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Adjoints of rationally induced composition operators on Bergman and Dirichlet spaces

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Abstract: We will state a connection between the adjoints of a vast variety of bounded operators on 2 different weighted Hardy spaces. We will apply it to determine the adjoints of rationally induced composition operators on Dirichlet and Bergman spaces.

Key words: Weighted composition operator, adjoint, weighted Hardy space

1. Introduction

Let $\mathbb{U}$ denote the open unit disk of the complex plane. For each sequence $\beta = \{\beta_n\}$ of positive numbers, the weighted Hardy space $H^2(\beta)$ consists of analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on $\mathbb{U}$ for which the norm

$$\|f\|_\beta = \left(\sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 \right)^{\frac{1}{2}}$$

is finite. Notice that the above norm is induced by the following inner product:

$$\left\langle \sum_{n=0}^{\infty} a_n z^n, \sum_{n=0}^{\infty} b_n z^n \right\rangle_\beta = \sum_{n=0}^{\infty} a_n \bar{b_n} \beta_n^2,$$

and that the monomials z^n form a complete orthogonal system for $H^2(\beta)$. Consequently, the polynomials are dense in $H^2(\beta)$ (see [4, Section 2.1]). Observe that particular instances of the sequence $\beta = \{\beta_n\}$ yield well-known Hilbert spaces of analytic functions. Indeed, $\beta_n = 1$ corresponds to the Hardy space $H^2(\mathbb{U})$. If $\beta_0 = 1$ and $\beta_n = n^{1/2}$ for $n \geq 1$, the resulting space is the classical Dirichlet space $\mathcal{D}$, and if $\beta_n = (n+1)^{-1/2}$, we have the Bergman space $A^2(\mathbb{U})$.

If u is analytic on the open unit disk $\mathbb{U}$ and φ is an analytic map of the unit disk into itself, the weighted composition operator on $H^2(\beta)$ with symbols u and φ is the operator $(W_{u,\varphi} f)(z) = u(z) f(\varphi(z))$ for f in $H^2(\beta)$. When $u(z) \equiv 1$ we call the operator a composition operator and denote it by C_φ. For general information in this context one can refer to excellent monographs [4, 12, 13]. One of the most fundamental questions related to composition and weighted composition operators is how to obtain a reasonable

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representation for their adjoints. The problem of computing the adjoint of a composition operator induced by a linear fractional symbol on the Hardy space was solved by Cowen [2]. Hurst [7] used an analogous argument to obtain the solution in the weighted Bergman space $A^2_\alpha(\mathbb{U})$. In both of these cases, the adjoint consists of a product of a composition operator and 2 Toeplitz operators. In 2003, Gallardo and Montes [5] computed the adjoint of a composition operator acting on the Dirichlet space by a different method from those used by Cowen and Hurst. Hammond et al. [6] solved the case for rationally induced composition operators on the Hardy space, $H^2(\mathbb{U})$. Bourdon and Shapiro [1] reproduced the Hammond–Moorhouse–Robbins formula in a straightforward algebraic fashion. For more information, we refer interested readers to [3, 10, 11].

In this paper we will show that the adjoint problem for weighted composition operators on different weighted Hardy spaces can be reduced, at least for the classical Hardy, Dirichlet, and Bergman spaces, to solving the problem in one specific weighted Hardy space. Among all these specific spaces it is natural to choose the most simple space, $H^2(\mathbb{U})$. Specifically, we will obtain the adjoint of rationally induced composition operators on Dirichlet and Bergman spaces by using the adjoint formula for a composition operator on Hardy space.

2. Weighted Hardy spaces

Let $H^2(\gamma)$ and $H^2(\beta)$ denote the weighted Hardy spaces with weight sequences $\{\gamma_n\}$ and $\{\beta_n\}$, respectively. Then $H^2(\gamma)\cap H^2(\beta)$ contains all polynomials and hence is not empty. Our main theorem is the following.

Theorem 2.1 *Let T_0 and T_1 be bounded operators on $H^2(\gamma)$ and $H^2(\beta)$, respectively, such that for any polynomial p, $T_0p = T_1p$ and $T : H^2(\beta) \to H^2(\gamma)$ is defined by $T(\sum_{n=0}^{\infty} a_n z^n) = \sum_{n=0}^{\infty} a_n \frac{\beta_n}{\gamma_n} z^n$. Then T is invertible and*

(i) $T^{-1}T_0^*Tp = TT_1^*T^{-1}p$ for any polynomial p.

