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Generalized higher commutators generated by the multilinear fractional integrals and Lipschitz functions

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Abstract: Let $l \in \mathbb{N}$ and $\vec{A} = (A_1, \dots, A_l)$ and $\vec{f} = (f_1, \dots, f_l)$ be 2 finite collections of functions, where every function A_i has derivatives of order m_i and $f_1, \dots, f_l \in L_c^\infty(\mathbb{R}^n)$. Let $x \notin \cap_{i=1}^l \mathrm{Supp} f_i$. The generalized higher commutator generated by the multilinear fractional integral is then given by

$$I_{\alpha,m}^{\vec{A}}(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} \frac{\prod_{i=1}^{l} R_{m_i+1}(A_i; x, y_i) f_i(y_i)}{|(x-y_1, \dots, x-y_m)|^{ln+(m_1+m_2+\cdots+m_l)-\alpha}} dy_1 \dots dy_l.$$

When $D^\gamma A_i \in \dot{\Lambda}_{\beta_i}(0 < \beta_i < 1, |\gamma| = m_i)$, $i = 1, \cdots, m$, the authors establish the boundedness of $I_{\alpha,m}^{\vec{A}}$ on the product Lebesgue space, Triebel–Lizorkin space, and Lipschitz space.

Key words: Multilinear fractional integral, commutator, Triebel–Lizorkin space, Lipschitz function space

1. Introduction

In the 1970s, Coifman and Meyer [2] were the first to introduce the definition of the multilinear integral. The study of the multilinear singular integral is motivated not only by a quest to generalize the theory of linear operators but also by their natural appearance in analysis. In recent years, the research of the multilinear integral has received much attention and great developments have been achieved. Authors such as Grafakos and Kalton and Grafakos and Torres [5, 6, 7, 4] gave the systematic treatment of the multilinear Calderón–Zygmund operator. The multilinear fractional integral operators were also investigated by Kenig and Stein [9]. Recently many people have been studying these operators from various points of view [8, 13, 15, 18].

On the other hand, the commutators generated by the multilinear singular integrals and bounded mean oscillation functions or Lipschitz functions also attract much attention, since the commutator is more singular than the singular integral operator itself.

Lian and Wu [10] and Xu [17] established the boundedness of commutators associated with the multilinear Calderón–Zygmund singular integral or multilinear fractional integral in product Lebesgue spaces. In [11] and [12], the boundedness of commutators generated by Lipschitz functions and multilinear fractional integrals or multilinear Calderón–Zygmund type singular integrals on product Lebesgue spaces, Triebel–Lizorkin spaces, and Lipschitz spaces were obtained.

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Motivated by the works of Mo and Lu and Mo and Zhang [12, 11], we study the boundedness of generalized higher commutators generated by multilinear fractional integrals and Lipschitz functions. These generalized commutators can be regarded as some extensions of classical commutators and have important applications in partial differential equations; see [16] for an example.

Now we give the definition of the generalized higher commutators generated by multilinear fractional integrals.

Let $\mathbb{R}^n$ be the n-dimension Euclidean space. Let $l \in \mathbb{N}\backslash\{0\}$, $x_i \in \mathbb{R}^n$, $i = 1, 2, \ldots, l$. Then $|(x_1, x_2, \ldots, x_l)|$ denotes the norm of $(x_1, x_2, \ldots, x_l)$ in $(\mathbb{R}^n)^l$.

Denote $\vec{f}$ by the l-tube $(f_1, \cdots, f_l)$ and $I_{\alpha,l}$ by the lth fractional integral operator, defined as follows:

$$I_{\alpha,l}(\vec{f})(x) = \int_{(\mathbb{R}^n)^l} \frac{f_1(y_1)\ldots f_l(y_l)}{|(x-y_1,\ldots,x-y_l)|^{ln-\alpha}} dy_1 \ldots dy_l, \;\; 0 < \alpha < ln,$$

whenever $f_i, i = 1, \ldots, l$ are smooth functions with compact support and $x \notin \cap_{i=1}^{l} \mathrm{supp} f_i$. When $l = 1$, we denote it by $I_\alpha(f)$.

It is easy to see that for $|x-z| \leq 1/2 \max_{1\leq k\leq l} |x-y_k|$,

$$\left| \frac{1}{|(x-y_1,\ldots,x-y_l)|^{ln-\alpha}} - \frac{1}{|(z-y_1,\ldots,z-y_l)|^{ln-\alpha}} \right| \leq \frac{C|x-z|}{|(x-y_1,\ldots,x-y_l)|^{ln-\alpha+1}}.$$

Let us now assume that $m \in \mathbb{N}$ and A is a function with derivatives of order m on $\mathbb{R}^n$. We denote

$$R_{m+1}(A;x,y) = A(x) - \sum_{|\gamma|\leq m} \frac{1}{\gamma!} D^\gamma A(y)(x-y)^\gamma.$$

Let $\vec{A} = (A_1, \ldots, A_l)$ be a finite collection of functions, where every A_i has derivatives of order m_i and $f_1, \ldots, f_l \in L_c^\infty(\mathbb{R}^n)$. Let $x \notin \cap_{i=1}^{l} \mathrm{Supp} f_i$, and then the lth generalized higher commutator generated by the multilinear fractional integral is defined by

$$I_{\alpha,l}^{\vec{A}}(\vec{f})(x) = \int_{(\mathbb{R}^n)^l} \frac{\prod_{i=1}^{l} R_{m_i+1}(A_i;x,y_i) f_i(y_i)}{|(x-y_1,\ldots,x-y_l)|^{ln+(m_1+m_2+\cdots+m_l)-\alpha}} dy_1 \ldots dy_l.$$

Note that when every $m_i = 0$, then

$$I_{\alpha,l}^{\vec{A}}(\vec{f})(x) = \int_{(\mathbb{R}^n)^l} \frac{\prod_{i=1}^{l} (A_j(x) - A_j(y_i)) f_i(y_i)}{|(x-y_1\ldots,x-y_l)|^{ln-\alpha}} dy_1 \ldots dy_l,$$

is the commutator generated by the multilinear fractional integral. For $\beta > 0$, the homogeneous Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}^n)$ is the space of function f such that

$$\|f\|_{\dot{\Lambda}_\beta} = \sup_{x,h\in\mathbb{R}^n, h\neq 0} \frac{|\Delta_h^{[\beta]+1} f(x)|}{|h|^\beta} < \infty,$$

where Δ_h^k denotes the kth difference operator (see [14]).

