Coverings and crossed modules of topological groups with operations

Osman MUCUK[∗]**, Tunçar ŞAHAN**

Department of Mathematics, Erciyes University, Kayseri, Turkey

Abstract: It is a well-known result of the covering groups that a subgroup G of the fundamental group at the identity of a semilocally simply connected topological group determines a covering morphism of topological groups with characteristic group G. In this paper we generalize this result to a large class of algebraic objects called topological groups with operations, including topological groups. We also prove that the crossed modules and internal categories within topological groups with operations are equivalent. This equivalence enables us to introduce the cover of crossed modules within topological groups with operations. Finally, we draw relations between the coverings of an internal groupoid within topological groups with operations and those of the corresponding crossed module.

Key words: Covering groups, universal cover, crossed module, group with operations, topological groups with operations

1. Introduction

The theory of covering spaces is one of the most interesting theories in algebraic topology. It is well known that if X is a topological group, say additive, $p \colon \widetilde{X} \to X$ is a simply connected covering map and $\tilde{0} \in \widetilde{X}$ is such that $p(\tilde{0}) = 0$, then $\widetilde{X}$ becomes a topological group with identity $\tilde{0}$ such that p is a morphism of topological groups (see, for example, [9]).

The problem of universal covers of nonconnected topological groups was first studied in [25]. Taylor proved that a topological group X determines an obstruction class k_X in $H^3(\pi_0(X), \pi_1(X, 0))$, and that the vanishing of k_X is a necessary and sufficient condition for the lifting of the group structure to a universal cover. In [17], an analogous algebraic result was given in terms of crossed modules and group-groupoids, i.e. group objects in the category of groupoids (see also [7] for a revised version, which generalizes these results and shows the relation with the theory of obstructions to extension for groups, and [19] for the recently developed notion of monodromy for topological group-groupoids).

In [8, Theorem 1], Brown and Spencer proved that the category of internal categories within the groups, i.e. group-groupoids, is equivalent to the category of crossed modules of groups. In [20], considering this equivalence of the categories, normality and quotient concepts are related in 2 categories. In [23, Section 3], Porter then proved that a similar result to [8, Theorem 1] holds for certain algebraic categories **C**, introduced by Orzech [21], whose definition was adapted by him and called the category of groups with operations. Applying Porter's result, the study of internal category theory in **C** was continued in the works of Datuashvili, i.e. [11]

and [13]. Moreover, she developed a cohomology theory of internal categories, equivalently, crossed modules, in categories of groups with operations [10, 12]. In a similar way, the results of [8] and [23] enabled us to prove that some properties of covering groups can be generalized to topological groups with operations.

If X is a connected topological space that has a universal cover, $x_0 \in X$, and G is a subgroup of the fundamental group $\pi_1(X, x_0)$ of X at the point x_0, then by [24, Theorem 10.42] we know that there is a covering map $p \colon (\widetilde{X}_G, \tilde{x}_0) \to (X, x_0)$ of pointed spaces, with characteristic group G. In particular, if G is singleton, then p becomes the universal covering map. Furthermore, if X is a topological group, then $\widetilde{X}_G$ becomes a topological group such that p is a morphism of topological groups. Recently in [2], this method was applied to topological R-modules and a more general result was obtained (see also [3] and [18] for groupoid setting).

The object of this paper is to prove that this result can be generalized to a wide class of algebraic categories, which include categories of topological groups, topological rings, topological R-modules, and alternative topological algebras. This is conveniently handled by working in a category TC. The method we use is based on that used by Rotman in [24, Theorem 10.42].

We also prove that the crossed modules and internal categories in TC are equivalent. Finally, we introduce the cover of crossed modules in TC and draw relations between the covers of an internal groupoid in TC and those of the corresponding crossed module.

2. Preliminaries on groupoids and covering groups

As defined in [4, 16], a groupoid G has a set G of morphisms, which we call just *elements* of G, a set G_0 of *objects* together with maps $d_0, d_1 \colon G \to G_0$ and $\epsilon \colon G_0 \to G$ such that $d_0\epsilon = d_1\epsilon = 1_{G_0}$. The maps d_0, d_1 are called *initial* and *final* point maps respectively and the map ϵ is called *object inclusion*. If $a, b \in G$ and $d_1(a) = d_0(b)$, then the *composite* $a \circ b$ exists such that $d_0(a \circ b) = d_0(a)$ and $d_1(a \circ b) = d_1(b)$. Thus, there exists a partial composition defined by $G_{d_1} \times_{d_0} G \to G, (a, b) \mapsto a \circ b$, where $G_{d_1} \times_{d_0} G$ is the pullback of d_1 and d_0. Furthermore, this partial composition is associative, for $x \in G_0$ the element $\epsilon(x)$ acts as the identity, and each element a has an inverse a^{-1} such that $d_0(a^{-1}) = d_1(a)$, $d_1(a^{-1}) = d_0(a)$, $a \circ a^{-1} = \epsilon d_0(a)$ and $a^{-1} \circ a = \epsilon d_1(a)$. The map $G \to G$, $a \mapsto a^{-1}$ is called the *inversion*.

In a groupoid G for $x, y \in G_0$ we write $G(x, y)$ for the set of all morphisms with initial point x and final point y. According to [4] for $x \in G_0$, the *star* of x is defined as $\{a \in G \mid d_0(a) = x\}$ and denoted as $\mathsf{St}_G x$.

Let G and H be groupoids. A *morphism* from H to G is a pair of maps $f \colon H \to G$ and $f_0 \colon H_0 \to G_0$ such that $d_0 f = f_0 d_0$, $d_1 f = f_0 d_1$, $f\epsilon = \epsilon f_0$, and $f(a \circ b) = f(a) \circ f(b)$ for all $(a, b) \in H_{d_1} \times_{d_0} H$. For such a morphism we simply write $f \colon H \to G$.