(ii) If $H^2(\gamma) \subset H^2(\beta)$ with continuous inclusion, then $T^{-1}T_0^*Tg = TT_1^*T^{-1}g$ for any $g \in H^2(\gamma)$.

Proof (i) Let m and n be nonnegative integers and put $f_n(z) = z^n$, $e_n(z) = \frac{1}{\beta_n}z^n$, and $u_n(z) = \frac{1}{\gamma_n}z^n$. Then $\{e_n\}$ and $\{u_n\}$ are bases of $H^2(\beta)$ and $H^2(\gamma)$, respectively. Additionally,

$$\langle T_0^* f_n, f_m\rangle_\gamma = \langle f_n, T_0 f_m\rangle_\gamma = \langle f_n, \sum_{j=0}^{\infty} \frac{(T_0 f_m)^{(j)}(0)}{j!} f_j\rangle_\gamma = \frac{\overline{(T_0 f_m)^{(n)}(0)}}{n!}\gamma_n^2. \tag{1}$$

Similarly,

$$\langle T_1^* f_n, f_m\rangle_\beta = \frac{\overline{(T_1 f_m)^{(n)}(0)}}{n!}\beta_n^2. \tag{2}$$

Since $T_0 f_m = T_1 f_m$, comparing (1) and (2) we have

$$\frac{1}{\gamma_n^2}\langle T_0^* f_n, f_m\rangle_\gamma = \frac{1}{\beta_n^2}\langle T_1^* f_n, f_m\rangle_\beta.$$

Therefore,

$$\begin{aligned}
\langle T_1^* e_n, e_m\rangle_\beta e_m &= \frac{1}{\beta_n\beta_m}\langle T_1^* f_n, f_m\rangle_\beta \frac{\gamma_m}{\beta_m}u_m \\
&= \frac{\gamma_m}{\beta_n\beta_m^2}.\frac{\beta_n^2}{\gamma_n^2}\langle T_0^* f_n, f_m\rangle_\gamma u_m \\
&= \frac{\gamma_m\beta_n}{\beta_m^2\gamma_n^2}\langle T_0^*(\gamma_n u_n), \gamma_m u_m\rangle_\gamma u_m \\
&= \frac{\gamma_m^2\beta_n}{\beta_m^2\gamma_n}\langle T_0^* u_n, u_m\rangle_\gamma u_m. \qquad (3)
\end{aligned}$$

It is not difficult to verify that T is an isometric isomorphism that maps the basis of $H^2(\beta)$ to the basis of $H^2(\gamma)$. Furthermore,

$$\langle e_n, T^* u_m\rangle_\beta = \langle Te_n, u_m\rangle_\gamma = \langle u_n, u_m\rangle_\gamma = \langle e_n, e_m\rangle_\beta = \langle e_n, T^{-1}u_m\rangle_\beta.$$

Thus, $T^*u_m = T^{-1}u_m$ and hence $T^* = T^{-1}$. Since $TT_1^* f_n \in H^2(\gamma)$, using (3) we have

$$\begin{aligned}
TT_1^* f_n &= \sum_{m=0}^{\infty}\langle TT_1^* f_n, u_m\rangle_\gamma u_m = \sum_{m=0}^{\infty}\langle T_1^* f_n, T^{-1}u_m\rangle_\beta u_m \\
&= \sum_{m=0}^{\infty}\langle T_1^*(\beta_n e_n), e_m\rangle_\beta \frac{\beta_m}{\gamma_m}e_m = \beta_n\sum_{m=0}^{\infty}\frac{\beta_m}{\gamma_m}\langle T_1^* e_n, e_m\rangle_\beta e_m \\
&= \beta_n\sum_{m=0}^{\infty}\frac{\beta_m}{\gamma_m}\frac{\gamma_m^2\beta_n}{\beta_m^2\gamma_n}\langle T_0^* u_n, u_m\rangle_\gamma u_m \\
&= \frac{\beta_n^2}{\gamma_n}\sum_{m=0}^{\infty}\frac{\gamma_m}{\beta_m}\langle T_0^* u_n, u_m\rangle_\gamma u_m.
\end{aligned}$$