We now turn to the precise statements of our results.

Theorem 1 *Let* $0<\alpha<\infty$, $0<\beta_i<1(1\le i\le l)$ *such that* $\sum_{i=1}^{l}\beta_i+\alpha=\beta+\alpha<ln$. *Suppose that* $1\le p,p_1,\cdots,p_l\le\infty$ *satisfy* $1/p=1/p_1+\cdots+1/p_l-(\beta+\alpha)/n>0$. *If* $D^\gamma A_i\in\dot\Lambda_{\beta_i}(0<\beta_i<1,|\gamma|=m_i)$, $i=1,\ldots,l$, *then we have the following conclusions:*

(1) if $p_i>1$, $i=1,\cdots,l$, *then*

$$\|I_{\alpha,l}^{\vec{A}}\vec{f}\|_{L^p}\le C\prod_{i=1}^{l}\|f_i\|_{L^{p_i}};$$

(2) if at least one p_i *equals 1, then*

$$\|I_{\alpha,l}^{\vec{A}}\vec{f}\|_{L^{p,\infty}}\le C\prod_{i=1}^{l}\|f_i\|_{L^{p_i}}.$$

To describe this simply, in the following text we will consider the bilinear case.

Theorem 2 *Let* $D^\gamma A_i\in\dot\Lambda_{\beta_i}(0<\beta_i<1,|\gamma|=m_i)$ *for* $i=1,2$. *Suppose that* $0<\alpha_i<n$, $0<\beta_i<1$, $1<p,p_i<\infty$ *such that* $\max\{(\alpha+\beta_1-m_2)/n-1,\alpha_1+\beta_1-1/2n\}<1/p_1$, $\max\{(\alpha+\beta_2-m_1)/n-1,\alpha_2+\beta_2-1/2n\}<1/p_2$ *and* $1/p=1/p_1+1/p_2$, *for* $i=1,2$, *where* $\alpha=\alpha_1+\alpha_2$, $\beta=\beta_1+\beta_2$ *and* $\alpha+\beta<2n$. *Then* $I_{\alpha,2}^{(A_1,A_2)}$ *is bounded from* $L^{p_1}\times L^{p_2}$ *to* $\dot\Lambda_{(\alpha+\beta)-n/p}$ *for* $0<(\alpha+\beta)-n/p<1$.

Theorem 3 *Let* $D^\gamma A_i\in\dot\Lambda_{\beta_i}(0<\beta_i<1/2,|\gamma|=m_i)$ *for* $i=1,2$ *and denote* $\beta_1+\beta_2=\beta$. *Suppose that* $0<\alpha_1,\alpha_2<n$, $\alpha_1+\alpha_2=\alpha$ *and* $1<p<\infty$, $1<p_i<n/\alpha_i, i=1,2$, *satisfy* $1/p=1/p_1+1/p_2-\alpha/n$. *Then* $I_{\alpha,2}^{(A_1,A_2)}$ *is bounded from* $L^{p_1}\times L^{p_2}$ *to the Triebel–Lizorkin space* $\dot F_p^{\beta,\infty}$; *that is, there exists a constant* $C>0$ *such that*

$$\|I_{\alpha,2}^{(A_1,A_2)}\vec{f}\|_{\dot F_p^{\beta,\infty}}\le C\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}.$$

Throughout this paper, the letter C always denotes a positive constant that may vary at each occurrence but is independent of the essential variable. We also denote

$$\sum_{|\gamma|=m_i}\|D^\gamma A_i\|_{\dot\Lambda_{\beta_i}}=\|A_i\|_*,$$

for simplicity, where $i=1,\ldots,l$.

2. Some basic lemmas

Lemma 2.1 *[3] Let* A *be a function with derivatives of order* m *in* $\dot\Lambda_\beta(0<\beta<1)$, *and then there exists a constant* $C>0$ *such that*

$$|R_{m+1}(A;x,y)|\le C\Big(\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot\Lambda_\beta}\Big)|x-y|^{m+\beta}; \tag{2.1}$$

$$|R_{m+1}(A;x,y)-R_{m+1}(A;x,z)|\le C\Big(\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot\Lambda_\beta}\Big)\sum_{i=0}^{m}|x-z|^i|z-y|^{m-i+\beta}; \tag{2.2}$$

$$|R_{m+1}(A;x,y)-R_{m+1}(A;z,y)|\le C\Big(\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot\Lambda_\beta}\Big)\Big(\sum_{i=1}^m|x-z|^i|z-y|^{m-i+\beta}+|x-z|^{m+\beta}\Big). \tag{2.3}$$

Lemma 2.2 [9] *Let $0<\alpha<mn$, $I_{\alpha,m}$ be an mth linear fractional integral operator. Suppose that $1\le p_1,\cdots,p_m\le\infty$ and $1/q=1/p_1+\cdots+1/p_m-\alpha/n>0$.*

(1) If $p_i>1, i=1,\cdots,m$, then $\|I_{\alpha,m}\vec f\|_{L^q}\le C\prod_{i=1}^m\|f_i\|_{L^{p_i}}$;

(2) If at least one p_i equals 1, then $\|I_{\alpha,m}\vec f\|_{L^{q,\infty}}\le C\prod_{i=1}^m\|f_i\|_{L^{p_i}}$.

Lemma 2.3 [14] *(a) For $0<\beta<1, 1\le q<\infty$, we have*

$$\|f\|_{\dot\Lambda_\beta}\approx\sup_Q\frac{1}{|Q|^{1+\beta/n}}\int_Q|f-f_Q|\approx\sup_Q\frac{1}{|Q|^{\beta/n}}\Big(\frac{1}{|Q|}\int_Q|f-f_Q|^q\Big)^{1/q}$$

for $q=\infty$, and the formula should be modified appropriately.

(b) For $0<\beta<1, 1<p<\infty$, we have

$$\|f\|_{\dot F_p^{\beta,\infty}}\approx\Big\|\sup_{Q\ni\cdot}\frac{1}{|Q|^{1+\beta/n}}\int_Q|f-f_Q|\Big\|_{L^p},$$

where $f_Q=\frac{1}{|Q|}\int_Q f(x)dx$.