We assume the usual theory of covering maps. All spaces X are assumed to be locally path-connected and semilocally 1-connected, so that each path component of X admits a simply connected cover. Recall that a covering map $p \colon \widetilde{X} \to X$ of connected spaces is called *universal* if it covers every covering of X in the sense that if $q \colon \widetilde{Y} \to X$ is another covering of X then there exists a map $r \colon \widetilde{X} \to \widetilde{Y}$ such that $p = qr$ (hence, r becomes a covering). A covering map $p \colon \widetilde{X} \to X$ is called *simply connected* if $\widetilde{X}$ is simply connected. Note that a simply connected covering is a universal covering.

A subset U of a space X, which has a universal cover, is called *liftable* if it is open and path-connected and it lifts to each covering of X; that is, if $p \colon \widetilde{X} \to X$ is a covering map, $\imath \colon U \to X$ is the inclusion map and

$\tilde{x} \in \widetilde{X}$ such that $p(\tilde{x}) = x \in U$, then there exists a map (necessarily unique) $\hat{\imath}: U \to \widetilde{X}$ such that $p\hat{\imath} = \imath$ and $\hat{\imath}(x) = \tilde{x}$. It is an easy application that U is liftable if and only if it is open and path-connected and for all $x \in U$, the fundamental group $\pi_1(U, x)$ is mapped to the singleton by the morphism $\imath_\star: \pi_1(U, x) \to \pi_1(X, x)$ induced by the inclusion $\imath: (U, x) \to (X, x)$.

A space X is called *semilocally simply connected* if each point has a liftable neighborhood and *locally simply connected* if it has a base of simply connected sets. Thus, a locally simply connected space is also semilocally simply connected.

For a covering map $p: (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ of pointed topological spaces, the subgroup $p_\star(\pi_1(\widetilde{X}, \tilde{x}_0))$ of $\pi_1(X, x_0)$ is called *characteristic group* of p, where $p_\star$ is the morphism induced by p (see [4, p.379] for the characteristic group of a covering map in terms of covering morphism of groupoids). Two covering maps $p: (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ and $q: (\widetilde{Y}, \tilde{y}_0) \to (X, x_0)$ are called *equivalent* if their characteristic groups are equal, and equivalently there is a homeomorphism $f: (\widetilde{X}, \tilde{x}_0) \to (\widetilde{Y}, \tilde{y}_0)$ such that $qf = p$.

We recall a construction from [24, p.295] as follows: let X be a topological space with a base point x_0 and G a subgroup of $\pi_1(X, x_0)$. Let $P(X, x_0)$ be the set of all paths of α in X with initial point x_0. Then the relation defined on $P(X, x_0)$ by $\alpha \simeq \beta$ if and only if $\alpha(1) = \beta(1)$ and $[\alpha \circ \beta^{-1}] \in G$ is an equivalence relation. Denote the equivalence class of α by $\langle \alpha \rangle_G$ and define $\widetilde{X}_G$ as the set of all such equivalence classes of the paths in X with initial point x_0. Define a function $p: \widetilde{X}_G \to X$ by $p(\langle \alpha \rangle_G) = \alpha(1)$. Let α_0 be the constant path at x_0 and $\tilde{x}_0 = \langle \alpha_0 \rangle_G \in \widetilde{X}_G$. If $\alpha \in P(X, x_0)$ and U is an open neighborhood of $\alpha(1)$, then a path of the form $\alpha \circ \lambda$, where λ is a path in U with $\lambda(0) = \alpha(1)$, is called a *continuation* of α. For an $\langle \alpha \rangle_G \in \widetilde{X}_G$ and an open neighborhood U of $\alpha(1)$, let $(\langle \alpha \rangle_G, U) = \{\langle \alpha \circ \lambda \rangle_G: \lambda(I) \subseteq U\}$. Then the subsets $(\langle \alpha \rangle_G, U)$ form a basis for a topology on $\widetilde{X}_G$ such that the map $p: (\widetilde{X}_G, \tilde{x}_0) \to (X, x_0)$ is continuous.

In Theorem 3.6 we generalize the following result to topological groups with operations.

Theorem 2.1 [24, Theorem 10.34] *Let (X, x_0) be a pointed topological space and G a subgroup of $\pi_1(X, x_0)$. If X is connected, locally path-connected, and semilocally simply connected, then $p: (\widetilde{X}_G, \tilde{x}_0) \to (X, x_0)$ is a covering map with characteristic group G.*

Remark 2.2 Let X be a connected, locally path-connected, and semilocally simply connected topological space and $q: (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ a covering map. Let G be the characteristic group of q. Then the covering map q is equivalent to the covering map $p: (\widetilde{X}_G, \tilde{x}_0) \to (X, x_0)$ corresponding to G.

From Theorem 2.1 the following result is obtained.

Theorem 2.3 [24, Theorem 10.42] *Suppose that X is a connected, locally path-connected, and semilocally simply connected topological group. Let $0 \in X$ be the identity element and $p: (\widetilde{X}, \tilde{0}) \to (X, 0)$ a covering map. Then the group structure of X lifts to $\widetilde{X}$, i.e. $\widetilde{X}$ becomes a topological group such that $\tilde{0}$ is identity and $p: (\widetilde{X}, \tilde{0}) \to (X, 0)$ is a morphism of topological groups.*

3. Universal covers of topological groups with operations

In this section we apply the methods of Section 2 to the topological groups with operations and obtain parallel results.

The idea of the definition of categories of groups with operations comes from [15] and [21] (see also [22]) and the definition below is from [23] and [14, p.21], which is adapted from [21].

Definition 3.1 Let C be a category of groups with a set of operations Ω and with a set E of identities such that E includes the group laws, and the following conditions hold for the set Ω_i of i-ary operations in Ω:

(a) $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$.

