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Covers and preenvelopes by V-Gorenstein flat modules

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Abstract: In this paper, we introduce and study V-Gorenstein flat modules and show the stability of the category of V-Gorenstein flat modules. We investigate the existence of V-Gorenstein flat covers and V-Gorenstein flat preenvelopes for any left R-module. Also we prove that $(V\text{-}\mathcal{GF}, V\text{-}\mathcal{GF}^{\perp})$ is a perfect hereditary cotorsion pair in $\mathcal{B}^l(R)$, where $V\text{-}\mathcal{GF}$ stands the class of V-Gorenstein flat left R-modules and $\mathcal{B}^l(R)$ is the left Bass class. Some applications are given.

Key words: Dualizing module, V-Gorenstein flat module, precover, preenvelope

1. Introduction and some basic facts

We use R-Mod (resp. $R^{\rm op}$-Mod) to denote the category of left (resp. right) R-modules. For any R-module M, $\mathrm{pd}(M)$ (resp. $\mathrm{id}(M)$, $\mathrm{fd}(M)$) denotes the projective (resp. injective, flat) dimension of M. The character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is denoted by M^+.

Since Auslander and Bridger [1] introduced the G-dimension of a finitely generated module, the study of Gorenstein dimensions of modules has been the subject of numerous publications. The use of equivalence introduced by Foxby has been shown to be of great utility in this study. Enochs and Jenda [7] studied V-Gorenstein modules relative to a dualizing module. These modules constitute a generalization of the well-known Gorenstein modules [4] and at the same time an extension to the noncommutative case of Ω-Gorenstein modules [5]. They proved that under certain condition on the finiteness of projective dimension for flat modules V-Gorenstein injectives and projectives form part of perfect cotorsion pairs. However, in that case, V-Gorenstein flat modules could not be introduced. Our aim in this paper is to introduce the notion of V-Gorenstein flat modules and study some of their properties.

We now summarize the layout of the paper. In Section 1, we give some notions and draw some basic consequences for use throughout this paper. In the first part of Section 2, we introduce the notion of V-Gorenstein flat modules and study some of their properties. In the second part of Section 2, we show the stability of the category of V-Gorenstein flat modules. In Section 3, we study the existence of V-Gorenstein flat covers and V-Gorenstein preenvelopes, and prove that $(V\text{-}\mathcal{GF}, V\text{-}\mathcal{GF}^{\perp})$ is a perfect hereditary cotorsion pair in $\mathcal{B}^l(R)$, where $V\text{-}\mathcal{GF}$ stands for the class of V-Gorenstein flat left R-modules and $\mathcal{B}^l(R)$ is the left Bass class. In Section 4, we characterize some rings in terms of Gorenstein and V-Gorenstein homological modules.

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Cotorsion pairs. Given a class $\mathcal{C}$ of R-modules, we let ${}^{\perp}\mathcal{C}$ be the class of R-modules F such that $\mathrm{Ext}^1_R(F,C)=0$ for all $C\in\mathcal{C}$ and let $\mathcal{C}^{\perp}$ be the class of R-modules F such that $\mathrm{Ext}^1_R(C,F)=0$ for all $C\in\mathcal{C}$. Following [4], a pair of classes of R-modules $(\mathcal{F},\mathcal{C})$ is called a cotorsion pair if $\mathcal{F}^{\perp}=\mathcal{C}$ and ${}^{\perp}\mathcal{C}=\mathcal{F}$. A cotorsion pair $(\mathcal{F},\mathcal{C})$ is said to be complete if it has enough projectives and injectives, that is, for any R-module M, there are exact sequences $0\to C\to F\to M\to 0$ and $0\to M\to C'\to F'\to 0$ respectively with $C,C'\in\mathcal{C}$ and $F,F'\in\mathcal{F}$. A cotorsion pair is said to be perfect if every R-module has an $\mathcal{F}$-cover and a $\mathcal{C}$-envelope. A cotorsion pair is said to be hereditary if $0\to F'\to F\to F''\to 0$ is an exact sequence with F, $F''\in\mathcal{F}$, then $F'\in\mathcal{F}$.

Resolutions. Let $\mathbf{A}$ be an abelian category and $\mathcal{F}$ a class of objects of $\mathbf{A}$. For an object M of $\mathbf{A}$, a left $\mathcal{F}$-resolution of M is a $\mathrm{Hom}(\mathcal{F},-)$ exact complex $\cdots\to F_1\to F_0\to M\to 0$ (not necessarily exact) with each $F_i\in\mathcal{F}$. A right $\mathcal{F}$-resolution of M is a $\mathrm{Hom}(-,\mathcal{F})$ exact complex $0\to M\to F^0\to F^1\to\cdots$ (not necessarily exact) with each $F^i\in\mathcal{F}$ (see [4]).

Balanced functors. Let $\mathbf{C},\mathbf{D}$, and $\mathbf{E}$ be abelian categories and $T:\mathbf{C}\times\mathbf{D}\to\mathbf{E}$ be an additive functor contravariant in the first variable and covariant in the second. Let $\mathcal{F}$ and $\mathcal{G}$ be classes of objects of $\mathbf{C}$ and $\mathbf{D}$ respectively. Then T is said to be right balanced by $\mathcal{F}\times\mathcal{G}$ if for each object M of $\mathbf{C}$ there is a $T(-,\mathcal{G})$ exact complex $\cdots\to F_1\to F_0\to M\to 0$ with each $F_i\in\mathcal{F}$, and if for each object N of $\mathbf{D}$ there is a $T(\mathcal{F},-)$ exact complex $0\to N\to G^0\to G^1\to\cdots$ with each $G^i\in\mathcal{G}$. If, on the other hand, the complex $\cdots\to F_1\to F_0\to M\to 0$ is $T(\mathcal{G},-)$ exact and the complex $0\to N\to G^0\to G^1\to\cdots$ is $T(-,\mathcal{F})$ exact, then T is said to be left balanced by $\mathcal{G}\times\mathcal{F}$.

The definitions above are easily modified to give the definition of a left or right balanced functor relative to $\mathcal{F}\times\mathcal{G}$ with other choices of variances and complexes (see [4]).

Dualizing module. Let R be a left and right Noetherian ring and let V be an (R,R)-bimodule such that $\mathrm{End}({}_RV)=R$ and $\mathrm{End}(V_R)=R$. Then V is said to be a dualizing module [8] if it satisfies the following 3 conditions:

(i) $\mathrm{id}({}_RV)\leqslant r$ and $\mathrm{id}(V_R)\leqslant r$ for some integer r;

(ii) $\mathrm{Ext}^i_R(V,V)=\mathrm{Ext}^i_{R^{\mathrm{op}}}(V,V)=0$ for all $i\geqslant 1$;

(iii) ${}_RV$ and V_R are finitely generated.

The preceding is given in [8] for a bimodule ${}_SV_R$, where S and R are left and right Noetherian rings respectively, but throughout this paper we consider the case $S=R$.

It is immediately seen that if R is a local Cohen–Macaulay ring admitting a dualizing module Ω or R is an n-Gorenstein ring (2-sided Noetherian ring with $\mathrm{id}({}_RR)\leqslant n$, $\mathrm{id}(R_R)\leqslant n$), then Ω and R are dualizing modules in this sense.

In what follows, R will always be a left and right Noetherian ring and V a dualizing module for R. Enochs et al. [9] introduced the left, right Auslander classes $\mathcal{A}^l(R)$, $\mathcal{A}^r(R)$ and the left, right Bass classes $\mathcal{B}^l(R)$, $\mathcal{B}^r(R)$. It is easily seen that

$$V\otimes_R-:\mathcal{A}^l(R)\leftrightarrows\mathcal{B}^l(R):\mathrm{Hom}_R(V,-),\quad -\otimes_R V:\mathcal{A}^r(R)\leftrightarrows\mathcal{B}^r(R):\mathrm{Hom}_{R^{\mathrm{op}}}(V,-)$$

give equivalences between the 2 categories.

