

Essential norms of weighted composition operators between Zygmund-type spaces and Bloch-type spaces

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Received: 02.01.2014 • Accepted: 17.02.2014 • Published Online: 01.07.2014 • Printed: 31.07.2014

Abstract: We investigate the boundedness of weighted composition operator uC_φ mapping the Zygmund-type space Z^α into the Bloch-type space B^β . Then we give essential norm estimates of such an operator in terms of u and φ .

Key words: Zygmund-type space, Bloch-type space, weighted composition operator, essential norm

1. Introduction and preliminaries

Let \mathbb{D} denote the open unit disc of the complex plane and $H(\mathbb{D})$ the space of analytic functions on \mathbb{D} . Let $u, \varphi \in H(\mathbb{D})$, where φ is a selfmap of \mathbb{D} . Then the well-known *weighted composition operator* uC_φ on $H(\mathbb{D})$ is defined by $(uC_\varphi)(f)(z) = u(z)f(\varphi(z))$, for all $f \in H(\mathbb{D})$ and $z \in \mathbb{D}$. Weighted composition operators can be regarded as generalizations of multiplication operators and composition operators. These operators appear in the study of dynamical systems. Moreover, it is known that isometries on many analytic function spaces are of the canonical forms of weighted composition operators. For more information about these types of operators we refer to the books [3, 18]. In this paper we consider weighted composition operators on certain spaces of analytic functions, defined as follows.

By a *weight function* ν we mean a continuous, strictly positive, and bounded function $\nu : \mathbb{D} \rightarrow \mathbb{R}_+$. For a weight ν , the *weighted Banach space of analytic functions* H_ν^∞ is the space of all analytic functions $f \in H(\mathbb{D})$ for which

$$\|f\|_\nu = \sup_{z \in \mathbb{D}} \nu(z)|f(z)| < \infty.$$

The weight ν is called *radial* if $\nu(z) = \nu(|z|)$ for all $z \in \mathbb{D}$. Also, for a weight ν , the associated weight $\tilde{\nu}$ is defined by

$$\tilde{\nu}(z) = (\sup\{|f(z)| : f \in H_\nu^\infty, \|f\|_\nu \leq 1\})^{-1},$$

for all $z \in \mathbb{D}$. It is known that for the *standard weights* ($0 < \alpha < \infty$)

$$\nu_\alpha(z) = (1 - |z|^2)^\alpha, \quad z \in \mathbb{D},$$

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2010 *AMS Mathematics Subject Classification*: Primary: 47B33; Secondary: 47B38.

and the *logarithmic weight*

$$\nu_{\log}(z) = \left(\log \frac{2}{1 - |z|^2} \right)^{-1}, \quad z \in \mathbb{D},$$

the associated weights and the weights are the same, i.e. $\tilde{\nu}_\alpha = \nu_\alpha$ and $\tilde{\nu}_{\log} = \nu_{\log}$.

For $0 < \alpha < \infty$, the *Bloch-type space* \mathcal{B}^α is the space of all analytic functions $f \in H(\mathbb{D})$ for which

$$\sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f'(z)| < \infty.$$

The Bloch-type space \mathcal{B}^α is a Banach space with the norm

$$\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f'(z)|,$$

for all $f \in \mathcal{B}^\alpha$. When $\alpha = 1$, we get the classic *Bloch space* $\mathcal{B} = \mathcal{B}^1$.

Recall that the *Zygmund space* \mathcal{Z} is the class of all functions $f \in H(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ with

$$\sup_{\substack{e^{i\theta} \in \partial\mathbb{D} \\ h > 0}} \frac{|f(e^{i(\theta+h)}) + f(e^{i(\theta-h)}) - 2f(e^{i\theta})|}{h} < \infty.$$

By [4, Theorem 5.3], an analytic function f belongs to \mathcal{Z} if and only if $f' \in \mathcal{B}^1$, or equivalently

$$\sup_{z \in \mathbb{D}} (1 - |z|) |f''(z)| < \infty.$$

For $0 < \alpha < \infty$ we denote by \mathcal{Z}^α the *Zygmund-type space* of those functions $f \in H(\mathbb{D})$ satisfying $\sup_{z \in \mathbb{D}} (1 - |z|)^\alpha |f''(z)| < \infty$. The Zygmund-type space \mathcal{Z}^α is a Banach space with the norm

$$\|f\|_{\mathcal{Z}^\alpha} = |f(0)| + |f'(0)| + \sup_{z \in \mathbb{D}} (1 - |z|)^\alpha |f''(z)|,$$

for all $f \in \mathcal{Z}^\alpha$.