(b) The group operations written additively $0, -$, and $+$ are the elements of Ω_0, Ω_1, and Ω_2, respectively. Let $\Omega_2' = \Omega_2 \backslash \{+\}$, $\Omega_1' = \Omega_1 \backslash \{-\}$ and assume that if $\star \in \Omega_2'$, then $\star^\circ$ defined by $a \star^\circ b = b \star a$ is also in Ω_2'. Also assume that $\Omega_0 = \{0\}$.

(c) For each $\star \in \Omega_2'$, E includes the identity $a \star (b + c) = a \star b + a \star c$.

(d) For each $\omega \in \Omega_1'$ and $\star \in \Omega_2'$, E includes the identities $\omega(a+b) = \omega(a) + \omega(b)$ and $\omega(a) \star b = \omega(a \star b)$.

The category C satisfying conditions (a)–(d) is called a *category of groups with operations.*

In the paper from now on, C will denote a category of groups with operations.

A *morphism* between any 2 objects of C is a group homomorphism, which preserves the operations of Ω_1' and Ω_2'.

Remark 3.2 The set Ω_0 contains exactly one element, the group identity; hence, for instance, the category of associative rings with unit is not a category of groups with operations.

Example 3.3 The categories of groups, rings not necessarily with identity, R-modules, associative, associative commutative, Lie, Leibniz, and alternative algebras are examples of categories of groups with operations.

The category of topological groups with operations is defined in [1] as follows:

Definition 3.4 A category TC of topological groups with a set Ω of continuous operations and with a set E of identities including the group laws such that conditions (a)–(d) in Definition 3.1 are satisfied is called a *category of topological groups with operations* and the objects of TC are called *topological groups with operations.*

In the rest of the paper, TC will denote a category of topological groups with operations.

A *morphism* between any 2 objects of TC is a continuous group homomorphism, which preserves the operations in Ω_1' and Ω_2'.

The categories of topological groups, topological rings, topological R-modules, and alternative topological algebras are examples of categories of topological groups with operations.

Proposition 3.5 *If X is a topological group with operations, then the fundamental group $\pi_1(X, 0)$ becomes a group with operations.*

Proof Let X be an object of TC and $P(X, 0)$ the set of all paths in X with initial point 0 as described in Section 2. There are binary operations on $P(X, 0)$ defined by

$$(\alpha \star \beta)(t) = \alpha(t) \star \beta(t) \tag{1}$$

for $\star \in \Omega_2$ and $t \in I$, unit interval, and unary operations defined by

$$(\omega\alpha)(t) = \omega(\alpha(t)) \tag{2}$$

for $\omega \in \Omega_1$. Hence, the operations (1) induce binary operations on $\pi_1(X, 0)$ defined by

$$[\alpha] \star [\beta] = [\alpha \star \beta] \tag{3}$$

for $[\alpha], [\beta] \in \pi_1(X, 0)$. Since the binary operations $\star$ in Ω_2 are continuous, it follows that the binary operations (3) are well defined. Similarly, the operations (2) reduce the unary operations defined by

$$\omega[\alpha] = [\omega\alpha]. \tag{4}$$

By the continuity of the unary operations $\omega \in \Omega_1$, the operations (4) are also well defined. The other details can be checked and so $\pi_1(X, 0)$ becomes a group with operations, i.e. an object of C. □

We now generalize Theorem 2.1 to topological groups with operations. We first make the following preparation:

Let X be a topological group with operations. By the evaluation of the compositions and operations of the paths in X such that $\alpha_1(1) = \beta_1(0)$ and $\alpha_2(1) = \beta_2(0)$, we have the following *interchange law*:

$$(\alpha_1 \circ \beta_1) \star (\alpha_2 \circ \beta_2) = (\alpha_1 \star \alpha_2) \circ (\beta_1 \star \beta_2) \tag{5}$$

for $\star \in \Omega_2$, where $\circ$ denotes the composition of paths, and

$$(\alpha \star \beta)^{-1} = \alpha^{-1} \star \beta^{-1} \tag{6}$$

for $\alpha, \beta \in P(X, 0)$ where, say, α^{-1} is the inverse path defined by $\alpha^{-1}(t) = \alpha(1 - t)$ for $t \in I$. Further, we have that

$$(\omega\alpha)^{-1} = \omega\alpha^{-1}, \tag{7}$$

$$\omega(\alpha \circ \beta) = (\omega\alpha) \circ (\omega\beta) \tag{8}$$

when $\alpha(1) = \beta(0)$.

Parallel to Theorem 2.1, in the following theorem we prove a general result for topological groups with operations.

Theorem 3.6 *Let X be a topological group with operations, i.e. an object of TC, and let G be a subobject of $\pi_1(X, 0)$. Suppose that the underlying space of X is connected, locally path-connected, and semilocally simply connected. Let $p\colon (\widetilde{X}_G, \tilde{0}) \to (X, 0)$ be the covering map corresponding to G as a subgroup of the additive group of $\pi_1(X, 0)$ by Theorem 2.1. Then the operations of X lift to $\widetilde{X}_G$, i.e. $\widetilde{X}_G$ is a topological group with operations and $p\colon \widetilde{X}_G \to X$ is a morphism of TC.*

Proof By the construction of $\widetilde{X}_G$ in Section 2, $\widetilde{X}_G$ is the set of equivalence classes defined via G. The binary operations on $P(X, 0)$ defined by (1) induce binary operations

$$\langle\alpha\rangle_G \star \langle\beta\rangle_G = \langle\alpha \star \beta\rangle_G \tag{9}$$

and the unary operations on $P(X,0)$ defined by (2) induce unary operations

$$\omega\langle\alpha\rangle_G = \langle\omega\alpha\rangle_G \tag{10}$$

on $\widetilde{X}_G$.