Denote

$$\mathcal{W}=\{W\cong V\otimes_R P\,|\,P \text{ is a projective left } R\text{-module}\},$$

$$\mathcal{X} = \{X \cong V \otimes_R F | F \text{ is a flat left } R\text{-module}\},$$

$$\mathcal{U} = \{U \cong \mathrm{Hom}_{R^{\mathrm{op}}}(V, E) | E \text{ is an injective right } R\text{-module}\}.$$

Then clearly $\mathcal{W} \subseteq \mathcal{X} \subseteq \mathcal{B}^l(R)$ and $\mathcal{U} \subseteq \mathcal{A}^r(R)$. Every right R-module has a $\mathcal{U}$-preenvelope and every left R-module has an $\mathcal{X}$-precover. Then the right $\mathcal{U}$-dimension of a right R-module and the left $\mathcal{X}$-dimension of a left R-module are defined as usual.

Gorenstein modules. A right R-module M is said to be V-Gorenstein injective [7] if there exists an exact sequence:

$$\mathbb{U} : \cdots \longrightarrow U_1 \longrightarrow U_0 \longrightarrow U^0 \longrightarrow U^1 \longrightarrow \cdots$$

of modules in $\mathcal{U}$ such that $M = \mathrm{Ker}(U^0 \to U^1)$ and such that $\mathrm{Hom}_{R^{\mathrm{op}}}(U, \mathbb{U})$ and $\mathrm{Hom}_{R^{\mathrm{op}}}(\mathbb{U}, U)$ are exact for every $U \in \mathcal{U}$. The class of V-Gorenstein injective right R-modules is denoted by V-$\mathcal{GI}$. Dually, the notion of V-Gorenstein projective left R-modules is defined. The class of V-Gorenstein projective left R-modules is denoted by V-$\mathcal{GP}$. A left R-module G is said to be Gorenstein flat [12] if there exists an exact sequence:

$$\mathbb{F} : \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots$$

of flat left R-modules such that $G = \mathrm{Ker}(F^0 \to F^1)$ and $I \otimes_R \mathbb{F}$ is exact for any injective right R-module I. The exact sequence $\mathbb{F}$ is called a complete flat resolution. The class of Gorenstein flat left R-modules is denoted by $\mathcal{GF}$.

Lemma 1.1 *Let $0 \to M' \to M \to M'' \to 0$ be exact in R^{op}-Mod (resp. R-Mod). If any 2 of M', M, M'' are in $\mathcal{A}^r(R)$ (resp. $\mathcal{B}^l(R)$), then so is the third.*

Proof By analogy with the proof of [8, Proposition 3.13]. □

Lemma 1.2 *Every left R-module has an $\mathcal{X}$-preenvelope.*

Proof By [14, Proposition 5.3]. □

Lemma 1.3 *Let $M \in R$-Mod and $0 \to M \to X^0 \to X^1 \to \cdots$ be a right $\mathcal{X}$-resolution of M. Then the complex $0 \to U \otimes_R M \to U \otimes_R X^0 \to U \otimes_R X^1 \to \cdots$ is exact for any $U \in \mathcal{U}$.*

Proof Let $0 \to M \to X^0 \to X^1 \to \cdots$ be a right $\mathcal{X}$-resolution of M and let $U \in \mathcal{U}$. Then $U^+ \in \mathcal{X}$. Consider the following commutative diagram:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \mathrm{Hom}_R(X^1, U^+) & \longrightarrow & \mathrm{Hom}_R(X^0, U^+) & \longrightarrow & \mathrm{Hom}_R(M, U^+) & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\ \cdots & \longrightarrow & (U \otimes_R X^1)^+ & \longrightarrow & (U \otimes_R X^0)^+ & \longrightarrow & (U \otimes_R M)^+ & \longrightarrow & 0 \end{array}$$

with the upper row exact. So $0 \to U \otimes_R M \to U \otimes_R X^0 \to U \otimes_R X^1 \to \cdots$ is exact. □

Proposition 1.4 *$- \otimes_R -$ is left balanced on $\mathcal{A}^r(R) \times \mathcal{B}^l(R)$ by $\mathcal{P}roj \times \mathcal{W}$ and $\mathcal{F}lat \times \mathcal{X}$.*

Proposition 1.5 *$- \otimes_R -$ is right balanced on $\mathcal{A}^r(R) \times \mathcal{B}^l(R)$ by $\mathcal{U} \times \mathcal{X}$.*

Proof Let $M \in \mathcal{A}^r(R)$ and $0 \to M \to U^0 \to U^1 \to \cdots$ be a right $\mathcal{U}$-resolution of M. Then $0 \to M \otimes_R V \to U^0 \otimes_R V \to U^1 \otimes_R V \to \cdots$ is exact by Lemma 1.1 since M and each U^i are in $\mathcal{A}^r(R)$, and so $0 \to M \otimes_R X \to U^0 \otimes_R X \to U^1 \otimes_R X \to \cdots$ is exact for all $X \in \mathcal{X}$. The result now follows from Lemma 1.3. □

Proposition 1.6 *Hom*$(-,-)$ *is left balanced on* R*-Mod* $\times$ R*-Mod by* $\mathcal{X} \times \mathcal{X}$.

Proposition 1.7 *Hom*$(-,-)$ *is right balanced on* $\mathcal{A}^r(R) \times \mathcal{A}^r(R)$ *by* $\mathcal{P}roj \times \mathcal{U}$ *and on* $\mathcal{B}^l(R) \times \mathcal{B}^l(R)$ *by* $\mathcal{W} \times \mathcal{I}nj$.

Proof Let $M \in \mathcal{A}^r(R)$ and $\cdots \to P_1 \to P_0 \to M \to 0$ be a projective resolution of M. Then $\cdots \to P_1 \otimes_R V \to P_0 \otimes_R V \to M \otimes_R V \to 0$ is exact. It is easy to check that $0 \to \mathrm{Hom}_{R^{\mathrm{op}}}(M, U) \to \mathrm{Hom}_{R^{\mathrm{op}}}(P_0, U) \to \mathrm{Hom}_{R^{\mathrm{op}}}(P_1, U) \to \cdots$ is exact for all $U \in \mathcal{U}$. On the other hand, let $N \in \mathcal{A}^r(R)$ and $0 \to N \to U^0 \to U^1 \to \cdots$ be a right $\mathcal{U}$-resolution of N. Then $0 \to \mathrm{Hom}_{R^{\mathrm{op}}}(P, N) \to \mathrm{Hom}_{R^{\mathrm{op}}}(P, U^0) \to \mathrm{Hom}_{R^{\mathrm{op}}}(P, U^1) \to \cdots$ is exact for any projective right R-module P. The second part follows dually. □

2. V-Gorenstein flat modules and stability

In this section, we introduce the notion of V-Gorenstein flat modules, characterize them in terms of the so-called left Auslander and left Bass classes, and generalize some results obtained in [5] and [16]. We also show the stability of the category of V-Gorenstein flat modules.

Definition 2.1 A left R-module M is called V-Gorenstein flat if there is an exact sequence:

$$\cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow X^0 \longrightarrow X^1 \longrightarrow \cdots$$

of modules in $\mathcal{X}$ such that $M = \mathrm{Ker}(X^0 \to X^1)$ and such that $\mathrm{Hom}_R(W, -)$ and $U \otimes_R -$ leave the sequence exact whenever $U \in \mathcal{U}$ and $W \in \mathcal{W}$.

The class of V-Gorenstein flat left R-modules is denoted by V-$\mathcal{GF}$.

Remark 2.2 Clearly, every module in $\mathcal{X}$ is V-Gorenstein flat. Moreover, if R is Gorenstein, then in this case the V-Gorenstein flat R-module is simply the usual Gorenstein flat R-module.