Lemma 2.4 [1] *For $1\le\gamma<\infty$ and $\beta>0$, let*

$$M_{\gamma,\beta}(f)(x)=\sup_{B\ni x}\Big(\frac{1}{|B|^{1-\beta\gamma/n}}\int_B|f(y)|^\gamma dy\Big)^{1/\gamma}.$$

Suppose that $\gamma<p<n/\beta$ and $1/q=1/p-\beta/n$, and then

$$\|M_{\gamma,\beta}(f)\|_{L^q}\le C\|f\|_{L^p}.$$

3. Proofs of theorems

3.1. Proof of Theorem 1

Since $|x-y_1|^{m_1+\beta_1}\cdots|x-y_l|^{m_l+\beta_l}\le|(x-y_1,\cdots,x-y_l)|^{(m_1+\cdots+m_l)+\beta}$, then by Lemma 2.2, we have the following pointwise estimate:

$$\begin{aligned}|I_{\alpha,l}^{\vec A}(\vec f)(x)|\le&\int_{(\mathbb R^n)^l}\frac{\prod_{i=1}^l|R_{m_i+1}(A_i;x,y_i)||f_i(y_i)|}{|(x-y_1,\ldots,x-y_l)|^{ln+(m_1+m_2+\cdots+m_l)-\alpha}}dy_1\ldots dy_l,\\ &\le C\prod_{i=1}^l\|A_i\|_*\int_{(\mathbb R^n)^l}\prod_{i=1}^l|f_i(y_i)|\frac{|x-y_1|^{m_1+\beta_1}\cdots|x-y_l|^{m_l+\beta_l}}{|(x-y_1,\ldots,x-y_l)|^{ln+(m_1+m_2+\cdots+m_l)-\alpha}}dy_1\cdots dy_l\\ &\le C\prod_{i=1}^l\|A_i\|_*\int_{(\mathbb R^n)^l}\frac{\prod_{i=1}^l|f_i(y_i)|}{|(x-y_1,\cdots,x-y_l)|^{ln-(\alpha+\beta)}}dy_1\cdots dy_l\\ &=C\prod_{i=1}^l\|A_i\|_*I_{\alpha+\beta,l}(|f_1|,\cdots,|f_l|)(x).\end{aligned}$$

Theorem 1 is therefore established from Lemma 2.2, for $0<\alpha+\beta<ln$.

3.2. Proof of Theorem 2

In fact, using Lemma 2.3, it suffices to show

$$\sup_{Q} \frac{1}{|Q|^{1+(\alpha+\beta)/n-1/p}} \int_{Q} |I_{\alpha,2}^{(A_1,A_2)}(f_1,f_2)(z) - (I_{\alpha,2}^{(A_1,A_2)}(f_1,f_2))_Q| dz \leq C\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}.$$

For any $x \in \mathbb{R}^n$, fix a cube $Q(x_Q, r) \ni x$ with its center at x_Q and denote the half side length of Q by r. For $f_i \in L^{p_i}(\mathbb{R}^n)$, let $f_i^0 = f\chi_{Q^*}, f_i^\infty = f - f_i^0, i = 1,2$, where $Q^* = 4\sqrt{n}Q$ denotes the $4\sqrt{n}$ times extensions of Q with its center at x_Q. It is obvious that there is $N \in \mathbb{N}$, such that $2^N \leq 4\sqrt{n} < 2^{N+1}$.

Take the constant

$$a = I_{\alpha,2}^{(A_1,A_2)}(f_1^0, f_2^\infty)(x_Q) + I_{\alpha,2}^{(A_1,A_2)}(f_1^\infty, f_2^0)(x_Q) + I_{\alpha,2}^{(A_1,A_2)}(f_1^\infty, f_2^\infty)(x_Q). \tag{3.1}$$

Then

$$\begin{aligned}
&\frac{1}{|Q|^{1+(\alpha+\beta)/n-1/p}} \int_{Q} |I_{\alpha,2}^{(A_1,A_2)}(f_1,f_2)(z) - (I_{\alpha,2}^{(A_1,A_2)}(f_1,f_2))_Q| dz \\
&\leq \frac{2}{|Q|^{1+(\alpha+\beta)/n-1/p}} \int_{Q} |I_{\alpha,2}^{(A_1,A_2)}(f_1,f_2)(z) - a| dz \\
&\leq \frac{2}{|Q|^{1+(\alpha+\beta)/n-1/p}} \int_{Q} |I_{\alpha,2}^{(A_1,A_2)}(f_1^0,f_2^0)(z)| dz \\
&\quad + \frac{2}{|Q|^{1+(\alpha+\beta)/n-1/p}} \int_{Q} |I_{\alpha,2}^{(A_1,A_2)}(f_1^0,f_2^\infty)(z) - I_{\alpha,2}^{(A_1,A_2)}(f_1^0,f_2^\infty)(x_Q)| dz \\
&\quad + \frac{2}{|Q|^{1+(\alpha+\beta)/n-1/p}} \int_{Q} |I_{\alpha,2}^{(A_1,A_2)}(f_1^\infty,f_2^0)(z) - I_{\alpha,2}^{(A_1,A_2)}(f_1^\infty,f_2^0)(x_Q)| dz \\
&\quad + \frac{2}{|Q|^{1+(\alpha+\beta)/n-1/p}} \int_{Q} |I_{\alpha,2}^{(A_1,A_2)}(f_1^\infty,f_2^\infty)(z) - I_{\alpha,2}^{(A_1,A_2)}(f_1^\infty,f_2^\infty)(x_Q)| dz \\
&:= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Take $1 < q < \infty$, such that $1/q = 1/p_1 + 1/p_2 - (\alpha+\beta)/n$, and then by Hölder's inequality and Theorem 1, we have

$$\begin{aligned}
I_1 &\leq \frac{C}{|Q|^{1+(\alpha+\beta)/n-1/p}} \left[\int_{Q} |I_{\alpha,2}^{(A_1,A_2)}(f_1^0,f_2^0)(z)|^q dz\right]^{1/q} |Q|^{1-1/q} \\
&\leq \frac{C}{|Q|^{(\alpha+\beta)/n-1/p+1/q}} \|f_1^0\|_{L^{p_1}}\|f_2^0\|_{L^{p_2}} \\
&\leq C\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}.
\end{aligned}$$