We now prove that operations (9) and (10) are well defined. For $\star \in \Omega_2$ and the paths $\alpha, \beta, \alpha_1, \beta_1 \in P(X,0)$ with $\alpha(1) = \alpha_1(1)$ and $\beta(1) = \beta_1(1)$, we have that

$$[(\alpha \star \beta) \circ (\alpha_1 \star \beta_1)^{-1}] = [(\alpha \star \beta) \circ (\alpha_1^{-1} \star \beta_1^{-1})] \tag{by 6}$$

$$= [(\alpha \circ \alpha_1^{-1}) \star (\beta \circ \beta_1^{-1})] \tag{by 5}$$

$$= [\alpha \circ \alpha_1^{-1}] \star [\beta \circ \beta_1^{-1}]. \tag{by 3}$$

Thus, if $\alpha_1 \in \langle\alpha\rangle_G$ and $\beta_1 \in \langle\beta\rangle_G$, then $[\alpha \circ \alpha_1^{-1}] \in G$ and $[\beta \circ \beta_1^{-1}] \in G$. Since G is a subobject of $\pi_1(X,0)$, we have that $[\alpha \circ \alpha_1^{-1}] \star [\beta \circ \beta_1^{-1}] \in G$. Therefore, the binary operations (9) are well defined.

Similarly, for the paths $\alpha, \alpha_1 \in P(X,0)$ with $\alpha(1) = \alpha_1(1)$ and $\omega \in \Omega_1$, we have that

$$[(\omega\alpha) \circ (\omega\alpha_1)^{-1})] = [(\omega\alpha) \circ (\omega\alpha_1^{-1})] \tag{by 7}$$

$$= [(\omega(\alpha \circ \alpha_1^{-1})] \tag{by 8}$$

$$= \omega[\alpha \circ \alpha_1^{-1}]. \tag{by 4}$$

Since G is a subobject of $\pi_1(X,0)$, if $[\alpha \circ \alpha_1^{-1}] \in G$ and $\omega \in \Omega_1$, then $\omega[\alpha \circ \alpha_1^{-1}] \in G$. Hence, the unary operations (10) are also well defined.

The axioms (a)–(d) of Definition 3.1 for $\widetilde{X}_G$ are satisfied and therefore $\widetilde{X}_G$ becomes a group with operations. Further, by Theorem 2.1, $p: (\widetilde{X}_G, \tilde{0}) \to (X,0)$ is a covering map, $\widetilde{X}_G$ is a topological group, and p is a morphism of topological groups. In addition to this, we need to prove that $\widetilde{X}_G$ is an object of TC and p is a morphism of TC. To prove that the operations (9) for $\star \in \Omega_2'$ are continuous, let $\langle\alpha\rangle_G, \langle\beta\rangle_G \in \widetilde{X}_G$ and $(W, \langle\alpha\star\beta\rangle_G)$ be a basic open neighborhood of $\langle\alpha\star\beta\rangle_G$. Here W is an open neighborhood of $(\alpha\star\beta)(1) = \alpha(1)\star\beta(1)$. Since the operations $\star: X \times X \to X$ are continuous, there are open neighborhoods U and V of $\alpha(1)$ and $\beta(1)$ respectively in X such that $U \star V \subseteq W$. Therefore, $(U, \langle\alpha\rangle_G)$ and $(V, \langle\beta\rangle_G)$ are respectively base open neighborhoods of $\langle\alpha\rangle_G$ and $\langle\beta\rangle_G$, and

$$(U, \langle\alpha\rangle_G) \star (V, \langle\beta\rangle_G) \subseteq (W, \langle\alpha \star \beta\rangle_G).$$

Therefore, the binary operations (9) are continuous

We now prove that the unary operations (10) for $\omega \in \Omega_1'$ are continuous. If $(V, \langle\omega\alpha\rangle)$ is a base open neighborhood of $\langle\omega\alpha\rangle$, then V is an open neighborhood of $\omega\alpha(1)$, and since the unary operations $\omega: X \to X$ are continuous, there is an open neighborhood U of $\alpha(1)$ such that $\omega(U) \subseteq V$. Therefore, $(U, \langle\alpha\rangle)$ is an open neighborhood of $\langle\alpha\rangle$ and $\omega(U, \langle\alpha\rangle) \subseteq (V, \langle\omega\alpha\rangle)$.

Moreover, the map $p: \widetilde{X}_G \to X$ defined by $p(\langle\alpha\rangle_G) = \alpha(1)$ preserves the operations of Ω_2 and Ω_1. □

From Theorem 3.6, the following result can be restated.

Theorem 3.7 *Suppose that X is a topological group with operations whose underlying space is connected, locally path-connected, and semilocally simply connected. Let $p\colon (\widetilde{X}, \tilde{0}) \to (X, 0)$ be a covering map such that $\widetilde{X}$ is path connected and the characteristic group G of p is a subobject of $\pi_1(X, 0)$. Then the operations on X lift to $\widetilde{X}$.*

Proof By assumption, the characteristic group G of the covering map $p\colon (\widetilde{X}, \tilde{0}) \to (X, 0)$ is a subobject of $\pi_1(X, 0)$. Thus, by Remark 2.2, we can assume that $\widetilde{X} = \widetilde{X}_G$, and hence by Theorem 3.6, the group operations of X lift to $\widetilde{X}$ as required. □

In particular, in Theorem 3.6 if the subobject G of $\pi_1(X, 0)$ is chosen to be the singleton, then the following corollary is obtained.

Corollary 3.8 *Let X be a topological group with operations such that the underlying space of X is connected, locally path-connected, and semilocally simply connected. Let $p\colon (\widetilde{X}, \tilde{0}) \to (X, 0)$ be a universal covering map. Then the operations of X lift to $\widetilde{X}$.*

The following proposition is useful for Theorem 3.11.