Theorem 2.3 *The following are equivalent for* $M \in R$*-Mod:*

(1) M *is* V*-Gorenstein flat;*

(2) $M \in \mathcal{B}^l(R)$ *and* $\mathrm{Tor}_i^R(U, M) = 0$ *for all* $i \geqslant 1$ *and any* $U \in \mathcal{U}$*;*

(3) $M \in \mathcal{B}^l(R)$ *and* $\mathrm{Tor}_i^R(U, M) = 0$ *for* $1 \leqslant i \leqslant r$ *and any* $U \in \mathcal{U}$*;*

(4) $M \in \mathcal{B}^l(R)$ *and* $\mathrm{Tor}_i^R(L, M) = 0$ *for all* $i \geqslant 1$ *and any* L *of finite right* $\mathcal{U}$*-dimension;*

(5) $M \in \mathcal{B}^l(R)$ *and* $\mathrm{Tor}_1^R(L, M) = 0$ *for any* L *of finite right* $\mathcal{U}$*-dimension;*

(6) $M \in \mathcal{B}^l(R)$ *and* $\mathrm{Hom}_R(V, M)$ *is Gorenstein flat;*

(7) *There exists an exact sequence* $0 \to M \to X^0 \to \cdots \to X^{r-1} \to C \to 0$ *with each* $X^i \in \mathcal{X}$ *and* $C \in \mathcal{B}^l(R)$*;*

(8) M^+ *is* V*-Gorenstein injective.*

Proof (1) $\Rightarrow$ (2). By analogy with the proof of [7, Proposition 3.2], we see that $M \in \mathcal{B}^l(R)$. Let $U_i \cong \mathrm{Hom}_{R^{\mathrm{op}}}(V, E_i) \in \mathcal{U}$ for $i = 1, 2$. Then [4, Theorem 3.2.13] implies that

$$\begin{aligned}\mathrm{Ext}^n_{R^{\mathrm{op}}}(U_1, U_2) &\cong \mathrm{Hom}_{R^{\mathrm{op}}}(\mathrm{Tor}^R_n(\mathrm{Hom}_{R^{\mathrm{op}}}(V, E_1), V), E_2)\\ &\cong \mathrm{Hom}_{R^{\mathrm{op}}}(\mathrm{Hom}_{R^{\mathrm{op}}}(\mathrm{Ext}^n_{R^{\mathrm{op}}}(V, V), E_1), E_2) = 0,\ \forall\ n \geqslant 1,\end{aligned}$$

and so $\mathrm{Tor}^R_n(U, X)^+ \cong \mathrm{Ext}^n_{R^{\mathrm{op}}}(U, X^+) = 0$ for all $U \in \mathcal{U}, X \in \mathcal{X}$ and all $n \geqslant 1$ since $X^+ \in \mathcal{U}$. Thus $\mathrm{Tor}^R_i(U, M) = 0$ for any $U \in \mathcal{U}$ and all $i \geqslant 1$.

(2) $\Rightarrow$ (4) follows from dimension shifting.

(4) $\Rightarrow$ (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) are obvious.

(3) $\Rightarrow$ (2). Let $U \cong \mathrm{Hom}_{R^{\mathrm{op}}}(V, E) \in \mathcal{U}$. Since $\mathrm{id}(V_R) \leqslant r$, $\mathrm{fd}(U_R) \leqslant r$ by [4, Theorem 3.2.16]. Then $\mathrm{Tor}^R_i(U, M) = 0$ for all $i \geqslant r+1$, and so $\mathrm{Tor}^R_i(U, M) = 0$ for all $i \geqslant 1$.

(5) $\Rightarrow$ (4) and (2) $\Rightarrow$ (1). By analogy with the proof of [5, Proposition 4.5].

(1) $\Leftrightarrow$ (6). By analogy with the proof of [16, Proposition 2.9].

(1) $\Rightarrow$ (7). Since M is V-Gorenstein flat, there exists an exact sequence $0 \to M \to X^0 \to \cdots \to X^{r-1} \to C \to 0$ with each $X^i \in \mathcal{X}$. However, $M \in \mathcal{B}^l(R)$ and each $X^i \in \mathcal{B}^l(R)$; it follows from Lemma 1.1 that $C \in \mathcal{B}^l(R)$.

(7) $\Rightarrow$ (2). Let $0 \to M \to X^0 \to \cdots \to X^{r-1} \to C \to 0$ be exact with each $X^i \in \mathcal{X}$ and $C \in \mathcal{B}^l(R)$. Then $M \in \mathcal{B}^l(R)$ by Lemma 1.1. Let $U \in \mathcal{U}$. Then $\mathrm{fd}(U_R) \leqslant r$ since $\mathrm{id}(V_R) \leqslant r$, and hence $\mathrm{Tor}^R_i(U, M) \cong \mathrm{Tor}^R_{r+i}(U, C) = 0$ for all $i \geqslant 1$.

(1) $\Rightarrow$ (8). Since M is V-Gorenstein flat, there is a $\mathcal{U} \otimes_R -$ exact exact sequence:

$$\mathbb{X}:\ \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow X^0 \longrightarrow X^1 \longrightarrow \cdots$$

of modules in $\mathcal{X}$ such that $M = \mathrm{Ker}(X^0 \to X^1)$. Therefore, $M^+ \in \mathcal{A}^r(R)$ and

$$\mathbb{X}^+:\ \cdots \longrightarrow (X^1)^+ \longrightarrow (X^0)^+ \longrightarrow (X_0)^+ \longrightarrow (X_1)^+ \longrightarrow \cdots$$

is an exact sequence of modules in $\mathcal{U}$ such that $M^+ = \mathrm{Coker}((X^1)^+ \to (X^0)^+)$. To prove that $M^+ \in V\text{-}\mathcal{GI}(R)$, it suffices to show that $\mathrm{Hom}_{R^{\mathrm{op}}}(U, \mathbb{X}^+)$ is exact for any $U \in \mathcal{U}$ by [7, Theorem 2.4]. Note that $\mathrm{Hom}_{R^{\mathrm{op}}}(U, \mathbb{X}^+) \cong (U \otimes_R \mathbb{X})^+$ is exact for any $U \in \mathcal{U}$, which implies that $M^+ \in V\text{-}\mathcal{GI}$.

(8) $\Rightarrow$ (2). Since M^+ is V-Gorenstein injective, we have $M \in \mathcal{B}^l(R)$ and $\mathrm{Tor}^R_i(U, M)^+ \cong \mathrm{Ext}^i_{R^{\mathrm{op}}}(U, M^+) = 0$ for any $U \in \mathcal{U}$ and all $i \geqslant 1$, as desired. □

By Proposition 1.5, we can define right derived functors of $- \otimes -$ by using right $\mathcal{U}$-resolutions and right $\mathcal{X}$-resolutions in the first and second variables respectively. These new derived functors are denoted by Tor^n_R. We also note that there exists a natural morphism

$$M \otimes_R N \longrightarrow \mathrm{Tor}^0_R(M, N).$$

We let $\overline{\mathrm{Tor}}_0(N, M)$ and $\overline{\mathrm{Tor}}^0(N, M)$ be the kernel and the cokernel of the above morphism.

Theorem 2.4 *The following are equivalent for $M \in R$-Mod:*

(1) *M is V-Gorenstein flat;*

(2) *$M \in \mathcal{B}^l(R)$ and $Tor_i^R(Q,M) = Tor_R^i(Q,M) = 0$ for all $i \geqslant 1$ and $\overline{Tor}_0(Q,M) = \overline{Tor}^0(Q,M) = 0$ for all $Q \in \mathcal{U} \cup \mathcal{P}roj$;*

(3) *$M \in \mathcal{B}^l(R)$ and $Tor_i^R(Q,M) = Tor_R^i(Q,M) = 0$ for all $i \geqslant 1$ and $\overline{Tor}_0(Q,M) = \overline{Tor}^0(Q,M) = 0$ for all $Q \in \mathcal{A}^r(R)$ of finite right $\mathcal{U}$-dimension or projective dimension.*

Proof (1) $\Rightarrow$ (2). Since M is V-Gorenstein flat, there is a $\mathcal{U} \otimes_R -$ exact exact sequence:

$$\cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow X^0 \longrightarrow X^1 \longrightarrow \cdots$$

of modules in $\mathcal{X}$ such that $M = \mathrm{Ker}(X^0 \to X^1)$. If Q is projective, then the homology groups vanish. Let $U \in \mathcal{U}$. Since R is right Noetherian, $\mathrm{Hom}_R(V,M)$ has an exact right flat resolution, and hence M has an exact right $\mathcal{X}$-resolution $0 \to M \to X^0 \to X^1 \to \cdots$ that may be used to calculate $\mathrm{Tor}_R^i(U,M)$. Thus $\mathrm{Tor}_R^i(U,M) = 0$ for all $i \geqslant 1$. Note that $\mathrm{Tor}_i^R(U,X)^+ \cong \mathrm{Ext}^i_{R^{\mathrm{op}}}(U,X^+) = 0$ for any $X \in \mathcal{X}$. Then $\mathrm{Tor}_i^R(U,M) = 0$ for all $i \geqslant 1$. Finally $U \otimes_R M \cong \mathrm{Ker}(U \otimes_R X^0 \to U \otimes_R X^1) \cong \mathrm{Tor}_R^0(U,M)$.