Hence,

$$TT_1^*T^{-1}f_n = TT_1^*\left(\frac{\gamma_n}{\beta_n}f_n\right) = \frac{\gamma_n}{\beta_n}TT_1^* f_n = \beta_n\sum_{m=0}^{\infty}\frac{\gamma_m}{\beta_m}\langle T_0^* u_n, u_m\rangle_\gamma u_m. \qquad (4)$$

Also, $T^{-1}T_0^* f_n \in H^2(\beta)$. Thus,

$$\begin{aligned}
T^{-1}T_0^* f_n &= \sum_{m=0}^{\infty}\langle T^{-1}T_0^* f_n, e_m\rangle_\beta e_m = \sum_{m=0}^{\infty}\langle T_0^* f_n, Te_m\rangle_\gamma e_m \\
&= \sum_{m=0}^{\infty}\langle T_0^*(\gamma_n u_n), u_m\rangle_\gamma \frac{\gamma_m}{\beta_m}u_m = \gamma_n\sum_{m=0}^{\infty}\frac{\gamma_m}{\beta_m}\langle T_0^* u_n, u_m\rangle_\gamma u_m.
\end{aligned}$$

Hence,

$$T^{-1}T_0^*Tf_n = \frac{\beta_n}{\gamma_n}T^{-1}T_0^* f_n = \beta_n\sum_{m=0}^{\infty}\frac{\gamma_m}{\beta_m}\langle T_0^* u_n, u_m\rangle_\gamma u_m. \qquad (5)$$

Comparing (4) and (5), for every nonnegative integer n, we have $T^{-1}T_0^*Tf_n = TT_1^*T^{-1}f_n$, and so the first statement of the theorem holds.
(ii) Let $U_0 = TT_1^*T^{-1} : H^2(\gamma) \longrightarrow H^2(\gamma)$ and $U_1 = T^{-1}T_0^*T : H^2(\beta) \longrightarrow H^2(\beta)$. Then for any polynomial p, $U_0p = U_1p$. Furthermore, for arbitrary $g \in H^2(\gamma) \subseteq H^2(\beta)$ and $\varepsilon > 0$, there exists a polynomial p_0 such that $\|g - p_0\|_\gamma < \varepsilon$. Hence, for some constant $C > 0$,

$$\|U_0g - U_0p_0\|_\beta \leq C\|U_0g - U_0p_0\|_\gamma \leq C\|U_0\|\|g - p_0\|_\gamma < \varepsilon C\|U_0\|,$$

$$\|U_1g - U_1p_0\|_\beta \leq \|U_1\|\|g - p_0\|_\beta \leq C\|U_1\|\|g - p_0\|_\gamma < \varepsilon C\|U_1\|.$$

Therefore,

$$\begin{aligned} \|U_1g - U_0g\|_\beta &= \|U_1g - U_1p_0 + U_0p_0 - U_0g\|_\beta \\ &\leq \|U_1g - U_1p_0\|_\beta + \|U_0g - U_0p_0\|_\beta \\ &< \varepsilon C(\|U_0\| + \|U_1\|). \end{aligned}$$

Since ε is arbitrary, we conclude $U_1g = U_0g$.

□

Corollary 2.2 *For T_0, T_1, and T as in the statement of Theorem 2.1, we have*

$$T_0^*p = T(T(T_1^*(T^{-1}(T^{-1}p))))$$

for any polynomial p.

Proof The statement is clear from Theorem 2.1(i) since Tp and $T^{-1}p$ are polynomials whenever p is a polynomial. Note that for any polynomial p,

$$T^{-1}T_0^*p = TT_1^*T^{-1}(T^{-1}p) \in H^2(\gamma) \cap H^2(\beta).$$

□

Corollary 2.3 *Let $H^2(\gamma) \subset H^2(\beta)$ with continuous inclusion. Then for T_0, T_1, and T as in the statement of Theorem 2.1, we have*

$$T_1^*f = T^{-1}(T^{-1}(T_0^*(T(Tf)))), \quad (f \in H^2(\beta)).$$

Proof By Theorem 2.1(ii), $T^{-1}T_0^*Tg = TT_1^*T^{-1}g$, for $g \in H^2(\gamma)$. Thus, for $f \in H^2(\beta)$, putting $g = Tf$, we have $T^{-1}T_0^*T(Tf) = TT_1^*f \in H^2(\gamma)$. Hence, $T_1^*f = T^{-1}(T^{-1}(T_0^*(T(Tf))))$.