Proposition 3.9 *Let X be a topological group with operations and V a liftable neighborhood of 0 in X. Then there is a liftable neighborhood U of 0 in X such that $U \star U \subseteq V$ for $\star \in \Omega_2$.*

Proof Since X is a topological group with operations and hence the binary operations $\star \in \Omega_2$ are continuous, there is an open neighborhood U of 0 in X such that $U \star U \subseteq V$. Further, if V is liftable, then U can be chosen as liftable, for if V is liftable, then for each $x \in U$, the fundamental group $\pi_1(U, x)$ is mapped to the singleton by the morphism induced by the inclusion map $\imath\colon U \to X$. Here U is not necessarily path-connected and hence not necessarily liftable. However, since the path component $C_0(U)$ of 0 in U is liftable and satisfies these conditions, U can be replaced by the the path component $C_0(U)$ of 0 in U and it is assumed that U is liftable. □

Definition 3.10 Let X and Y be topological groups with operations and U an open neighborhood of 0 in X. A continuous map $\phi\colon U \to S$ is called a *local morphism* in TC if $\phi(a \star b) = \phi(a) \star \phi(b)$ when $a, b \in U$ such that $a \star b \in U$ for $\star \in \Omega_2$.

Theorem 3.11 *Let X and $\widetilde{X}$ be topological groups with operations and $q\colon \widetilde{X} \to X$ a morphism of TC, which is a covering map. Let U be an open, path-connected neighborhood of 0 in X such that for each $\star \in \Omega_2$, the set $U \star U$ is contained in a liftable neighbourhood V of 0 in X. Then the inclusion map $i\colon U \to X$ lifts to a local morphism $\hat{\imath}\colon U \to \widetilde{X}$ in TC.*

Proof Since V lifts to $\widetilde{X}$, then U lifts to $\widetilde{X}$ by $\hat{\imath}\colon U \to \widetilde{X}$. We now prove that $\hat{\imath}$ is a local morphism of topological groups with operations. We know by the lifting theorem that $\hat{\imath}\colon U \to \widetilde{X}$ is continuous. Let $a, b \in U$ be such that for each $\star \in \Omega_2$, $a \star b \in U$. Let α and β be the paths from 0 to a and b respectively in U. Let $\gamma = \alpha \star \beta$. Thus, γ is a path from 0 to $a \star b$. Since $U \star U \subseteq V$, the paths γ is in V. Thus, the paths α, β, and γ lift to $\widetilde{X}$. Suppose that $\tilde{\alpha}$, $\tilde{\beta}$, and $\tilde{\gamma}$ are the liftings of α, β, and γ in $\widetilde{X}$, respectively. Then we have

$$q(\widetilde{\gamma}) = \gamma = \alpha \star \beta = q(\widetilde{\alpha}) \star q(\widetilde{\beta}).$$

However, q is a morphism of a topological group with operations and so we have

$$q(\widetilde{\alpha} \star \widetilde{\beta}) = q(\widetilde{\alpha}) \star q(\widetilde{\beta})$$

for $\star \in \Omega_2$. Since the paths $\widetilde{\gamma}$ and $\widetilde{\alpha} \star \widetilde{\beta}$ have the initial point $\widetilde{0} \in \widetilde{X}$, by the unique path lifting

$$\widetilde{\gamma} = \widetilde{\alpha} \star \widetilde{\beta}.$$

On evaluating these paths at $1 \in I$, we have

$$\hat{\imath}(a \star b) = \hat{\imath}(a) \star \hat{\imath}(b).$$

$\square$

4. Covers of crossed modules within topological groups with operations

If A and B are objects of C, an *extension* of A by B is an exact sequence

$$0 \longrightarrow A \overset{\imath}{\longrightarrow} E \overset{p}{\longrightarrow} B \longrightarrow 0 \tag{11}$$

in which p is surjective and $\imath$ is the kernel of p. It is *split* if there is a morphism $s \colon B \to E$ such that $ps = \imath d_B$. A split extension of A by B is called a B-*structure* on A. Given such a B-structure on A we get actions of B on A corresponding to the operations in C. For any $b \in B$, $a \in A$, and $\star \in \Omega_2'$ we have actions called *derived actions* by Orzech [21, p. 293]:

$$\begin{aligned} b \cdot a &= s(b) + a - s(b) \\ b \star a &= s(b) \star a. \end{aligned} \tag{12}$$

In addition to this, we note that topologically if an exact sequence (11) in TC is a split extension, then the derived actions (12) are continuous. Thus, we can state Theorem [21, Theorem 2.4] in a topological case, which is useful for the proof of Theorem 4.7, as follows.

Theorem 4.1 *A set of actions (one for each operation in Ω_2) is a set of continuous derived actions if and only if the semidirect product $B \ltimes A$ with underlying set $B \times A$ and operations*

$$\begin{aligned} (b, a) + (b', a') &= (b + b', a + (b \cdot a')) \\ (b, a) \star (b', a') &= (b \star b', a \star a' + b \star a' + a \star b') \end{aligned}$$

is an object in TC.

The internal category in C is defined in [23] as follows. We follow the notations of Section 2 for groupoids.

Definition 4.2 An internal category C in C is a category in which the initial and final point maps $d_0, d_1 \colon C \to C_0$, the object inclusion map $\epsilon \colon C_0 \to C$, and the partial composition $\circ \colon C_{d_1} \times_{d_0} C \to C, (a, b) \mapsto a \circ b$ are morphisms in the category C.

Note that since ϵ is a morphism in C, $\epsilon(0) = 0$, and that the operation $\circ$ being a morphism implies that for all $a, b, c, d \in C$ and $\star \in \Omega_2$,

$$(a \star b) \circ (c \star d) = (a \circ c) \star (b \circ d) \tag{13}$$

whenever one side makes sense. This is called the *interchange law* [23].

We also note from [23] that any internal category in C is an internal groupoid since, given $a \in C$, $a^{-1} = \epsilon d_1(a) - a + \epsilon d_0(a)$ satisfies $a^{-1} \circ a = \epsilon d_1(a)$ and $a \circ a^{-1} = \epsilon d_0(a)$. Thus, we use the term *internal groupoid* rather than internal category and write G for an internal groupoid. For the category of internal groupoids in C we use the same notation, $\mathsf{Cat}(\mathsf{C})$, as in [23]. Here a *morphism* $f \colon H \to G$ in $\mathsf{Cat}(\mathsf{C})$ is a morphism of underlying groupoids and a morphism in C.