(2) $\Leftrightarrow$ (3) follows from dimension shifting.

(3) $\Rightarrow$ (1). Let $\cdots \to X_1 \to X_0 \to M \to 0$ be an exact left $\mathcal{X}$-resolution of M and let $0 \to M \to X^0 \to X^1 \to \cdots$ be a right $\mathcal{X}$-resolution of M. Then $0 \to M \to X^0 \to X^1 \to \cdots$ is exact since $\mathrm{Tor}_R^i(R,M) = 0$ for all $i \geqslant 1$ and $M \cong R \otimes_R M \cong \mathrm{Tor}_R^0(R,M) = \mathrm{Ker}(R \otimes_R X^0 \to U \otimes_R X^1)$. Hence

$$\cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow X^0 \longrightarrow X^1 \longrightarrow \cdots$$

is an exact sequence of modules in $\mathcal{X}$ and the sequence is $\mathcal{U} \otimes_R -$ exact since the homology groups vanish, which implies that M is V-Gorenstein flat. □

[4, Exercise 10, p.318] proved that an R-module M is Gorenstein flat if and only if $M \in \mathcal{G}_0(R)$ and $\Omega \otimes_R M$ is Ω-Gorenstein flat. Here we have the following result.

Proposition 2.5 *M is a Gorenstein flat left R-module if and only if $M \in \mathcal{A}^l(R)$ and $V \otimes_R M$ is a V-Gorenstein flat left R-module.*

Proof "$\Rightarrow$" Since M is Gorenstein flat, there exists a complete flat resolution:

$$\mathbb{F}: \ \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F^0 \longrightarrow F^1 \to \cdots$$

such that $M \cong \mathrm{Ker}(F^0 \to F^1)$. Since $\mathrm{id}(V_R) \leqslant r$, we have $\mathrm{Tor}_i^R(V,M) = 0$ for all $i \geqslant 1$. Consider the exact sequence $0 \to M \to F^0 \to F^1$. Note that $M^1 = \mathrm{Im}(F^0 \to F^1)$ and $M^2 = \mathrm{Im}(F^1 \to F^2)$ are in $\mathcal{GF}$ and so $\mathrm{Tor}_1^R(V,M^1) = 0 = \mathrm{Tor}_1^R(V,M^2)$. Hence $0 \to V \otimes_R M \to V \otimes_R F^0 \to V \otimes_R F^1$ is exact, which implies that

$$0 \longrightarrow \mathrm{Hom}_R(V, V \otimes_R M) \longrightarrow \mathrm{Hom}_R(V, V \otimes_R F^0) \longrightarrow \mathrm{Hom}_R(V, V \otimes_R F^1)$$

is exact. Thus $M \cong \mathrm{Hom}_R(V, V \otimes_R M)$ since each $F^i \in \mathcal{A}^l(R)$. Analogously, it can be seen that $C^i \cong \mathrm{Hom}_R(V, V \otimes_R C^i)$ for every cosyzygy C^i of $0 \to M \to F^0 \to F^1 \to \cdots$ for all $i = 1, 2 \cdots$. Since $0 \to V \otimes_R C^i \to V \otimes_R F^i \to V \otimes_R C^{i+1} \to 0$ is exact, we have

$$\mathrm{Hom}_R(V, V \otimes_R F^i) \longrightarrow \mathrm{Hom}_R(V, V \otimes_R C^{i+1}) \longrightarrow \mathrm{Ext}^1_R(V, V \otimes_R C^i) \longrightarrow 0$$

is exact, and so $\mathrm{Ext}_R^1(V, V\otimes_R C^i)=0$; it follows that $\mathrm{Ext}_R^i(V, V\otimes_R M)\cong \mathrm{Ext}_R^1(V, V\otimes_R C^{i-1})=0$ for all $i\geqslant 1$. Thus $M\in\mathcal{A}^l(R)$ and

$$V\otimes_R \mathbb{F}:\ \cdots\longrightarrow V\otimes_R F_1\longrightarrow V\otimes_R F_0\longrightarrow V\otimes_R F^0\longrightarrow V\otimes_R F^1\longrightarrow\cdots$$

is exact such that $V\otimes_R M\cong \mathrm{Ker}(V\otimes_R F^0\to V\otimes_R F^1)$. Let $U\cong \mathrm{Hom}_{R^{\mathrm{op}}}(V,E)\in\mathcal{U}$. Then $U\otimes_R(V\otimes_R\mathbb{F})\cong E\otimes_R\mathbb{F}$ is exact, and hence $V\otimes_R M\in V\text{-}\mathcal{GF}$.

"$\Leftarrow$" $M\cong \mathrm{Hom}_R(V, V\otimes_R M)\in\mathcal{GF}$ by Theorem 2.3. □

Proposition 2.6 *Let M be a right R-module. If M is V-Gorenstein injective, then M^+ is V-Gorenstein flat.*

Proof Since M is V-Gorenstein injective, we see that $M^+\in\mathcal{B}^l(R)$ and $M\otimes_R V\in\mathcal{B}^r(R)$ is Gorenstein injective by [8, Theorem 4.5], and hence there is an exact sequence $0\to K\to E_{r-1}\to\cdots\to E_0\to M\otimes_R V\to 0$ with each E_i injective and $K\in\mathcal{B}^r(R)$. However, $0\to \mathrm{Hom}_{R^{\mathrm{op}}}(V,K)\to \mathrm{Hom}_{R^{\mathrm{op}}}(V,E_{r-1})\to\cdots\to \mathrm{Hom}_{R^{\mathrm{op}}}(V,E_0)\to M\to 0$ is exact, and so

$$0\to M^+\to \mathrm{Hom}_{R^{\mathrm{op}}}(V,E_0)^+\to\cdots\to \mathrm{Hom}_{R^{\mathrm{op}}}(V,E_{r-1})^+\to \mathrm{Hom}_{R^{\mathrm{op}}}(V,K)^+\to 0$$

is exact and $\mathrm{Hom}_{R^{\mathrm{op}}}(V,K)^+\in\mathcal{B}^l(R)$. Therefore, Theorem 2.3 implies that $M^+\in V\text{-}\mathcal{GF}$. □

Proposition 2.7 *Let $M\in\mathcal{B}^l(R)$. Then the following hold:*

(i) *If $\cdots\to X_1\to X_0\to M\to 0$ is an exact left $\mathcal{X}$-resolution and $C_i=$ Coker$(X_{i+1}\to X_i)$, then C_i is V-Gorenstein flat for $i\geqslant r$;*

(ii) *If $0\to M\to X^0\to X^1\to\cdots$ is a right $\mathcal{X}$-resolution and $C^i=$ Ker$(X^i\to X^{i+1})$, then C^i is V-Gorenstein flat for $i\geqslant r-1$.*

Proof (i) By assumption and Lemma 1.1, we get every $C_i\in\mathcal{B}^l(R)$. Let $U\in\mathcal{U}$. Then $\mathrm{fd}(U_R)\leqslant r$ since $\mathrm{id}(V_R)\leqslant r$, and so $\mathrm{Tor}_i^R(U,M)=0$ for all $i\geqslant r+1$. Thus $\mathrm{Tor}_1^R(U,C_i)=0$ for $i\geqslant r$. Consider the exact sequence

$$0\longrightarrow C_{2r}\longrightarrow X_{2r-1}\longrightarrow\cdots\longrightarrow X_r\longrightarrow C_r\longrightarrow 0.$$

Then $\mathrm{Tor}_n^R(U,C_{2r})\cong \mathrm{Tor}_{n+r}^R(U,C_r)=0$ for all $n\geqslant 1$ and $U\in\mathcal{U}$, and hence $C_{r+i}\in V\text{-}\mathcal{GF}$ for $i\geqslant r$ by Theorem 2.3. Let E be an injective right R-module. Then

$$\mathrm{Ext}_R^1(C_{2r-1}, V\otimes_R E^+)\cong \mathrm{Ext}_R^1(C_{2r-1}, \mathrm{Hom}_{R^{\mathrm{op}}}(V,E)^+)\cong \mathrm{Tor}_1^R(\mathrm{Hom}_{R^{\mathrm{op}}}(V,E), C_{2r-1})^+=0,$$

and so $\mathrm{Tor}_1^R(E,\mathrm{Hom}_R(V,C_{2r-1}))^+\cong \mathrm{Ext}_R^1(\mathrm{Hom}_R(V,C_{2r-1}),E^+)=0$ by Lemma 3.2, which implies that $\mathrm{Hom}_R(V,C_{2r-1})\in\mathcal{GF}$ by [12, Proposition 3.8]. Therefore, $C_{2r-1}\in V\text{-}\mathcal{GF}$. Now repeat the process to get that $C_i\in V\text{-}\mathcal{GF}$ for $i\geqslant r$ by Theorem 2.3.