□

Note that Theorem 2.1 and its corollaries cover a wide class of well-known operators including weighted composition operators on Hilbert spaces of analytic functions and hence may be used to translate results relative to the adjoint problem (at least in cases of Dirichlet, Hardy, and Bergman spaces) from one case to another, as we will see in next section.

3. Applications to Dirichlet and Bergman spaces

Let $Rat(\mathbb{U})$ denote the collection of all rational functions of one complex variable defined on $\mathbb{U}$ that map $\mathbb{U}$ into itself and $\varphi \in Rat(\mathbb{U})$. The degree of φ is the larger of the degrees of its numerator and denominator. Define $\varphi_e := \rho \circ \varphi \circ \rho$ where ρ is defined on extended complex plane $\hat{\mathbb{C}}$ by $\rho(z) = 1/\bar{z}$. If the degree of φ is d, then for each point $w \in \hat{\mathbb{C}}$ the inverse image $\varphi^{-1}(\{w\})$ has, counting multiplicities, exactly d points. If $\varphi^{-1}(\{w\})$ has d distinct points we will say that w is a regular value of φ. For any rational function, all but finitely many points of $\hat{\mathbb{C}}$ are regular values. The collection of points in $\hat{\mathbb{C}}$ that are regular values of φ is denoted by $reg(\varphi)$. Let $\{\sigma_j\}_{j=1}^d$ be d distinct branches of φ_e^{-1}, which are defined on a suitable neighborhood of any regular point of the open unit disk. Furthermore, let B be the backward shift operator on $H^2(\mathbb{U})$ defined for $f \in H^2(\mathbb{U})$ by

$$(Bf)(z) = \begin{cases} \frac{f(z)-f(0)}{z} & \text{if } z \in \mathbb{U} \setminus \{0\}, \\ f'(0) & \text{if } z = 0. \end{cases}$$

For more details on the above concepts, one can see [1]. In this section we will obtain the adjoint formula for rationally induced composition operators on the Dirichlet and Bergman spaces. We need the following result of [1].

Theorem 3.1 *If $\varphi \in Rat(\mathbb{U})$, then for each $f \in H^2(\mathbb{U})$*

$$C_\varphi^* f(z) = \frac{f(0)}{1 - \overline{\varphi(0)}z} + z \sum_{j=1}^{d} \sigma_j^{'}(z)(Bf)(\sigma_j(z)), \quad (z \in reg(\varphi_e) \cap \mathbb{U}). \tag{6}$$

3.1. Dirichlet space

Let C_φ be a bounded rationally induced composition operator on Dirichlet space $\mathcal{D}$. We will use the results of Section 2 to identify the adjoint of a rationally induced composition operator on Dirichlet space. Put $H^2(\gamma) = \mathcal{D}$, $H^2(\beta) = H^2(\mathbb{U})$ and let $T : H^2(\mathbb{U}) \longrightarrow \mathcal{D}$ be the operator introduced in the main theorem for this particular choice of spaces.

Proposition 3.2 *For any polynomial f,*

$$T^{-1}(T^{-1}f)(z) = f(0) + zf'(z), \tag{7}$$

and for any $f \in H^2(\mathbb{U})$,

$$T(Tf)(z) = \begin{cases} f(0) + \int_0^z (Bf)(w)dw & \text{if } z \neq 0, \\ f(0) & \text{if } z = 0. \end{cases} \tag{8}$$

Proof The statement is easily verified using the definition of T and the Maclaurin series expansion of f.

□

The following is a generalization of the result obtained in [5] for the adjoint of a composition operator with linear fractional symbol on Dirichlet space $\mathcal{D}$.