In particular, if C is the category of groups, then an internal groupoid G in C becomes a group-groupoid and, in the case where C is the category of rings, an internal groupoid in C is a ring object in the category of groupoids [18].

Definition 4.3 An internal groupoid in the category TC of topological groups with operations is called a *topological internal groupoid*.

Thus, a topological internal groupoid is a topological groupoid G in which the set of morphisms and the set G_0 of objects are objects of TC and all structural maps of G, i.e the source and target maps $d_0, d_1 \colon G \to G_0$, the object inclusion map $\epsilon \colon G_0 \to G$, and the composition map $\circ \colon G_{d_1} \times_{d_0} G \to G$, are morphisms of TC.

If TC is the category of topological groups, then a topological internal groupoid becomes a topological group-groupoid.

For the category of topological internal groupoids in TC we use the notation $\mathsf{Cat}(\mathsf{TC})$. Here a *morphism* $f \colon H \to G$ in $\mathsf{Cat}(\mathsf{TC})$ is morphism of underlying groupoids and a morphism in TC.

Theorem 4.4 *Let X be an object of TC such that the underlying space is locally path-connected and semilocally simply connected. Then the fundamental groupoid πX is a topological internal groupoid.*

Proof Let X be a topological group with operations as assumed. By [5, Theorem 1], πX has a topology such that it is a topological groupoid. We know by [5, Proposition 3] that when X and Y are endowed with such topologies, for a continuous map $f \colon X \to Y$, the induced morphism $\pi(f) \colon \pi X \to \pi Y$ is also continuous. Hence, the continuous binary operations $\star \colon X \times X \to X$ for $\star \in \Omega_2$ and the unary operations $\omega \colon X \to X$ for $\omega \in \Omega_1$ respectively induce continuous binary operations $\widetilde{\star} \colon \pi X \times \pi X \to \pi X$ and unary operations $\widetilde{\omega} \colon \pi X \to \pi X$. The set of morphisms thus becomes a topological group with operations. The groupoid structural maps are morphisms of topological groups with operations, i.e. preserve the operations. Therefore, πX becomes a topological internal groupoid. $\square$

Proposition 4.5 *Let X and Y be topological groups with operations such that the underlying spaces are locally path-connected and semilocally simply connected. Then $\pi(X \times Y)$ and $\pi X \times \pi Y$ are isomorphic as topological internal groupoids.*

Proof By Theorem 4.4, $\pi(X \times Y)$ and $\pi X \times \pi Y$ are topological internal groupoids, and by [5, Proposition 5], they are isomorphic as topological groupoids. The other details follow. $\square$

Similar to the crossed module in C formulated in [23, Proposition 2], we define a crossed module in TC as follows:

Definition 4.6 A *crossed module* in TC is a morphism $\alpha\colon A \to B$ in TC, where B acts topologically on A (i.e. we have continuous derived actions in TC) with the following conditions for any $b \in B$, $a, a' \in A$, and $\star \in \Omega_2'$:

CM1 $\alpha(b \cdot a) = b + \alpha(a) - b$;

CM2 $\alpha(a) \cdot a' = a + a' - a$;

CM3 $\alpha(a) \star a' = a \star a'$;

CM4 $\alpha(b \star a) = b \star \alpha(a)$ and $\alpha(a \star b) = \alpha(a) \star b$.

A *morphism* from $\alpha\colon A \to B$ to $\alpha'\colon A' \to B'$ is a pair $f_1\colon A \to A'$ and $f_2\colon B \to B'$ of the morphisms in TC such that

1. $f_2\alpha(a) = \alpha' f_1(a)$,

2. $f_1(b \cdot a) = f_2(b) \cdot f_1(a)$,

3. $f_1(b \star a) = f_2(b) \star f_1(a)$,

for any $x \in B$, $a \in A$ and $\star \in \Omega_2'$. Thus, we have a category $\mathsf{XMod(TC)}$ of crossed modules in TC.

The algebraic case of the following theorem was proven in C in [23, Theorem 1]. We now prove the topological version as follows.

Theorem 4.7 *The category $\mathsf{XMod(TC)}$ of crossed modules in TC and the category $\mathsf{Cat(TC)}$ of internal groupoids in TC are equivalent.*

Proof We give a sketch of a proof based on that of the algebraic case. A functor $\delta\colon \mathsf{Cat(TC)} \to \mathsf{XMod(TC)}$ is defined as follows: for a topological internal groupoid G, let $\delta(G)$ be the topological crossed module (A, B, d_1) in TC, where $A = \mathsf{Ker}d_0$, $B = G_0$, and $d_1\colon A \to B$ is the restriction of the target point map. Here A and B inherit the structures of a topological group with operations from that of G, and the target point map $d_1\colon A \to B$ is a morphism in TC. Further, the actions $B \times A \to A$ on the topological group with operations A given by

$$\begin{aligned} b \cdot a &= \epsilon(b) + a - \epsilon(b) \\ b \star a &= \epsilon(b) \star a \end{aligned}$$

for $a \in A$, $b \in B$ are continuous by the continuities of ϵ and the operations in Ω_2, and the axioms of Definition 4.6 are satisfied. Thus, (A, B, d_1) becomes a crossed module in TC.

Conversely, define a functor $\eta\colon \mathsf{XMod(TC)} \to \mathsf{Cat(TC)}$ in the following way. For a crossed module (A, B, α) in TC, define a topological internal groupoid $\eta(A, B, \alpha)$ whose set of objects is the topological group with operations B and set of morphisms is the semidirect product $B \ltimes A$, which is a topological

group with operations by Theorem 4.1. The source and target point maps are defined to be $d_0(b, a) = b$ and $d_1(b, a) = \alpha(a) + b$ while the object inclusion map and groupoid composition is given by $\epsilon(b) = (b, 0)$ and

$$(b, a) \circ (b_1, a_1) = (b, a_1 + a)$$

whenever $b_1 = \alpha(a) + b$. These structural maps are all continuous and therefore $\eta(A, B, \alpha)$ is a topological internal groupoid.