(ii) If $0\to M\to X^0\to X^1\to\cdots$ is a right $\mathcal{X}$-resolution of M, then $0\to \mathrm{Hom}_R(V,M)\to \mathrm{Hom}_R(V,X^0)\to \mathrm{Hom}_R(V,X^1)\to\cdots$ is a right flat resolution of $\mathrm{Hom}_R(V,M)$. Hence [4, Theorem 8.4.36] implies that the sequence $X^{r-1}\to X^r\to X^{r+1}\to X^{r+2}\to\cdots$ is exact. Now by analogy with the proof of (i), we get the desired result. □

Define by $V\text{-}\mathcal{GF}^2$ the subcategory of R-Mod for which there exists an exact sequence:

$$\mathbb{G}: \ \cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow G^0 \longrightarrow G^1 \longrightarrow \cdots$$

of modules in V-$\mathcal{GF}$ such that $M = \mathrm{Ker}(G^0 \to G^1)$, and such that $\mathrm{Hom}_R(H, \mathbb{G})$ and $H' \otimes_R \mathbb{G}$ are exact for any $H \in V$-$\mathcal{GP}$ and $H' \in V$-$\mathcal{GI}$. It is routine to check that V-$\mathcal{GF} \subseteq V$-$\mathcal{GF}^2$. Next we establish the stability of the category of V-Gorenstein flat left R-modules.

Theorem 2.8 V-$\mathcal{GF} = V$-$\mathcal{GF}^2$.

Proof It suffices to prove that V-$\mathcal{GF}^2 \subseteq V$-$\mathcal{GF}$. Let $M \in V$-$\mathcal{GF}^2$. Note that each module in $\mathcal{W}$ is V-Gorenstein projective; it follows from the proof of [7, Proposition 3.2] that $M \in \mathcal{B}^l(R)$. On the other hand, there is an exact sequence:

$$\mathbb{G}: \ \cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow G^0 \longrightarrow G^1 \longrightarrow \cdots$$

of modules in V-$\mathcal{GF}$ such that $M = \mathrm{Ker}(G^0 \to G^1)$ and $H \otimes_R \mathbb{G}$ is exact for any $H \in V$-$\mathcal{GI}$. Note that G_i and G^i are in $\mathcal{B}^l(R)$ for all i. Then we get the following exact sequence:

$$\mathrm{Hom}_R(V, \mathbb{G}) : \cdots \to \mathrm{Hom}_R(V, G_1) \to \mathrm{Hom}_R(V, G_0) \to \mathrm{Hom}_R(V, G^0) \to \mathrm{Hom}_R(V, G^1) \to \cdots$$

of modules in $\mathcal{GF}$ such that $\mathrm{Hom}_R(V, M) = \mathrm{Ker}(\mathrm{Hom}_R(V, G^0) \to \mathrm{Hom}_R(V, G^1))$. Let L be a Gorenstein injective right R-module. Then $L \in \mathcal{B}^r(R)$ and $\mathrm{Hom}_{R^{\mathrm{op}}}(V, L) \in V$-$\mathcal{GI}$ by [8, Proposition 3.8 and Theorem 4.5]. Thus $L \otimes_R \mathrm{Hom}_R(V, \mathbb{G}) \cong \mathrm{Hom}_{R^{\mathrm{op}}}(V, L) \otimes_R V \otimes_R \mathrm{Hom}_R(V, \mathbb{G}) \cong \mathrm{Hom}_{R^{\mathrm{op}}}(V, L) \otimes_R \mathbb{G}$ is exact; it follows that $\mathrm{Hom}_R(V, M) \in \mathcal{GF}$ by [2, Theorem 1.2]. Consequently, $M \in V$-$\mathcal{GF}$, as claimed. □

3. V-Gorenstein flat (pre)covers and preenvelopes

In this section, we study the existence of V-Gorenstein flat covers and preenvelopes of modules. It is shown that $(V$-$\mathcal{GF}, V$-$\mathcal{GF}^{\perp})$ is a perfect hereditary cotorsion pair in $\mathcal{B}^l(R)$.

Lemma 3.1 *Let $M \in \mathcal{A}^l(R)$ and $0 \to G \to F_{r-1} \to \cdots \to F_0 \to M \to 0$ be exact with every F_i flat. Then G is Gorenstein flat.*

Proof By assumption, we have $0 \to V \otimes_R G \to V \otimes_R F_{r-1} \to \cdots \to V \otimes_R F_0 \to V \otimes_R M \to 0$ is exact with each $V \otimes_R F_i \in \mathcal{X}$ and $V \otimes_R M \in \mathcal{B}^l(R)$. Hence, Theorem 2.3 implies that $V \otimes_R G \in V$-$\mathcal{GF}$. Consequently, $G \in \mathcal{GF}$ by Proposition 2.5. □

Lemma 3.2 *Let $M, N \in \mathcal{A}^l(R)$. Then $\mathrm{Ext}^i_R(M, N) = 0$ if and only if $\mathrm{Ext}^i_R(V \otimes_R M, V \otimes_R N) = 0$ for all $i \geqslant 1$.*

Proof This follows from [14, Theorem 6.4]. □

It was shown in [12, Theorem 3.23] that if M is an R-module with finite Gorenstein flat dimension n, then M admits a surjective $\mathcal{GF}$-precover $\varphi : T \to M$ with $\mathrm{fd}({}_R\mathrm{Ker}\varphi) = n - 1$.

Theorem 3.3 *Every left R-module M in $\mathcal{B}^l(R)$ has a V-Gorenstein flat precover $\varphi : F \to M$ such that left $\mathcal{X}$-dim $\mathrm{Ker}\varphi \leqslant r - 1$.*

Proof Let $M \in \mathcal{B}^l(R)$. Then $\mathrm{Hom}_R(V,M) \in \mathcal{A}^l(R)$, and so $\mathrm{Hom}_R(V,M)$ has Gorenstein flat dimension less than or equal to r by Lemma 3.1. Therefore, [12, Theorem 3.23] yields an exact sequence $0 \to K \to G \to \mathrm{Hom}_R(V,M) \to 0$, where $G \to \mathrm{Hom}_R(V,M)$ is a $\mathcal{GF}$-precover and $\mathrm{fd}({}_RK) \leqslant r-1$. However, $M \in \mathcal{B}^l(R)$; it follows that $0 \to V \otimes_R K \to V \otimes_R G \xrightarrow{\varphi} M \to 0$ is exact with $V \otimes_R G \in V\text{-}\mathcal{GF}$ and left $\mathcal{X}$-dim $(V \otimes_R K) \leqslant r-1$. Next we show that $\varphi : V \otimes_R G \to M$ is a V-Gorenstein flat precover of M. Let $Q \in V\text{-}\mathcal{GF}$. Then $\mathrm{Hom}_R(V,Q) \in \mathcal{GF}$. Consider the following commutative diagram:

$$\begin{array}{ccccc}
\mathrm{Hom}_R(\mathrm{Hom}_R(V,Q),G) & \longrightarrow & \mathrm{Hom}_R(\mathrm{Hom}_R(V,Q),\mathrm{Hom}_R(V,M)) & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow \cong & & \\
\mathrm{Hom}_R(Q, V \otimes_R G) & \longrightarrow & \mathrm{Hom}_R(Q,M) & &
\end{array}$$

with the upper row exact. Thus $V \otimes_R G \to M$ is the desired V-Gorenstein flat precover. □

Let $\mathcal{L}$ denote the class of left R-modules L such that $L \cong V \otimes_R K$ for some $K \in R$-Mod cotorsion. If T is a Gorenstein flat R module, then $\mathrm{Ext}_R^i(T,K) = 0$ for all $i \geqslant 1$ and all cotorsion R-modules K with finite flat dimension by [12, Proposition 3.22].