Now we rewrite I_2 as follows:

$$\begin{aligned}
I_2 \quad &\leq \frac{2}{|Q|^{1+(\alpha+\beta)/n-1/p}} \int_Q \int_{\mathbb{R}^{2n}} \Bigg| \frac{R_{m_1+1}(A_1;z,y_1)R_{m_2+1}(A_2;z,y_2)}{|(z-y_1,z-y_2)|^{2n+(m_1+m_2)-\alpha}} \\
&\quad - \frac{R_{m_1+1}(A_1;x_Q,y_1)R_{m_2+1}(A_2;x_Q,y_2)}{|(x_Q-y_1,x_Q-y_2)|^{2n+(m_1+m_2)-\alpha}} \Bigg| |f_1^0(y_1)||f_2^\infty(y_2)|dy_1dy_2dz \\
&\leq \frac{2}{|Q|^{1+(\alpha+\beta)/n-1/p}} \int_Q \int_{\mathbb{R}^{2n}} \frac{|R_{m_1+1}(A_1;z,y_1)||R_{m_2+1}(A_2;z,y_2)-R_{m_2+1}(A_2;x_Q,y_2)|}{|(z-y_1,z-y_2)|^{2n+(m_1+m_2)-\alpha}} \\
&\quad \times |f_1^0(y_1)||f_2^\infty(y_2)|dy_1dy_2dz \\
&\quad + \frac{2}{|Q|^{1+(\alpha+\beta)/n-1/p}} \int_Q \int_{\mathbb{R}^{2n}} \frac{|R_{m_1+1}(A_1;z,y_1)-R_{m_1+1}(A_1;x_Q,y_1)||R_{m_2+1}(A_2;x_Q,y_2)|}{|(z-y_1,z-y_2)|^{2n+(m_1+m_2)-\alpha}} \\
&\quad \times |f_1^0(y_1)||f_2^\infty(y_2)|dy_1dy_2dz \\
&\quad + \frac{2}{|Q|^{1+(\alpha+\beta)/n-1/p}} \int_Q \int_{\mathbb{R}^{2n}} \Bigg| \frac{1}{|(z-y_1,z-y_2)|^{2n+(m_1+m_2)-\alpha}} \\
&\quad - \frac{1}{|(x_Q-y_1,x_Q-y_2)|^{2n+(m_1+m_2)-\alpha}} \Bigg| |R_{m_1+1}(A_1;x_Q,y_1)||R_{m_2+1}(A_2;x_Q,y_2)| \\
&\quad \times |f_1^0(y_1)||f_2^\infty(y_2)|dy_1dy_2dz \\
&:= I_{21}+I_{22}+I_{23}.
\end{aligned}$$

Let us estimate every part for I_2.

Since for any $y_2 \in (Q^*)^c$, $|y_2-x_Q| \sim |y_2-z|$, then Lemma 2.1 (3), Hölder's inequality, and the conditions $(\alpha+\beta_2-1-m_1)/n-1 < 1/p_2$ for Theorem 2 yield the following:

$$\begin{aligned}
I_{21} \quad &\leq \frac{C\|A_1\|_*\|A_2\|_*}{|Q|^{1+(\alpha+\beta)/n-1/p}} \int_Q \int_{Q^*} \sum_{k=N}^{\infty} \int_{Q_{k+1}\setminus Q_k} |z-y_1|^{m_1+\beta_1} \\
&\quad \times \frac{\left(\sum_{i=1}^{m_2} |z-x_Q|^i |y_2-x_Q|^{m_2-i+\beta_2} + |z-x_Q|^{m_2+\beta_2} \right)}{|(z-y_1,z-y_2)|^{2n+(m_1+m_2)-\alpha}} |f_1(y_1)||f_2(y_2)|dy_1dy_2dz \\
&\leq \frac{C\|A_1\|_*\|A_2\|_*}{|Q|^{(\alpha+\beta)/n-1-1/p_2}} \|f_1\|_{L^{p_1}} r^{m_1+\beta_1} \sum_{k=N}^{\infty} \frac{\left(\sum_{i=1}^{m_2} |r|^i |2^{k+1}r|^{m_2-i+\beta_2} + |r|^{m_2+\beta_2} \right) |Q_k|^{1-1/p_2} \|f_2\|_{L^{p_2}}}{|2^k r|^{2n+(m_1+m_2)-\alpha}} \\
&\leq C\|A_1\|_*\|A_2\|_*\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}} \sum_{k=N}^{\infty} 2^{k(\alpha+\beta_2-1-m_1-n-n/p_2)} \\
&\leq C\|A_1\|_*\|A_2\|_*\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
I_{22} \quad &\leq \frac{C\|A_1\|_*\|A_2\|_*}{|Q|^{1+(\alpha+\beta)/n-1/p}} \int_Q \int_{Q^*} \sum_{k=N}^{\infty} \int_{Q_{k+1}\setminus Q_k} \frac{\left(\sum_{i=1}^{m_1} |z-x_Q|^i |y_1-x_Q|^{m_1-i+\beta_1} + |z-x_Q|^{m_1+\beta_1} \right)}{|(z-y_1,z-y_2)|^{2n+(m_1+m_2)-\alpha}} \\
&\quad \times |z-y_2|^{m_2+\beta_2} |f_1(y_1)||f_2(y_2)|dy_1dy_2dz \\
&\leq \frac{C\|A_1\|_*\|A_2\|_*}{|Q|^{(\alpha+\beta)/n-1-1/p_2}} \|f_1\|_{L^{p_1}} r^{m_1+\beta_1} \sum_{k=N}^{\infty} \frac{(2^{k+1}r)^{m_2+\beta_2} |Q_k|^{1-1/p_2} \|f_2\|_{L^{p_2}}}{|2^k r|^{2n+(m_1+m_2)-\alpha}} \\
&\leq C\|A_1\|_*\|A_2\|_*\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}} \sum_{k=N}^{\infty} 2^{k(\alpha+\beta_2-m_1-n-n/p_2)} \\
&\leq C\|A_1\|_*\|A_2\|_*\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}.
\end{aligned}$$