The other details of the proof are obtained from that of [23, Theorem 1].

$\square$

A morphism of groupoids $p \colon H \to G$ is called a *covering morphism* and H a *covering groupoid* of G if for each $\tilde{x} \in H_0$ the restriction $\mathsf{St}_H \tilde{x} \to \mathsf{St}_G p(\tilde{x})$ of p is a bijection. Let $G_{d_0} \times_p H_0$ be the pullback of $d_0 \colon G \to G_0$ and $p \colon H_0 \to G_0$. If $p \colon H \to G$ is a covering morphism, there is a *lifting function* $s_p \colon G_{d_0} \times_p H_0 \to H$ assigning to the pair $(a, \tilde{x})$ the unique element $b \in \mathsf{St}_H \tilde{x}$ such that $p(b) = a$ and s_p is inverse to $(p, d_0) \colon H \to G_{d_0} \times_p H_0$. Thus, it can be stated that p is a covering morphism if and only if $(p, d_0) \colon H \to G_{d_0} \times_p H_0$ is a bijection. In terms of this function (p, d_0), the notion of of topological covering morphism is defined in [6, p.144] as follows:

A morphism of topological groupoids $p \colon H \to G$ is called a *topological covering morphism* if the function $(p, d_0) \colon H \to G_{d_0} \times_p H_0$ is a homeomorphism. Therefore, we can generalize this concept to the more general topological internal groupoids and call a morphism $p \colon H \to G$ of $\mathsf{Cat}(\mathsf{TC})$ a *covering morphism* if it is a topological groupoid covering on the underlying topological groupoids.

Proposition 4.8 If $f \colon \widetilde{X} \to X$ is a covering map in TC such that the underlying spaces are locally path-connected and semilocally simply connected, then $\pi f \colon \pi \widetilde{X} \to \pi X$ is a covering morphism in $\mathsf{Cat}(\mathsf{TC})$.

Proof By Theorem 4.4, πX and πY are topological internal groupoids, and by [5, Theorem 4], $\pi f \colon \pi X \to \pi Y$ is a topological covering morphism of topological groupoids. The other details follow. $\square$

Thus, by the equivalence of the categories proven in Theorem 4.7, evaluating this notion in the crossed modules in TC we obtain the cover of a crossed module in TC as follows.

If $f \colon H \to G$ is a covering morphism in $\mathsf{Cat}(\mathsf{TC})$ and (f_1, f_2) is the morphism of crossed modules corresponding to f, then $f_1 \colon \widetilde{A} \to A$ is an isomorphism in TC, where $\widetilde{A} = \mathsf{St}_H \tilde{0}$, $A = \mathsf{St}_G 0$, and f_1 is the restriction of f. Therefore, we call a morphism (f_1, f_2) of crossed modules in TC a *cover* if $f_1 \colon \widetilde{A} \to A$ is an isomorphism in TC.

Let G be a topological internal groupoid, i.e. an object of $\mathsf{Cat}(\mathsf{TC})$. Let $\mathsf{Cov}_{\mathsf{Cat}(\mathsf{TC})}/G$ be the category of covers of G in the category $\mathsf{Cat}(\mathsf{TC})$. Thus, the objects of $\mathsf{Cov}_{\mathsf{Cat}(\mathsf{TC})}/G$ are the covering morphisms $p \colon H \to G$ over G in $\mathsf{Cat}(\mathsf{TC})$ and a morphism from $p \colon H \to G$ to $q \colon K \to G$ is a morphism $f \colon H \to K$ in $\mathsf{Cat}(\mathsf{TC})$ such that $qf = p$.

The algebraic case of the following theorem was proven in [1, Theorem 5.3]. We have the topological version of this theorem for TC as follows.

Theorem 4.9 *Let G be an object of $\mathsf{Cat}(\mathsf{TC})$ and $\alpha \colon A \to B$ the crossed module in TC corresponding to G by Theorem 4.7. Let $\mathsf{Cov}_{\mathsf{XMod}(\mathsf{TC})}/(\alpha \colon A \to B)$ be the category of covers of $\alpha \colon A \to B$ in TC. Then the categories $\mathsf{Cov}_{\mathsf{Cat}(\mathsf{TC})}/G$ and $\mathsf{Cov}_{\mathsf{XMod}(\mathsf{TC})}/(\alpha \colon A \to B)$ are equivalent.*

Proof If $f \colon H \to G$ is a covering morphism in $\mathsf{Cat}(\mathsf{TC})$ and

$$(f_1, f_2) \colon (\widetilde{A}, \widetilde{B}, \tilde{\alpha}) \to (A, B, \alpha)$$

is the morphism of crossed modules in TC corresponding to f by Theorem 4.7, then $f_1\colon \widetilde{A} \to A$ becomes an isomorphism in the category TC, where $\widetilde{A} = \mathsf{St}_H\tilde{0}$, $A = \mathsf{St}_G 0$, and f_1 is the restriction of f. Therefore, (f_1, f_2) is a covering morphism in $\mathsf{XMod}(\mathsf{TC})$ and so in this way we have a functor

$$\delta\colon \mathsf{Cov}_{\mathsf{Cat}(\mathsf{TC})}/G \to \mathsf{Cov}_{\mathsf{XMod}(\mathsf{TC})}/(\alpha\colon A \to B).$$