Lemma 3.4 *If $\mathbb{X} : \cdots \to X_1 \to X_0 \to X^0 \to X^1 \to \cdots$ is an exact sequence of modules in $\mathcal{X}$ such that $U \otimes_R \mathbb{X}$ is exact for all $U \in \mathcal{U}$, then $\mathrm{Hom}_R(\mathbb{X},L)$ is exact for all $L \in \mathcal{L}$ with left $\mathcal{X}$-dim $L < \infty$.*

Proof Let $L \cong V \otimes_R K$ for some $K \in R$-Mod cotorsion. If left $\mathcal{X}$-dim $L = 0$, then K is flat and cotorsion, and so K^+ is injective, K^{++} is flat, and K^{++}/K is flat, which implies that $0 \to K \to K^{++} \to K^{++}/K \to 0$ is split. Therefore, L is a direct summand of $V \otimes_R K^{++}$. Note that

$$\mathrm{Hom}_R(\mathbb{X}, V \otimes_R K^{++}) \cong \mathrm{Hom}_R(\mathbb{X}, \mathrm{Hom}_{R^{\mathrm{op}}}(V,K^+)^+) \cong (\mathrm{Hom}_{R^{\mathrm{op}}}(V,K^+) \otimes_R \mathbb{X})^+$$

is exact. Thus $\mathrm{Hom}_R(\mathbb{X},L)$ is exact. Now let left $\mathcal{X}$-dim $L = n$ with $n \geqslant 1$. Then $\mathrm{fd}({}_RK) = n$. Consider the exact sequence $0 \to K' \to F \to K \to 0$, where $F \to K$ is a flat cover of K. Then K' is cotorsion with $\mathrm{fd}({}_RK') = n-1$ and F is cotorsion, which implies that

$$0 \longrightarrow \mathrm{Hom}_R(\mathbb{X}, V \otimes_R K') \longrightarrow \mathrm{Hom}_R(\mathbb{X}, V \otimes_R F) \longrightarrow \mathrm{Hom}_R(\mathbb{X}, V \otimes_R K) \longrightarrow 0$$

is exact since $\mathrm{Ext}_R^1(V \otimes_R Q, V \otimes_R K') = 0$ for all $V \otimes_R Q \in \mathcal{X}$ by Lemma 3.2. Hence $\mathrm{Hom}_R(\mathbb{X},L)$ is exact by the induction hypothesis. □

The following claim is an immediate consequence of Theorem 2.3.

Lemma 3.5 *The class V-$\mathcal{GF}$ is closed under direct limits.*

Lemma 3.6 *The class V-$\mathcal{GF}$ is closed under pure submodules and pure quotient modules.*

Proof Let $0 \to L \to M \to N \to 0$ be pure exact in R-Mod with $M \in V\text{-}\mathcal{GF}$. Then $0 \to N^+ \to M^+ \to L^+ \to 0$ is split with $M^+ \in V\text{-}\mathcal{GI}$ by Theorem 2.3. Again by Theorem 2.3, we get $L, N \in V\text{-}\mathcal{GF}$, as desired. □

It is well known that $(\mathcal{GF}, \mathcal{GF}^\perp)$ is a perfect hereditary cotorsion pair by [6, Theorem 2.12]. We do not know if it is true that $(V\text{-}\mathcal{GF}, V\text{-}\mathcal{GF}^\perp)$ is a perfect hereditary cotorsion pair in R-Mod. However, we have the following result.

Theorem 3.7 *$(V\text{-}\mathcal{GF}, V\text{-}\mathcal{GF}^{\perp})$ is a perfect hereditary cotorsion pair in $\mathcal{B}^l(R)$.*

Proof Let $0 \to M' \to M \to M'' \to 0$ be exact with $M'' \in V\text{-}\mathcal{GF}$. Then $M' \in \mathcal{B}^l(R)$ if and only if $M \in \mathcal{B}^l(R)$, and so $0 \to \mathrm{Hom}_R(V, M') \to \mathrm{Hom}_R(V, M) \to \mathrm{Hom}_R(V, M'') \to 0$ is exact. Note that $\mathrm{Hom}_R(V, M'') \in \mathcal{GF}$. Then $\mathrm{Hom}_R(V, M') \in \mathcal{GF}$ if and only if $\mathrm{Hom}_R(V, M) \in \mathcal{GF}$ by [12, Theorem 3.6], and hence $M' \in V\text{-}\mathcal{GF}$ if and only if $M \in V\text{-}\mathcal{GF}$ by Theorem 2.3. It follows that $(V\text{-}\mathcal{GF}, V\text{-}\mathcal{GF}^{\perp})$ is hereditary.

Let $F \in V\text{-}\mathcal{GF}$. Then there is a cardinal κ such that F can be written as the direct union of a continuous chain of submodules $(F_\alpha)_{\alpha<\lambda}$ with λ an ordinal number such that $F_0 \in V\text{-}\mathcal{GF}$, $F_{\alpha+1}/F_\alpha \in V\text{-}\mathcal{GF}$ when $\alpha+1 < \lambda$ with $|F_0|, |F_{\alpha+1}/F_\alpha| \leqslant \kappa$ by Lemma 3.6. Therefore if B is the direct sum of all representatives in $V\text{-}\mathcal{GF}$ such that their cardinals are less than or equal to κ, then $M \in V\text{-}\mathcal{GF}^{\perp}$ if and only if $\mathrm{Ext}^1_R(B, M) = 0$.

Let $N \in \mathcal{B}^l(R)$. Then $\mathrm{Hom}_R(V, N) \in \mathcal{A}^l(R)$ and there exists an exact sequence $0 \to \mathrm{Hom}_R(V, N) \to A \to G \to 0$ such that $A \in \mathcal{GF}^{\perp}$ and $G \in \mathcal{GF}$ by [6, Theorem 2.11], and so $0 \to N \to V \otimes_R A \to V \otimes_R G \to 0$ is exact with $V \otimes_R G \in V\text{-}\mathcal{GF}$. Let $W \in V\text{-}\mathcal{GF}$. Then $\mathrm{Ext}^1_R(\mathrm{Hom}_R(V, W), A) = 0$ by Theorem 2.3. Thus $\mathrm{Ext}^1_R(W, V \otimes_R A) = 0$ by Lemma 3.2, and hence $V \otimes_R A \in V\text{-}\mathcal{GF}^{\perp}$. Let $M \in {}^{\perp}(V\text{-}\mathcal{GF}^{\perp}) \cap \mathcal{B}^l(R)$ and $0 \to K \to X \to M \to 0$ be exact with $X \in \mathcal{X}$. Then $K \in \mathcal{B}^l(R)$ and there exists an exact sequence $0 \to K \to D \to F \to 0$ with $D \in V\text{-}\mathcal{GF}^{\perp}$ and $F \in V\text{-}\mathcal{GF}$ by the preceding proof. Consider the pushout of $K \to X$ and $K \to D$:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & K & \longrightarrow & X & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & D & \longrightarrow & C & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & F & = & F & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}
$$

Since X, $F \in V\text{-}\mathcal{GF}$, we have $C \in V\text{-}\mathcal{GF}$. Since $D \in V\text{-}\mathcal{GF}^{\perp}$, the central row splits, and so $M \in V\text{-}\mathcal{GF} \cap \mathcal{B}^l(R)$. This gives us that $V\text{-}\mathcal{GF} = {}^{\perp}(V\text{-}\mathcal{GF}^{\perp})$ in $\mathcal{B}^l(R)$, and hence $(V\text{-}\mathcal{GF}, V\text{-}\mathcal{GF}^{\perp})$ is a cotorsion pair in $\mathcal{B}^l(R)$ with enough projectives and injectives. Finally, the cotorsion pair in $\mathcal{B}^l(R)$ is perfect by [4, Theorem 7.2.6] and Lemma 3.5. □

From the preceding theorem, we get that every module in $\mathcal{B}^l(R)$ has a V-Gorenstein flat cover. However, for any left R-module, we get the following general result.