Theorem 3.3 *Let $\varphi \in Rat(\mathbb{U})$ and suppose $\mathbb{U} \subseteq reg(\varphi_e)$. Then for each $f \in \mathcal{D}$ and $z \in \mathbb{U}$,*

$$C_\varphi^* f(z) = f(0)K_{\varphi(0)}(z) + \sum_{j=1}^{d} C_{\sigma_j} f(z) - \sum_{j=1}^{d} C_{\sigma_j} f(0).$$

Proof Any self-map of $\mathbb{U}$ with bounded multiplicity induces a bounded composition operator on the Dirichlet space (see [8, Proposition 1.1]). Therefore, C_φ is a bounded composition operator on $\mathcal{D}$. It also follows from $\mathbb{U} \subseteq reg(\varphi_e)$ that each σ_j is well defined and analytic on $\mathbb{U}$. Furthermore, each σ_j is also univalent (see [1]) and hence C_{σ_j} is necessarily bounded on $\mathcal{D}$.

Let S_0 and S_1 be the adjoint of C_φ on $\mathcal{D}$ and $H^2(\mathbb{U})$, respectively, and let $f \in \mathcal{D}$ be a polynomial so that $f(0) = 0$. Then by (7) for $z \neq 0$,

$$\begin{aligned} B(T^{-1}(T^{-1}f))(z) &= \frac{T^{-1}(T^{-1}f)(z) - T^{-1}(T^{-1}f)(0)}{z} \\ &= \frac{f(0) + zf'(z) - f(0)}{z} = f'(z). \end{aligned}$$

Thus, for all $z \in \mathbb{U}$,

$$B(T^{-1}(T^{-1}f))(z) = f'(z). \tag{9}$$

Using (6), (7), and (9), we have

$$\begin{aligned} S_1(T^{-1}(T^{-1}f))(z) &= \frac{T^{-1}(T^{-1}f)(0)}{1 - \overline{\varphi(0)}z} + z\sum_{j=1}^{d} \sigma_j'(z) B(T^{-1}(T^{-1}f))(\sigma_j(z)) \\ &= z\sum_{j=1}^{d} {\sigma'}_j(z) f'(\sigma_j(z)) = z\sum_{j=1}^{d} (f(\sigma_j(z)))'. \end{aligned} \tag{10}$$

By corollary 2.2 and equations (8) and (10) for $z \in reg(\varphi_e) \cap \mathbb{U}$, we have

$$\begin{aligned} (S_0 f)(z) &= T(T(S_1(T^{-1}(T^{-1}f))))(z) \\ &= \begin{cases} \int_0^z (B(S_1(T^{-1}(T^{-1}f))))(w)dw & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases} \end{aligned}$$

Thus, using (10) it follows that for $0 \neq z \in reg(\varphi_e) \cap \mathbb{U}$,

$$\begin{aligned}
(S_0 f)(z) &= \lim_{z_0 \to 0} \int_{z_0}^{z} (B(S_1(T^{-1}(T^{-1}f))))(w)dw \\
&= \lim_{z_0 \to 0} \int_{z_0}^{z} \frac{S_1(T^{-1}(T^{-1}f))(w) - S_1(T^{-1}(T^{-1}f))(0)}{w} dw \\
&= \lim_{z_0 \to 0} \int_{z_0}^{z} \frac{S_1(T^{-1}(T^{-1}f))(w)}{w} dw \\
&= \lim_{z_0 \to 0} \int_{z_0}^{z} \frac{1}{w}(w \sum_{j=1}^{d} (f(\sigma_j(w)))')dw \\
&= \lim_{z_0 \to 0} \int_{z_0}^{z} \sum_{j=1}^{d} (f(\sigma_j(w)))' dw = \lim_{z_0 \to 0} \sum_{j=1}^{d} (f(\sigma_j(z)) - f(\sigma_j(z_0))) \\
&= \sum_{j=1}^{d} (f(\sigma_j(z)) - f(\sigma_j(0))). \qquad (11)
\end{aligned}$$