Conversely, let $(f_1, f_2)\colon (\widetilde{A}, \widetilde{B}, \tilde{\alpha}) \to (A, B, \alpha)$ be a covering morphism of crossed modules in $\mathsf{XMod}(\mathsf{TC})$ and $f\colon H \to G$ the morphism of $\mathsf{Cat}(\mathsf{TC})$ corresponding to (f_1, f_2), where $H = \widetilde{B} \ltimes \widetilde{A}$ and $G = B \ltimes A$. Since $f_1\colon \widetilde{A} \to A$ is an isomorphism in TC as stated above $(p, d_0)\colon H \to G_{d_0} \times_p H_0$ is a bijection, which is inverse to the lifting function $s_p\colon G_{d_0} \times_p H_0 \to H$. Since f_1 and f_2 are continuous, the maps s_p and (p, d_0) are continuous, i.e. f is a covering morphism in $\mathsf{Cat}(\mathsf{TC})$. Therefore, we have another functor

$$\eta\colon \mathsf{Cov}_{\mathsf{XMod}(\mathsf{TC})}/(\alpha\colon A \to B) \to \mathsf{Cov}_{\mathsf{Cat}(\mathsf{TC})}/G.$$

The other details of the equivalence of categories follow the proof of Theorem 4.7

$\square$

As a result of Theorem 4.9, the following corollary can be stated.

Corollary 4.10 *Let X be an object of TC such that the underlying space is locally path-connected and semilocally simply connected. Then the category $\mathsf{Cov}_{\mathsf{Cat}(\mathsf{TC})}/\pi X$ of coverings of πX in the category $\mathsf{Cat}(\mathsf{TC})$ and the category*

$$\mathsf{Cov}_{\mathsf{XMod}}/(d_1\colon \mathsf{St}_{\pi X}0 \to X)$$

of coverings of the crossed module $d_1\colon \mathsf{St}_{\pi X}0 \to X$ in $\mathsf{XMod}(\mathsf{TC})$ are equivalent.

Proof If X is an object of TC such that the underlying space is locally path-connected and semilocally simply connected, then by Theorem 4.4 πX becomes an object of $\mathsf{Cat}(\mathsf{TC})$. Therefore, the proof follows by Theorem 4.9. $\square$

Acknowledgment

References

[1] Akız HF, Alemdar N, Mucuk O, Şahan T. Coverings of internal groupoids and crossed modules in the category of groups with operations. Georgian Math J 2013; 20: 223–238.

[2] Alemdar N, Mucuk O. Existence of covering topological R-modules. Filomat 2013; 27: 1121–1126.

[3] Alemdar N, Mucuk O. The liftings of R-modules to covering groupoids. Hacet J Math Stat 2012; 41: 813–822.

[4] Brown R. Topology and Groupoids. 3rd ed. Charleston, SC, USA: BookSurge, 2006.

[5] Brown R, Danesh-Naruie G. The fundamental groupoid as a topological groupoid. Proc Edinburgh Math Soc 1975; 19: 237–244.

[6] Brown R, Danesh-Naruie G, Hardy JPL. Topological groupoids II: Covering morphisms and G-spaces. Math Nachr 1976; 74: 143–156.

[7] Brown R, Mucuk O. Covering groups of non-connected topological groups revisited. Math Proc Camb Phill Soc 1994; 115: 97–110.

[8] Brown R, Spencer CB. G-groupoids, crossed modules and the fundamental groupoid of a topological group. Proc Konn Ned Akad v Wet 1976; 79: 296–302.

[9] Chevalley C. Theory of Lie Groups. Princeton, NJ, USA: Princeton University Press, 1946.

[10] Datuashvili T. Cohomology of internal categories in categories of groups with operations. In: Adamek J, MacLane S, editors. Categorical Topology and Its Relation to Analysis, Algebra and Combinatorics. Teaneck, NJ, USA: World Scientific Publishing, 1989, pp. 270–283.

[11] Datuashvili T. Whitehead homotopy equivalence and internal category equivalence of crossed modules in categories of groups with operations. Proc A Razmadze Math Inst 1995; 113: 3–30.

[12] Datuashvili T. Cohomologically trivial internal categories in categories of groups with operations. Appl Categor Struct 1995; 3: 221–237.

[13] Datuashvili T. Kan extensions of internal functors. Nonconnected case. J Pure Appl Algebra 2002; 167: 195–202.

[14] Datuashvili T. Categorical, homological and homotopical properties of algebraic objects. PhD, Georgian Academy of Science, Tbilisi, 2006.

[15] Higgins PJ. Groups with multiple operators. Proc London Math Soc 1956; 6: 366–416.

[16] Mackenzie KCH. Lie groupoids and Lie algebroids in differential geometry. London Mathematical Society Lecture Note Series 124. Cambridge, UK: Cambridge University Press, 1987.

[17] Mucuk O. Covering groups of non-connected topological groups and the monodromy groupoid of a topological groupoid. PhD, University of Wales, Cardiff, UK, 1993.

[18] Mucuk O. Coverings and ring-groupoids. Georgian Math J 1998; 5: 475482.

[19] Mucuk O, Kılıçarslan B, Şahan T, Alemdar N. Group-groupoid and monodromy groupoid. Topol Appl 2011; 158: 2034–2042.

[20] Mucuk O, Şahan T, Alemdar N. Normality and quotients in crossed modules and group-groupoids. Appl Categor Struct DOI 10.1007/s10485-013-9335-6.

[21] Orzech G. Obstruction theory in algebraic categories, I. J Pure Appl Algebra 1972; 2: 287–314.

[22] Orzech G. Obstruction theory in algebraic categories, II. J Pure Appl Algebra 1972; 2: 315–340.

[23] Porter T. Extensions, crossed modules and internal categories in categories of groups with operations. Proc Edinb Math Soc 1987; 30: 373–381.

[24] Rotman JJ. An Introduction to Algebraic Topology. Graduate Texts in Mathematics 119. New York, NY, USA: Springer, 1988.

[25] Taylor RL. Covering groups of non-connected topological groups. Proc Amer Math Soc 1954; 5: 753–768.