For I_{23}, by the kernel conditions, Lemma 2.1 (1), and $(\alpha+\beta_2-1-m_1)/n-1<1/p_2$, we obtain

$$
\begin{aligned}
I_{23} \quad &\leq \frac{C\|A_1\|_*\|A_2\|_*}{|Q|^{1+(\alpha+\beta)/n-1/p}}\int_Q\int_{Q*}\int_{\mathbb{R}^n\setminus Q*}\frac{|z-x_Q||x_Q-y_1|^{m_1+\beta_1}|x_Q-y_2|^{m_2+\beta_2}}{|(x_Q-y_1,x_Q-y_2)|^{2n+(m_1+m_2)-\alpha+1}}\\
&\quad\times|f_1(y_1)||f_2(y_2)|dy_1dy_2dz\\
&\leq \frac{C\|A_1\|_*\|A_2\|_*|Q|^{2-1/p_1}}{|Q|^{1+(\alpha+\beta)/n-1/p}}\sum_{k=N}^{\infty}\frac{r^{1+m_1+\beta_1}|2^{k+1}r|^{m_2+\beta_2}}{(2^kr)^{2n+(m_1+m_2)-\alpha+1}}|Q_k|^{1-1/p_2}\|f_2\|_{L^{p_2}}\|f_1\|_{L^{p_1}}\\
&\leq C\|A_1\|_*\|A_2\|_*\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}.
\end{aligned}
$$

Combining the above estimates, we have $I_2\leq C\|A_1\|_*\|A_2\|_*\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}$. Similarly, we have the same estimate for I_3.

It is analogous to I_2 that part I_4 can be divided into 4 parts, as follows:

$$
\begin{aligned}
I_4 \quad &\leq \frac{2}{|Q|^{1+(\alpha+\beta)/n-1/p}}\int_Q\int_{(\mathbb{R}^{2n})}\frac{|R_{m_1+1}(A_1;z,y_1)||R_{m_2+1}(A_2;z,y_2)-R_{m_2+1}(A_2;x_Q,y_2)|}{|(z-y_1,z-y_2)|^{2n+(m_1+m_2)-\alpha}}\\
&\quad\times|f_1^\infty(y_1)||f_2^\infty(y_2)|dy_1dy_2dz\\
&\quad+\frac{2}{|Q|^{1+(\alpha+\beta)/n-1/p}}\int_Q\int_{(\mathbb{R}^{2n})}\frac{|R_{m_1+1}(A_1;z,y_1)-R_{m_1+1}(A_1;x_Q,y_1)||R_{m_2+1}(A_2;x_Q,y_2)|}{|(z-y_1,z-y_2)|^{2n+(m_1+m_2)-\alpha}}\\
&\quad\times|f_1^\infty(y_1)||f_2^\infty(y_2)|dy_1dy_2dz\\
&\quad+\frac{2}{|Q|^{1+(\alpha+\beta)/n-1/p}}\int_Q\int_{(\mathbb{R}^{2n})}\left|\frac{1}{|(z-y_1,z-y_2)|^{2n+(m_1+m_2)-\alpha}}\right.\\
&\quad\left.-\frac{1}{|(x_Q-y_1,x_Q-y_2)|^{2n+(m_1+m_2)-\alpha}}\right||R_{m_1+1}(A_1;x_Q,y_1)||R_{m_2+1}(A_2;x_Q,y_2)|\\
&\quad\times|f_1^\infty(y_1)||f_2^\infty(y_2)|dy_1dy_2dz\\
&:=I_{41}+I_{42}+I_{43}.
\end{aligned}
$$

From Lemma 2.1 (1) and the conditions $\alpha_i+\beta_i-1/2n<1/p_i$ for $i=1,2$, we conclude that

$$
\begin{aligned}
I_{41} \quad &\leq \frac{C\|A_1\|_*\|A_2\|_*}{|Q|^{1+(\alpha+\beta)/n-1/p}}\int_Q\sum_{j=N}^{\infty}\int_{Q_{j+1}\setminus Q_j}\sum_{k=N}^{\infty}\int_{Q_{k+1}\setminus Q_k}|z-y_1|^{m_1+\beta_1}\\
&\quad\times\frac{\left(\sum_{i=1}^{m_2}|z-x_Q|^i|y_2-x_Q|^{m_2-i+\beta_2}+|z-x_Q|^{m_2+\beta_2}\right)}{|z-y_1|^{n+m_1+1/2-\alpha_1}|z-y_2|^{n+m_2-1/2-\alpha_2}}|f_1(y_1)||f_2(y_2)|dy_1dy_2dz\\
&\leq \frac{C\|A_1\|_*\|A_2\|_*}{|Q|^{(\alpha+\beta)/n-1/p}}\sum_{j=N}^{\infty}\frac{|2^jQ|^{1-1/p_1}(2^jr)^{m_1+\beta_1}}{(2^jr)^{n+m_1+1/2-\alpha_1}}\\
&\quad\times\sum_{k=N}^{\infty}\frac{\left(\sum_{i=1}^{m_2}r^i(2^{k+1}r)^{m_2-i+\beta_2}+r^{m_2+\beta_2}\right)|Q_k|^{1-1/p_2}}{(2^kr)^{n+m_2-1/2-\alpha_2}}\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}\\
&\leq C\|A_1\|_*\|A_2\|_*\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}.
\end{aligned}
$$

Similarly,

$$I_{42}\leq C\|A_1\|_*\|A_2\|_*\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}.$$

Also,

$$\begin{aligned} I_{43} &\leq \frac{C\|A_1\|_*\|A_2\|_*}{|Q|^{1+(\alpha+\beta)/n-1/p}} \int_Q \sum_{j=N}^{\infty} \int_{Q_{j+1}\setminus Q_j} \sum_{k=N}^{\infty} \int_{Q_{k+1}\setminus Q_k} \\ &\quad \times \frac{|z-x_Q||x_Q-y_1|^{m_1+\beta_1}|x_Q-y_2|^{m_2+\beta_2}}{|z-y_1|^{n+m_1+1/2-\alpha_1}|z-y_2|^{n+m_2+1/2-\alpha_2}} |f_1(y_1)||f_2(y_2)|dy_1dy_2dz \\ &\leq \frac{C\|A_1\|_*\|A_2\|_*}{|Q|^{(\alpha+\beta)/n-1/p}} \sum_{j=N}^{\infty} \frac{|2^jQ|^{1-1/p_1}(2^jr)^{m_1+\beta_1}r^{1/2}}{(2^jr)^{n+m_1+1/2-\alpha_1}} \sum_{k=N}^{\infty} \frac{|2^kQ|^{1-1/p_2}(2^kr)^{m_2+\beta_2}r^{1/2}}{(2^kr)^{n+m_2+1/2-\alpha_2}} \|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}} \\ &\leq C\|A_1\|_*\|A_2\|_*\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}. \end{aligned}$$

Hence, we complete the proof of Theorem 2.