Continuity of S_0 and C_{σ_j} for $j = 1, ..., d$ and density of the polynomials in $\mathcal{D}$ implies that (11) holds for any $f \in \mathcal{D}$ with $f(0) = 0$. Now let $f \in \mathcal{D}$ be arbitrary. It follows that

$$\begin{aligned}
(S_0(f(0)))(w) &= \langle S_0(f(0)), K_w \rangle_{\mathcal{D}} = \langle f(0), C_\varphi K_w \rangle_{\mathcal{D}} = \langle f(0), K_w \circ \varphi \rangle_{\mathcal{D}} \\
&= f(0)\overline{K_w(\varphi(0))} = f(0)K_{\varphi(0)}(w),
\end{aligned}$$

therefore

$$\begin{aligned}
(S_0 f)(z) &= S_0(f(0))(z) + S_0(f - f(0))(z) \\
&= f(0)K_{\varphi(0)}(z) + \sum_{j=1}^{d} ((f - f(0))(\sigma_j(z)) - (f - f(0))(\sigma_j(0))) \\
&= f(0)K_{\varphi(0)}(z) + \sum_{j=1}^{d} (f(\sigma_j(z)) - f(\sigma_j(0))) \\
&= f(0)K_{\varphi(0)}(z) + \sum_{j=1}^{d} C_{\sigma_j} f(z) - \sum_{j=1}^{d} C_{\sigma_j} f(0).
\end{aligned}$$

□

3.2. Bergman space

Here we will apply the results of Section 2 to obtain the adjoint of a rationally induced composition operator on the Bergman space. Put $H^2(\gamma) = H^2(\mathbb{U})$, $H^2(\beta) = A^2(\mathbb{U})$ and let $T : A^2(\mathbb{U}) \longrightarrow H^2(\mathbb{U})$ be the operator introduced in the main theorem for these spaces.

Proposition 3.4 *For any $f \in H^2(\mathbb{U})$ with $T^{-1}f \in H^2(\mathbb{U})$,*

$$T^{-1}(T^{-1}f)(z) = (zf(z))' = f(z) + zf'(z). \qquad (12)$$

Also, for any $f \in A^2(\mathbb{U})$*,*

$$T(Tf)(z) = \begin{cases} \frac{1}{z}\int_0^z f(w)dw & \text{if } z \neq 0, \\ f(0) & \text{if } z = 0. \end{cases} \tag{13}$$

Hence, $T(T(f)) = B(F)$ *where* F *is the antiderivative of* f*.*

Proof It can be easily obtained from definition of T and Maclaurin series expansion of f. Note that for any $f \in A^2(\mathbb{U})$ we have $F \in \mathcal{D} \subset H^2(\mathbb{U})$.

□

Let $\mathbb{U}_0 = \bigcap_{j=1}^d \{z \in \mathbb{U} : z \neq 1/\overline{\varphi(\infty)},\ \sigma_j(z) \neq 0\}$. Clearly, $\mathbb{U}_0$ is an open subset of $\mathbb{U}$ containing all points of $\mathbb{U}$ except finitely many points. Let $Q : A^2(\mathbb{U}) \longrightarrow A^2(\mathbb{U})$ be the operator defined by $Qf = F$ where F is the antiderivative of f with $F(0) = 0$. Q is norm decreasing and hence bounded on $A^2(\mathbb{U})$.

Theorem 3.5 *Let* $\varphi \in Rat(\mathbb{U})$ *and* $f \in A^2(\mathbb{U})$*. For any* $z \in reg(\varphi_e) \cap \mathbb{U}_0$*,*

$$C_\varphi^* f(z) = \frac{f(0)}{(1-\overline{\varphi(\infty)}z)^2} + \sum_{j=1}^d W_{u_j', \sigma_j} Qf(z) + \sum_{j=1}^d W_{u_j\sigma_j', \sigma_j} f(z),$$

where $u_j(z) = \frac{z^2\sigma_j'(z)}{(\sigma_j(z))^2}$*.*

Proof Note that all composition operators on Bergman space are bounded (see [9, Proposition 3.4]). Let S_0 and S_1 be the adjoint of C_φ on $H^2(\mathbb{U})$ and $A^2(\mathbb{U})$, respectively, and let F be the antiderivative of f with $F(0) = 0$. For $z \neq 0$,

$$\begin{aligned} (B^2F)(z) &= B(BF)(z) = \frac{(BF)(z) - (BF)(0)}{z} = \frac{\frac{F(z)}{z} - F'(0)}{z} \\ &= \frac{F(z) - zf(0)}{z^2}. \end{aligned} \tag{14}$$