Theorem 3.8 *Every left R-module has a V-Gorenstein flat cover.*

Proof This follows from Theorem 2.3, Lemma 3.6, and [13, Theorem 3.1]. □

It is well known that if M is a left R-module of finite Gorenstein flat dimension, then there is an exact sequence $0 \to M \to H \to A \to 0$ with $A \in \mathcal{GF}$ and $\mathrm{fd}({}_RH) = \mathrm{Gfd}({}_RM)$.

Proposition 3.9 *Let $M \in \mathcal{B}^l(R)$. Then there exists an exact sequence of left R-modules $0 \to M \to H \to A \to 0$, where $A \in V\text{-}\mathcal{GF}$ and left $\mathcal{X}$-dim $H \leqslant r$.*

Proof Note that $\mathrm{Hom}_R(V,M)\in\mathcal{A}^l(R)$. Therefore, $\mathrm{Hom}_R(V,M)$ has Gorenstein flat dimension less than or equal to r by Lemma 3.1. Hence there is an exact sequence $0\to\mathrm{Hom}_R(V,M)\to H'\to A'\to 0$ with $A'\in\mathcal{GF}$ and $\mathrm{fd}({}_RH')\leqslant r$. However, $M\in\mathcal{B}^l(R)$; it follows that $0\to M\to V\otimes_R H'\to V\otimes_R A'\to 0$ is the desired exact sequence. □

Theorem 3.10 *Every left R-module has a V-Gorenstein flat preenvelope.*

Proof By Theorem 2.3(7), one easily checks that the class V-$\mathcal{GF}$ is closed under direct products. Therefore every left R-module has a V-Gorenstein flat preenvelope by Theorem 2.3, Lemma 3.6, and [13, Theorem 3.1]. □

Proposition 3.11 $-\otimes-$ *is right balanced by V-$\mathcal{GI}\times V$-$\mathcal{GF}$ on R^{op}-Mod $\times$ R-Mod.*

Proof Let $M\in R$-Mod. Then there exists a right V-Gorenstein flat resolution $0\to M\to F^0\to F^1\to\cdots$ by Theorem 3.10. Let $G\in V$-$\mathcal{GI}$. Then $0\to G\otimes_R M\to G\otimes_R F^0\to G\otimes_R F^1\to\cdots$ is exact if and only if $\cdots\to\mathrm{Hom}_R(F^1,G^+)\to\mathrm{Hom}_R(F^0,G^+)\to\mathrm{Hom}_R(M,G^+)\to 0$ is exact. However, the last sequence is exact by Proposition 2.6.

Let $N\in R^{\mathrm{op}}$-Mod. There exists a right V-Gorenstein injective resolution $0\to N\to E^0\to E^1\to\cdots$ by [17, Proposition 3.13]. Let $G\in V$-$\mathcal{GF}$. By analogy with the preceding proof, we have $0\to G\otimes_R N\to G\otimes_R E^0\to G\otimes_R E^1\to\cdots$ is exact. □

Proposition 3.12 *Hom$(-,-)$ is left balanced by V-$\mathcal{GF}\times V$-$\mathcal{GF}$ on R-Mod $\times$ R-Mod.*

4. Applications

In this section, we characterize some rings in terms of Gorenstein and V-Gorenstein homological modules.

Proposition 4.1 *The following are equivalent for a ring R:*

(1) *R is a right self-injective ring;*

(2) *Every left R-module is Gorenstein flat;*

(3) *Every finitely generated left R-module is Gorenstein flat;*

(4) *Every left R-module is V-Gorenstein flat;*

(5) *Every finitely generated left R-module is V-Gorenstein flat.*

Proof (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) are trivial.

(1) $\Rightarrow$ (2). Let M be a left R-module. Then M has a right flat resolution $\mathbb{F}:0\to M\to F^0\to F^1\to\cdots$ since R is right Noetherian, which is exact since $(R_R)^+$ is flat. Let I be an injective right R-module. Then I is flat, and so $\mathrm{Tor}_i^R(I,M)=0$ for all $i\geqslant 1$. Thus $M\in\mathcal{GF}$.

(3) $\Rightarrow$ (1). Let E be any injective left R-module. Then E is flat by the proof of [3, Theorem 6], and so R is right self-injective by [15, Proposition 3.7] since R is right Noetherian.

(2) $\Rightarrow$ (4). Let M be a left R-module. Then $\mathrm{Hom}_R(V,M)\in\mathcal{GF}$, and so $V\otimes_R\mathrm{Hom}_R(V,M)\in\mathcal{B}^l(R)$. Let $0\to M\to E^0\to E^1\to\cdots$ be an injective resolution of M. Consider the exact sequence $0\to\mathrm{Hom}_R(V,M)\to\mathrm{Hom}_R(V,E^0)\to C\to 0$. Since $\mathrm{Hom}_R(V,M)$, $\mathrm{Hom}_R(V,E^0)\in\mathcal{A}^l(R)$, $C\in\mathcal{A}^l(R)$. Now we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 \longrightarrow & V\otimes_R \operatorname{Hom}_R(V,M) & \longrightarrow & V\otimes_R \operatorname{Hom}_R(V,E^0) & \longrightarrow & V\otimes_R C & \longrightarrow 0 \\ & \downarrow{\scriptstyle \sigma_M} & & \downarrow{\scriptstyle \cong} & & & \\ 0 \longrightarrow & M & \longrightarrow & E^0. & & & \end{array}$$

Therefore, σ_M is monic. Consider the exact sequence $0 \to V \otimes_R \operatorname{Hom}_R(V, M) \to M \to L \to 0$. Then

$$0 \longrightarrow \operatorname{Hom}_R(V, V \otimes_R \operatorname{Hom}_R(V, M)) \longrightarrow \operatorname{Hom}_R(V, M) \longrightarrow \operatorname{Hom}_R(V, L) \longrightarrow 0$$

is exact since $\operatorname{Ext}^1_R(V, V \otimes_R \operatorname{Hom}_R(V, M)) = 0$, and so $\operatorname{Hom}_R(V, L) = 0$ since $\operatorname{Hom}_R(V, V \otimes_R \operatorname{Hom}_R(V, M)) \cong \operatorname{Hom}_R(V, M)$, which implies that $L = 0$. Thus $M \cong V \otimes_R \operatorname{Hom}_R(V, M) \in \mathcal{B}^l(R)$ and M is V-Gorenstein flat by Proposition 2.5.

(4) $\Rightarrow$ (2). The proof is dual to that of (2) $\Rightarrow$ (4).

(5) $\Rightarrow$ (4). Let M be any left R-module. Then $M = \varinjlim M_i$, where M_i is a finitely generated submodule of M. Hence M is V-Gorenstein flat by (5) and Lemma 3.5. □

A ring R is said to be left (resp. right) n-perfect [8] if every flat left (resp. right) R-module has projective dimension less than or equal to n.

Proposition 4.2 *Let R be left n-perfect. The following are equivalent:*

(1) *R is left perfect;*

(2) *Every V-Gorenstein flat left R-module is V-Gorenstein projective;*

(3) *The class of V-Gorenstein projective left R-modules is closed under direct limits.*

Proof (1) $\Rightarrow$ (2). Let $M \in V\text{-}\mathcal{GF}$ and $W \cong V \otimes_R P \in \mathcal{W}$. Then $M \in \mathcal{B}^l(R)$ and $W^{++} \cong V \otimes_R P^{++}$; it follows the fact that $\operatorname{Ext}^i_R(M, W^{++}) \cong \operatorname{Tor}^R_i(W^+, M)^+ = 0$ for all $i \geqslant 1$. Consequently $\operatorname{Ext}^i_R(M, W) = 0$ for all $i \geqslant 1$ and $M \in V\text{-}\mathcal{GP}$ by [7, Theorem 3.4].