3.3. Proof of Theorem 3

Set Q, x_Q, $f_i^0, f_i^\infty, i=1,2,$ and Q^* as in the proof of Theorem 2. Take the constant a as in (3.1), and then

$$\begin{aligned} &\frac{1}{|Q|^{1+\beta/n}} \int_Q |I_{\alpha,2}^{(A_1,A_2)}(f_1,f_2)(z) - (I_{\alpha,2}^{(A_1,A_2)}(f_1,f_2))_Q|\, dz \\ &\leq \frac{2}{|Q|^{1+\beta/n}} \int_Q |I_{\alpha,2}^{(A_1,A_2)}(f_1,f_2)(z) - a|\, dz \\ &\leq \frac{2}{|Q|^{1+\beta/n}} \int_Q |I_{\alpha,2}^{(A_1,A_2)}(f_1^0,f_2^0)(z)|dz \\ &\quad + \frac{2}{|Q|^{1+\beta/n}} \int_Q |I_{\alpha,2}^{(A_1,A_2)}(f_1^0,f_2^\infty)(z) - I_{\alpha,2}^{(A_1,A_2)}(f_1^0,f_2^\infty)(x_Q)|dz \\ &\quad + \frac{2}{|Q|^{1+\beta/n}} \int_Q |I_{\alpha,2}^{(A_1,A_2)}(f_1^\infty,f_2^0)(z) - I_{\alpha,2}^{(A_1,A_2)}(f_1^\infty,f_2^0)(x_Q)|dz \\ &\quad + \frac{2}{|Q|^{1+\beta/n}} \int_Q |I_{\alpha,2}^{(A_1,A_2)}(f_1^\infty,f_2^\infty)(z) - I_{\alpha,2}^{(A_1,A_2)}(f_1^\infty,f_2^\infty)(x_Q)|dz \\ &:= J_1 + J_2 + J_3 + J_4. \end{aligned}$$

Choose $1 < q < \infty$, $1 < s_i < \min\left\{p_i, \frac{n}{\alpha_i+\beta_i}\right\}$ for $i=1,2,$ such that $1/q = \frac{1}{s_1} + \frac{1}{s_2}$. Then by Hölder's inequality and the boundedness of $I_{\alpha,2}^{A_1,A_2}$ (see Theorem 1), we have

$$\begin{aligned} J_1 &\leq \frac{2}{|Q|^{1+\beta/n}} |Q|^{1-1/q} \|I_{\alpha,2}^{A_1,A_2}(f_1^0,f_2^0)\|_{L^q} \\ &\leq C|Q|^{-1/p}\|f_1\|_{L^{s_1}}\|f_1\|_{L^{s_2}} \\ &\leq C\Big(\frac{1}{|Q|^{1-\alpha_1 s_1/n}} \int_{Q^*} |f_1(y)|^{s_1}dy\Big)^{1/s_1} \Big(\frac{1}{|Q|^{1-\alpha_2 s_2/n}} \int_{Q^*} |f_2(y)|^{s_2}dy\Big)^{1/s_2} \\ &\leq CM_{s_1,\alpha_1}(f_1)(x)M_{s_2,\alpha_2}(f_2)(x). \end{aligned}$$

It is analogous to the estimate of I_2 that we have $J_2 := J_{21} + J_{22} + J_{23}$, where,

$$\begin{aligned} J_{21} &= \frac{2}{|Q|^{1+\beta/n}} \int_Q \int_{(\mathbb{R}^{2n})} \frac{|R_{m_1+1}(A_1;z,y_1)||R_{m_2+1}(A_2;z,y_2) - R_{m_2+1}(A_2;x_Q,y_2)|}{|(z-y_1,z-y_2)|^{2n+(m_1+m_2)-\alpha}}, \\ &\quad \times |f_1^0(y_1)||f_2^\infty(y_2)|dy_1dy_2dz \\ J_{22} &= \frac{2}{|Q|^{1+\beta/n}} \int_Q \int_{(\mathbb{R}^{2n})} \frac{|R_{m_1+1}(A_1;z,y_1) - R_{m_1+1}(A_1;x_Q,y_1)||R_{m_2+1}(A_2;x_Q,y_2)|}{|(z-y_1,z-y_2)|^{2n+(m_1+m_2)-\alpha}}, \\ &\quad \times |f_1^0(y_1)||f_2^\infty(y_2)|dy_1dy_2dz \\ J_{23} &= \frac{2}{|Q|^{1+\beta/n}} \int_Q \int_{(\mathbb{R}^{2n})} \left| \frac{1}{|(z-y_1,z-y_2)|^{2n+(m_1+m_2)-\alpha}} - \frac{1}{|(x_Q-y_1,x_Q-y_2)|^{2n+(m_1+m_2)-\alpha}} \right| \\ &\quad \times |R_{m_1+1}(A_1;x_Q,y_1)||R_{m_2+1}(A_2;x_Q,y_2)||f_1^0(y_1)||f_2^\infty(y_2)|dy_1dy_2dz. \end{aligned}$$

