lambda(S_{1,1,1,1,1}) =$	$S_{3,1,1} + S_{2,1,1,1} + S_{1,2,1,1} + S_{1,1,1,1,1}$

By Tables 2 and 3, writing the basis matrix of $\pi^*(\mathcal{A}_2^*)$, which is denoted by $[M^{'}]_{Y'}$, and of $\mathrm{Ker}(\chi-1)$, which is denoted by $[N^{'}]_{Y'}$, relative to the basis $Y^{'}$, we see that the rank of $D = \left[\, [M^{'}]_{Y'} \ [N^{'}]_{Y'} \right]$ is 11, where $\mathrm{rank}\left([M^{'}]_{Y'}\right) = 2$ and $\mathrm{rank}\left([N^{'}]_{Y'}\right) = 10$. Therefore, $11 + \dim(M^{'} \cap N^{'}) = 12$. Following this, by Tables 2 and 3 we see $\pi^*(\xi_1^5)$ belong to $\mathrm{Ker}(\chi-1)$ and $\pi^*(\mathcal{A}_2^*)$. By the same argument in Example 5.7, we conclude that $\mathrm{Ker}(\chi_{\mathcal{A}_2^*}-1)$ has $\{\xi_1^5\}$ as a basis in degree 5.

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