Recall that a linear operator T between Banach spaces X and Y is *bounded* if it takes bounded sets into bounded sets, while T is *compact* if it takes bounded sets to sets with compact closure. The norm of the bounded operator $T : X \rightarrow Y$ is denoted by $\|T\|_{X \rightarrow Y}$. The space of all bounded operators and compact operators $T : X \rightarrow Y$ are denoted by $\mathcal{B}(X, Y)$ and $\mathcal{K}(X, Y)$, respectively. The *essential norm* $\|T\|_{e, X \rightarrow Y}$ of a bounded operator $T : X \rightarrow Y$ is defined as the distance from T to $\mathcal{K}(X, Y)$.

Function theoretic characterizations of boundedness, compactness, and essential norm estimates of a weighted composition operator uC_φ in terms of u and φ have been studied by many authors between different spaces of analytic functions. Boundedness and compactness of composition operators on Bloch spaces were first studied by Roan [17] and later by Madigan [13, 14] and Matheson [14]. Moreover, Ohno, Stroethoff, and Zhao studied weighted composition operators between Bloch-type spaces in [16].

Boundedness of composition operators on Zygmund spaces was first studied by Choe, Koo, and Smith in [1]. Later in [7], Hu and Ye characterized boundedness and compactness of weighted composition operators between Zygmund spaces. Recently, Esmaeili and Lindström [5] used an approach due to Hyvärinen and

Lindström [9] to obtain new characterizations for bounded weighted composition operators between Zygmund-type spaces, and to give similar estimates for the essential norms of such operators. In this paper, we obtain new characterizations for bounded weighted composition operators between Zygmund-type spaces and Bloch-type spaces and give estimates for the essential norms of such operators.

See [6, 10, 11, 12, 19, 21] for more related results concerning composition operators between different spaces of analytic functions.

The following theorems will be used in Section 2 and Section 3. Before stating these theorems we mention that in the rest of this paper for the real scalars A and B the notation $A \asymp B$ means that $cB \leq A \leq CB$ for some positive constants c and C .

The next theorem is due to Montes-Rodríguez [15, Theorem 2.1] and Hyvärinen et al. [8, Theorem 2.4]. See also the results of Contreras and Hernández-Díaz in [2].

Theorem 1.1 *Let ν and ω be radial, nonincreasing weights tending to zero at the boundary of \mathbb{D} . Then,*

(i) *the weighted composition operator uC_φ maps H_ν^∞ into H_ω^∞ if and only if*

$$\sup_{n \geq 0} \frac{\|u\varphi^n\|_\omega}{\|z^n\|_\nu} \asymp \sup_{z \in \mathbb{D}} \frac{\omega(z)}{\bar{\nu}(\varphi(z))} |u(z)| < \infty,$$

with norm comparable to the above supremum.

(ii) $\|uC_\varphi\|_{e, H_\nu^\infty \rightarrow H_\omega^\infty} = \limsup_{n \rightarrow \infty} \frac{\|u\varphi^n\|_\omega}{\|z^n\|_\nu} = \limsup_{|\varphi(z)| \rightarrow 1} \frac{\omega(z)}{\bar{\nu}(\varphi(z))} |u(z)|.$

Lemma 1.2 [9, Lemma 2.1] *For every $0 < \alpha < \infty$ we have*

(i) $\limsup_{n \rightarrow \infty} (n + 1)^\alpha \|z^n\|_{\nu_\alpha} = \left(\frac{2\alpha}{e}\right)^\alpha,$