Since for any $y_2 \in (Q^*)^c$, $|y_2 - x_Q| \sim |y_2 - z|$, then by Lemma 2.1(3) and the condition $0 < \beta_2 < 1/2$, we have

$$\begin{aligned}
J_{21} &\leq \frac{C\|A_1\|_*\|A_2\|_*}{|Q|^{1+\beta/n}} \int_Q \int_{Q^*} \sum_{k=N}^{\infty} \int_{Q_{k+1}\setminus Q_k} |z - y_1|^{m_1+\beta_1} \\
&\quad \times \frac{\left(\sum_{i=1}^{m_2} |z - x_Q|^i |y_2 - x_Q|^{m_2-i+\beta_2} + |z - x_Q|^{m_2+\beta_2}\right)}{|z - y_1|^{n+m_1-\alpha_1}|z - y_2|^{n+m_2-\alpha_2}} |f_1(y_1)||f_2(y_2)|dy_1dy_2dz \\
&\leq C\|A_1\|_*\|A_2\|_* \frac{1}{|Q|} \int_{Q^*} \frac{|f_1(y_1)|}{|z - y_1|^{n-\alpha_1}} dy_1 \sum_{k=N}^{\infty} 2^{-k(1-\beta_2)} \left\{ \frac{1}{|Q_{k+1}|^{1-\alpha_2/n}} \int_{Q_{k+1}} |f_2(y_2|dy_2 \right\} \\
&\leq C\|A_1\|_*\|A_2\|_* M(I_{\alpha_1}(|f_1|))(x) M_{1,\alpha_2}(f_2)(x).
\end{aligned}$$

Similarly, for J_{22} and J_{23} we have

$$\begin{aligned}
J_{22} &\leq \frac{C\|A_1\|_*\|A_2\|_*}{|Q|^{1+\beta/n}} \int_Q \int_{Q^*} \int_{\mathbb{R}^n\setminus Q^*} \frac{\left(\sum_{i=1}^{m_1} |z - x_Q|^i |y_1 - x_Q|^{m_1-i+\beta_1} + |z - x_Q|^{m_1+\beta_1}\right)}{|z - y_1|^{n+m_1-1-\alpha_1}} \\
&\quad \times \frac{|z - y_2|^{m_2+\beta_2}}{|z - y_2|^{n+m_2+1-\alpha_2}} |f_1(y_1)||f_2(y_2)|dy_1dy_2dz \\
&\leq C\|A_1\|_*\|A_2\|_* \frac{1}{|Q|} \int_{Q^*} \frac{|f_1(y_1)|}{|z - y_1|^{n-\alpha_1}} dy_1 \\
&\quad \times \sum_{k=N}^{\infty} 2^{-k(1-\beta_2)} \left\{ \frac{1}{|Q_{k+1}|^{1-\alpha_2/n}} \int_{Q_{k+1}} |f_2(y_2)dy_2 \right\} \\
&\leq C\|A_1\|_*\|A_2\|_* M(I_{\alpha_1}(|f_1|))(x) M_{1,\alpha_2}(f_2)(x).
\end{aligned}$$

$$\begin{aligned}
J_{23} &\leq \frac{2}{|Q|^{1+\beta/n}} \int_Q \int_{(\mathbb{R}^{2n})} \left| \frac{1}{|(z - y_1, z - y_2)|^{2n+(m_1+m_2)-\alpha}} - \frac{1}{|(x_Q - y_1, x_Q - y_2)|^{2n+(m_1+m_2)-\alpha}} \right| \\
&\quad \times |R_{m_1+1}(A_1; x_Q, y_1)||R_{m_2+1}(A_2; x_Q, y_2)||f_1^0(y_1)||f_2^\infty(y_2)|dy_1dy_2dz \\
&\leq \frac{C\|A_1\|_*\|A_2\|_*}{|Q|^{1+\beta/n}} \int_Q \int_{Q^*} \sum_{k=N}^{\infty} \int_{Q_{k+1}\setminus Q_k} \frac{c|z - x_Q||x_Q - y_1|^{m_1+\beta_1}|x_Q - y_2|^{m_2+\beta_2}}{|z - y_1|^{n+m_1-\alpha_1}|z - y_2|^{n+m_2+1-\alpha_2}} \\
&\quad \times |f_1(y_1)||f_2(y_2)|dy_1dy_2dz \\
&\leq C\|A_1\|_*\|A_2\|_* \frac{1}{|Q|} \int_{Q^*} \frac{|f_1(y_1)|}{|z - y_1|^{n-\alpha_1}} dy_1 \sum_{k=N}^{\infty} 2^{k(\beta_1-1)} \left\{ \frac{1}{|Q_{k+1}|^{1-\alpha_2/n}} \int_{Q_{k+1}} |f_2(y_2)dy_2 \right\} \\
&\leq C\|A_1\|_*\|A_2\|_* M(I_{\alpha_1}(|f_1|))(x) M_{1,\alpha_2}(f_2)(x).
\end{aligned}$$

Similarly,

$$J_3 \leq C\|A_1\|_*\|A_2\|_* M(I_{\alpha_2}(|f_2|))(x) M_{1,\alpha_1}(f_1)(x).$$

Moreover,

$$\begin{aligned}
J_4 &\leq \frac{2}{|Q|^{1+\beta/n}} \int_Q \int_{(\mathbb{R}^{2n})} \frac{|R_{m_1+1}(A_1; z, y_1)||R_{m_2+1}(A_2; z, y_2) - R_{m_2+1}(A_2; x_Q, y_2)|}{|(z - y_1, z - y_2)|^{2n+(m_1+m_2)-\alpha}} \\
&\quad \times |f_1^\infty(y_1)||f_2^\infty(y_2)|dy_1dy_2dz \\
&\quad + \frac{2}{|Q|^{1+\beta/n}} \int_Q \int_{(\mathbb{R}^{2n})} \frac{|R_{m_1+1}(A_1; z, y_1) - R_{m_1+1}(A_1; x_Q, y_1)||R_{m_2+1}(A_2; x_Q, y_2)|}{|(z - y_1, z - y_2)|^{2n+(m_1+m_2)-\alpha}} \\
&\quad + \frac{2}{|Q|^{1+\beta/n}} \int_Q \int_{(\mathbb{R}^{2n})} \left| \frac{1}{|(z - y_1, z - y_2)|^{2n+(m_1+m_2)-\alpha}} - \frac{1}{|(x_Q - y_1, x_Q - y_2)|^{2n+(m_1+m_2)-\alpha}} \right| \\
&\quad \times |R_{m_1+1}(A_1; x_Q, y_1)||R_{m_2+1}(A_2; x_Q, y_2)||f_1^\infty(y_1)||f_2^\infty(y_2)|dy_1dy_2dz \\
&:= J_{41} + J_{42} + J_{43}.
\end{aligned}$$