Hence by (6), (13), and (14) for any $z \in reg(\varphi_e) \cap \mathbb{U}_0$ and $f \in A^2(\mathbb{U})$,

$$\begin{aligned} S_0(T(Tf))(z) &= S_0(BF)(z) = \frac{(BF)(0)}{1-\overline{\varphi(0)}z} + z\sum_{j=1}^d \sigma_j'(z)(B^2F)(\sigma_j(z)) \\ &= \frac{f(0)}{1-\overline{\varphi(0)}z} + z\sum_{j=1}^d \sigma_j'(z)\frac{F(\sigma_j(z)) - f(0)\sigma_j(z)}{(\sigma_j(z))^2} \\ &= \frac{f(0)}{1-\overline{\varphi(\infty)}z} + z\sum_{j=1}^d \frac{\sigma_j'(z)}{(\sigma_j(z))^2}F(\sigma_j(z)). \end{aligned}$$

The last equality follows from the fact that

$$\frac{1}{1-\overline{\varphi(0)}z} - \frac{1}{1-\overline{\varphi(\infty)}z} = z\sum_{j=1}^d \frac{\sigma_j'(z)}{\sigma_j(z)},$$

from [1]. Therefore, by (12) and Corollary 2.3,

$$\begin{aligned}
(S_1 f)(z) &= T^{-1}(T^{-1}(S_0(T(T(f)))))(z) = (zS_0(T(Tf))(z))' \\
&= \frac{f(0)}{(1-\overline{\varphi(\infty)}z)^2} + \left(z^2 \sum_{j=1}^{d} \frac{\sigma_j'(z)}{(\sigma_j(z))^2} F(\sigma_j(z))\right)' \\
&= \frac{f(0)}{(1-\overline{\varphi(\infty)}z)^2} + \sum_{j=1}^{d} W_{u_j',\sigma_j} Qf(z) + \sum_{j=1}^{d} W_{u_j\sigma_j',\sigma_j} f(z).
\end{aligned}$$

□

Now we consider composition operators with linear fractional symbol. Let $\varphi(z) = \frac{az+b}{cz+d}$. Then $\sigma(z) := \sigma_1(z) = \varphi_e^{-1}(z) = \frac{\bar{a}z-\bar{c}}{-\bar{b}z+\bar{d}}$ and $u_1(z) = z^2 \frac{\sigma'(z)}{(\sigma(z))^2} = \frac{(\bar{a}\bar{d}-\bar{b}\bar{c})z^2}{(\bar{a}z-\bar{c})^2}$. Hurst's result [7] for C_φ^* on the weighted Bergman spaces yields that on the classical Bergman space by letting $g(z) = (-\bar{b}z+\bar{d})^{-2}$ and $h(z) = (cz+d)^2$ we have $C_\varphi^* = T_g C_\sigma T_h^*$ where T_g and T_h are Toeplitz operators on $A^2(\mathbb{U})$. Since T_h^* is not explicitly known on $A^2(\mathbb{U})$, this expression does not give explicit demonstration for C_φ^* on $A^2(\mathbb{U})$. As a consequence of Theorem 3.5, we obtain the following explicit formula for the adjoint of a composition operator on $A^2(\mathbb{U})$ induced by a linear fractional symbol.

Corollary 3.6 *Let C_φ be a composition operator on $A^2(\mathbb{U})$ with linear fractional symbol $\varphi(z) = \frac{az+b}{cz+d}$. Then for any $f \in A^2(\mathbb{U})$,*

$$\begin{aligned}
C_\varphi^* f(z) &= \frac{\bar{c}^2 f(0)}{(\bar{a}z-\bar{c})^2} + \frac{-2\bar{c}(\bar{a}\bar{d}-\bar{b}\bar{c})z}{(\bar{a}z-\bar{c})^3} F\left(\frac{\bar{a}z-\bar{c}}{-\bar{b}z+\bar{d}}\right) \\
&\quad + \left(\frac{(\bar{a}\bar{d}-\bar{b}\bar{c})z}{(-\bar{b}z+\bar{d})(\bar{a}z-\bar{c})}\right)^2 f\left(\frac{\bar{a}z-\bar{c}}{-\bar{b}z+\bar{d}}\right).
\end{aligned}$$

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