(2) $\Rightarrow$ (3). This follows from that $V\text{-}\mathcal{GP} \subseteq V\text{-}\mathcal{GF}$ and $V\text{-}\mathcal{GF}$ is closed under direct limits.

(3) $\Rightarrow$ (1). Let F be a flat left R-module. Then $F = \varinjlim F_i$, where each F_i is finitely generated projective. By (3), $V \otimes_R F \cong \varinjlim (V \otimes_R F_i) \in V\text{-}\mathcal{GP}$, and hence F is Gorenstein projective. However, $\operatorname{pd}({}_R F) \leqslant n$ and so F is projective by [12, Proposition 2.27]. □

Proposition 4.3 *The following are equivalent for a ring R:*

(1) *$\{fd(M) \leqslant r \,|\, M \in \mathcal{A}^l(R)\}$;*

(2) *$\{fd(M) < \infty \,|\, M \in \mathcal{A}^l(R)\}$;*

(3) *Every Gorenstein flat left R-module is flat;*

(3)′ *Every V-Gorenstein flat left R-module belongs to $\mathcal{X}$.*

If R is left n-perfect, then the above are also equivalent to:

(4) *Every Gorenstein projective left R-module is projective;*

(4)′ *Every V-Gorenstein projective left R-module belongs to $\mathcal{W}$;*

(5) *Every Gorenstein injective right R-module is injective;*

(5)′ *Every V-Gorenstein injective right R-module belongs to $\mathcal{U}$.*

Proof (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (3). Let $M \in \mathcal{GF}$. Then $M \in \mathcal{A}^l(R)$ and $\mathrm{fd}(M) < \infty$ by (2). Thus M is flat.

(3) $\Rightarrow$ (3)$'$. Let $M \in V\text{-}\mathcal{GF}$. Then $\mathrm{Hom}_R(V, M) \in \mathcal{GF}$, and so $\mathrm{Hom}_R(V, M)$ is flat. However, $M \cong V \otimes_R \mathrm{Hom}_R(V, M)$, as desired.

(3)$'$ $\Rightarrow$ (1). Let $M \in \mathcal{A}^l(R)$. Then $V \otimes_R M \in \mathcal{B}^l(R)$. Consider the exact sequence $0 \to K \to X_{r-1} \to \cdots \to X_0 \to V \otimes_R M \to 0$ with each $X_i \in \mathcal{X}$. Then Theorem 2.3 implies that $K \in V\text{-}\mathcal{GF}$, and so $K \in \mathcal{X}$. The exact sequence $0 \to \mathrm{Hom}_R(V, K) \to \mathrm{Hom}_R(V, X_{r-1}) \to \cdots \to \mathrm{Hom}_R(V, X_0) \to M \to 0$ gives that $\mathrm{fd}(_RM) \leqslant r$.

Next suppose that R is left n-perfect.

(2) $\Rightarrow$ (4). Let M be a Gorenstein projective left R-module. Then $M \in \mathcal{A}^l(R)$ and $\mathrm{fd}(_RM) < \infty$ by (2), and so $\mathrm{pd}(_RM) < \infty$. Consequently M is projective.

(4) $\Rightarrow$ (4)$'$. Let $M \in V\text{-}\mathcal{GP}$. Then $\mathrm{Hom}_R(V, M)$ is Gorenstein projective by [7, Theorem 3.4], and hence $\mathrm{Hom}_R(V, M)$ is projective by (4). However, $M \cong V \otimes_R \mathrm{Hom}_R(V, M)$, as desired.

(4)$'$ $\Rightarrow$ (1). Let $M \in \mathcal{A}^l(R)$. Consider the exact sequence $0 \to K \to P_{r-1} \to \cdots \to P_0 \to M \to 0$ with each P_i projective. Then [8, Theorem 3.20] shows that K is Gorenstein projective. However, $V \otimes_R K \in \mathcal{B}^l(R)$ and $K \cong \mathrm{Hom}_R(V, V \otimes_R K)$ is Gorenstein projective and so $V \otimes_R K \in V\text{-}\mathcal{GP}$ by [7, Theorem 3.4], which implies that K is projective, as desired.

(3) $\Rightarrow$ (5). Let M be a Gorenstein injective right R-module. Then M^+ is Gorenstein flat, and so M^+ is flat. Consequently M is injective.

(5) $\Rightarrow$ (5)$'$. Let $M \in V\text{-}\mathcal{GI}$. Then $M \otimes_R V$ is Gorenstein injective by [7, Theorem 2.4]. However, $M \cong \mathrm{Hom}_{R^{\mathrm{op}}}(V, M \otimes_R V)$, as desired.

(5)$'$ $\Rightarrow$ (1). Let $M \in \mathcal{A}^l(R)$. Consider the exact sequence $0 \to K \to F_{r-1} \to \cdots \to F_0 \to M \to 0$ with each F_i flat. Then $0 \to M^+ \to F_0^+ \to \cdots \to F_{r-1}^+ \to K^+ \to 0$ is exact and every F_i^+ is injective; it follows from [8, Theorem 3.17] that K^+ is Gorenstein injective. However, $K^+ \cong \mathrm{Hom}_{R^{\mathrm{op}}}(V, K^+) \otimes_R V$ and so $\mathrm{Hom}_{R^{\mathrm{op}}}(V, K^+) \in V\text{-}\mathcal{GI}$ and $\mathrm{Hom}_{R^{\mathrm{op}}}(V, K^+) \cong \mathrm{Hom}_{R^{\mathrm{op}}}(V, E)$ for some injective right R-module E by (5)$'$, and hence $K^+ \cong \mathrm{Hom}_{R^{\mathrm{op}}}(V, K^+) \otimes_R V \cong \mathrm{Hom}_{R^{\mathrm{op}}}(V, E) \otimes_R V \cong E$. This implies that K is flat, as desired. $\square$

Proposition 4.4 *The following are equivalent for a ring R:*

(1) *R is a QF ring;*

(2) *Every left R-module is Gorenstein projective;*

(2)$'$ *Every left R-module is V-Gorenstein projective;*

(3) *Every right R-module is Gorenstein injective;*

(3)$'$ *Every right R-module is V-Gorenstein injective;*

(4) *Every left R-module is Gorenstein flat;*

(4)$'$ *Every left R-module is V-Gorenstein flat.*

Proof (1) $\Rightarrow$ (2). Let M be a left R-module. Consider the projective resolution and injective resolution of M: $\cdots \to P_1 \to P_0 \to M \to 0$ and $0 \to M \to E^0 \to E^1 \to \cdots$. Then

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots$$

is a complete projective resolution of M. Thus M is Gorenstein projective.

(2) $\Rightarrow$ (2)$'$. Let M be a left R-module. Then $\mathrm{Hom}_R(V,M)$ is Gorenstein projective, and hence $V\otimes_R \mathrm{Hom}_R(V,M)\in \mathcal{B}^l(R)$. By analogy with the proof of Proposition 4.1, we get $M\cong V\otimes_R \mathrm{Hom}_R(V,M)$ and $M\in V$-$\mathcal{GP}$ by [17, Proposition 2.7].

(2)$'$ $\Rightarrow$ (2). Let M be a left R-module. Then $V\otimes_R M\in V$-$\mathcal{GP}$ by (2)$'$. By analogy with the proof of Proposition 4.1, $M\cong \mathrm{Hom}_R(V,V\otimes_R M)$ is Gorenstein projective.

(2) $\Rightarrow$ (1). Let I be an injective left R-module. Then I is Gorenstein projective, and so there is an exact sequence $0\to I\to P\to L\to 0$ with P projective. This implies that I is projective, as desired.

(1) $\Rightarrow$ (3) $\Leftrightarrow$ (3)$'$ and (1) $\Rightarrow$ (4) $\Leftrightarrow$ (4)$'$. By analogy with the proof of (1) $\Rightarrow$ (2) $\Leftrightarrow$ (2)$'$.

(3) $\Rightarrow$ (1). By analogy with the proof of (2) $\Rightarrow$ (1).

(4) $\Rightarrow$ (1). Let P be a projective right R-module. Then $P^+\in\mathcal{GF}$, and so there is an exact sequence $0\to P^+\to F\to L\to 0$ with F flat. This shows that P^+ is flat and P is injective, as desired. □

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