Also,

$$\begin{aligned} J_{41} &\leq \frac{C\|A_1\|_*\|A_2\|_*}{|Q|^{1+\beta/n}} \int_Q \int_{\mathbb{R}^n\setminus Q^*} \int_{\mathbb{R}^n\setminus Q^*} |z-y_1|^{m_1+\beta_1} \\ &\quad \times \frac{\left(\sum_{i=1}^{m_2} |z-x_Q|^i |y_2-x_Q|^{m_2-i+\beta_2} + |z-x_Q|^{m_2+\beta_2}\right)}{|z-y_1|^{n+m_1+1/2-\alpha_1}|z-y_2|^{n+m_2-1/2-\alpha_2}} |f_1(y_1)||f_2(y_2)|dy_1dy_2dz \\ &\leq C\|A_1\|_*\|A_2\|_* \sum_{j=N}^{\infty} 2^{-j(1/2-\beta_1)} \left\{\frac{1}{|Q_{j+1}|^{1-\alpha_1/n}} \int_{Q_{j+1}} |f_1(y_1)|dy_1\right\} \\ &\quad \times \sum_{k=N}^{\infty} 2^{-k(1/2-\beta_2)} \left\{\frac{1}{|Q_{k+1}|^{1-\alpha_2/n}} \int_{Q_{k+1}} |f_2(y_2)|dy_2\right\} \\ &\leq C\|A_1\|_*\|A_2\|_* M_{1,\alpha_1}(f_1)(x) M_{1,\alpha_2}(f_2)(x). \end{aligned}$$

It is analogous to J_{41} that

$$J_{42} \leq C\|A_1\|_*\|A_2\|_* M_{1,\alpha_1}(f_1)(x) M_{1,\alpha_2}(f_2)(x).$$

Also,

$$\begin{aligned} J_{43} &\leq \frac{C\|A_1\|_*\|A_2\|_*}{|Q|^{1+\beta/n}} \int_Q \int_{\mathbb{R}^n\setminus Q^*} \int_{\mathbb{R}^n\setminus Q^*} \frac{c|z-x_Q||x_Q-y_1|^{m_1+\beta_1}|x_Q-y_2|^{m_2+\beta_2}}{|z-y_1|^{n+m_1+1/2-\alpha_1}|z-y_2|^{n+m_2+1/2-\alpha_2}} \\ &\quad \times |f_1(y_1)||f_2(y_2)|dy_1dy_2dz \\ &\leq C\|A_1\|_*\|A_2\|_* \sum_{j=N}^{\infty} 2^{j(\beta_1-1/2)} \left\{\frac{1}{|Q_{j+1}|^{1-\alpha_1/n}} \int_{Q_{j+1}} |f_1(y_1|dy_1\right\} \\ &\quad \times \sum_{k=N}^{\infty} 2^{k(\beta_1-1/2)} \left\{\frac{1}{|Q_{k+1}|^{1-\alpha_2/n}} \int_{Q_{k+1}} |f_2(y_2|dy_2\right\} \\ &\leq C\|A_1\|_*\|A_2\|_* M_{1,\alpha_1}(f_1)(x) M_{1,\alpha_2}(f_2)(x). \end{aligned}$$

In conclusion, it follows from the estimates for J_1, J_2, and J_3 that

$$\begin{aligned} &\sup_{Q\ni\cdot} \frac{1}{|Q|^{1+\beta/n}} \int_Q |I_{\alpha,2}^{(A_1,A_2)}(f_1,f_2)(z) - (I_{\alpha,2}^{(A_1,A_2)}(f_1,f_2))_Q|dz \\ &\leq C\|A_1\|_*\|A_2\|_*[M_{s_1,\alpha_1}(f_1)(x)M_{s_2,\alpha_2}(f_2)(x) \\ &+M(I_{\alpha_1}(|f_1|))(x)M_{1,\alpha_2}(f_2)(x) + M(I_{\alpha_2}(|f_2|))(x)M_{1,\alpha_1}(f_1)(x) + M_{1,\alpha_1}(f_1)(x)M_{1,\alpha_2}(f_2)(x)]. \end{aligned}$$

Choose $q_1, q_2 > 1$ such that $1/q_1 = 1/p_1 - \alpha_1/n$ and $1/q_2 = 1/p_2 - \alpha_2/n$. It is obvious that $1/p = 1/q_1 + 1/q_2$. Then, by Hölder's inequality and Lemmas 2.3 and 2.4, we conclude that

$$\begin{aligned} &\|I_{\alpha,2}^{(A_1,A_2)}(f_1,f_2)\|_{\dot{F}_p^{\beta,\infty}} \\ &\approx \left\|\sup_{Q\ni\cdot} \frac{1}{|Q|^{1+\beta/n}} \int_Q |I_{\alpha,2}^{(A_1,A_2)}(f_1,f_2)(z) - (I_{\alpha,2}^{(A_1,A_2)}(f_1,f_2))_Q|dz\right\|_{L^p} \\ &\leq C\|A_1\|_*\|A_2\|_*[\|M_{s_1,\alpha_1}(f_1)M_{s_2,\alpha_2}(f_2)\|_{L^p} \\ &+\|M(I_{\alpha_1}(|f_1|))M_{1,\alpha_2}(f_2)\|_{L^p} + \|M(I_{\alpha_2}(|f_2|))M_{1,\alpha_1}(f_1)\|_{L^p} \\ &+\|M_{1,\alpha_1}(f_1)M_{1,\alpha_2}(f_2)\|_{L^p}] \\ &\leq C\|A_1\|_*\|A_2\|_*[\|M_{s_1,\alpha_1}(f_1)\|_{L^{q_1}}\|M_{s_2,\alpha_2}(f_2)\|_{L^{q_2}} \\ &+\|M(I_{\alpha_1}(|f_1|))\|_{L^{q_1}}\|M_{1,\alpha_2}(f_2)\|_{L^{q_2}} + \|M(I_{\alpha_2}(|f_2|))\|_{L^{q_1}}\|M_{1,\alpha_1}(f_1)\|_{L^{q_2}} \\ &+\|M_{1,\alpha_1}(f_1)\|_{L^{q_1}}\|M_{1,\alpha_2}(f_2)\|_{L^{q_2}}] \\ &\leq C\|A_1\|_*\|A_2\|_*\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}. \end{aligned}$$

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