(ii) $\limsup_{n \rightarrow \infty} (\log n) \|z^n\|_{\nu_{\log}} = 1.$

2. Boundedness

In this section we investigate the boundedness of weighted composition operator uC_φ mapping \mathcal{Z}^α into \mathcal{B}^β . Before stating the main results of this section, we mention the test functions

$$\begin{aligned} f_a(z) &= \frac{1}{a^2} \left[\frac{(1 - |a|^2)^2}{(1 - \bar{a}z)^\alpha} - \frac{1 - |a|^2}{(1 - \bar{a}z)^{\alpha-1}} \right], \\ h_a(z) &= \frac{1}{a} \int_0^z \frac{1 - |a|^2}{(1 - \bar{a}w)^\alpha} dw, \\ g_a(z) &= f_a(z) - h_a(z), \end{aligned}$$

where $0 < \alpha < \infty$ and $a, z \in \mathbb{D}$. It is easy to see that $f_a, h_a, g_a \in \mathcal{Z}^\alpha$, $\sup_{\frac{1}{2} < |a| < 1} \|f_a\|_{\mathcal{Z}^\alpha} < \infty$, and $\sup_{\frac{1}{2} < |a| < 1} \|h_a\|_{\mathcal{Z}^\alpha} < \infty$. Note that $f_a(a) = 0$, $f'_a(a) = h'_a(a) = \frac{1}{a}(1 - |a|^2)^{1-\alpha}$, $f''_a(a) = \frac{2\alpha}{(1 - |a|^2)^\alpha}$, and $h''_a(a) = \frac{\alpha}{(1 - |a|^2)^\alpha}$.

The following lemma collects some useful estimates for the functions in Zygmund type spaces.

Lemma 2.1 [5, Lemma 1.1] For every $f \in \mathcal{Z}^\alpha$, where $\alpha > 0$, we have

- (i) $|f'(z)| \leq \frac{2}{1-\alpha} \|f\|_{\mathcal{Z}^\alpha}$ and $|f(z)| \leq \frac{2}{1-\alpha} \|f\|_{\mathcal{Z}^\alpha}$ for every $0 < \alpha < 1$,
- (ii) $|f'(z)| \leq 2\|f\|_{\mathcal{Z}} \log \frac{2}{1-|z|}$ and $|f(z)| \leq \|f\|_{\mathcal{Z}}$ for $\alpha = 1$,
- (iii) $|f'(z)| \leq \frac{2}{\alpha-1} \frac{\|f\|_{\mathcal{Z}^\alpha}}{(1-|z|)^{\alpha-1}}$, for every $\alpha > 1$,
- (iv) $|f(z)| \leq \frac{2}{(\alpha-1)(2-\alpha)} \|f\|_{\mathcal{Z}^\alpha}$, for every $1 < \alpha < 2$,
- (v) $|f(z)| \leq 2\|f\|_{\mathcal{Z}^2} \log \frac{2}{1-|z|}$, for $\alpha = 2$,
- (vi) $|f(z)| \leq \frac{2}{(\alpha-1)(\alpha-2)} \frac{\|f\|_{\mathcal{Z}^\alpha}}{(1-|z|)^{\alpha-2}}$, for every $\alpha > 2$.

In the next theorems we characterize the boundedness of weighted composition operator $uC_\varphi : \mathcal{Z}^\alpha \rightarrow \mathcal{B}^\beta$ in different cases of $0 < \alpha < \infty$.

Theorem 2.2 If $0 < \alpha < 1$, then $uC_\varphi : \mathcal{Z}^\alpha \rightarrow \mathcal{B}^\beta$ is bounded if and only if $u \in \mathcal{B}^\beta$ and $u\varphi' \in H_{\nu_\beta}^\infty$.

Proof If the weighted composition operator $uC_\varphi : \mathcal{Z}^\alpha \rightarrow \mathcal{B}^\beta$ is bounded, then by considering $f(z) = 1$ and $f(z) = z$, we obtain $u \in \mathcal{B}^\beta$ and $u\varphi' \in H_{\nu_\beta}^\infty$. Now, suppose that $u \in \mathcal{B}^\beta$ and $u\varphi' \in H_{\nu_\beta}^\infty$. Using Lemma 2.1 (i), for all $z \in \mathbb{D}$, we have

$$(1 - |z|^2)^\beta |(uC_\varphi f)'(z)| \leq \frac{2}{1-\alpha} (1 - |z|^2)^\beta \|f\|_{\mathcal{Z}^\alpha} (|u'(z)| + |u(z)\varphi'(z)|).$$

Therefore, $uC_\varphi : \mathcal{Z}^\alpha \rightarrow \mathcal{B}^\beta$ is bounded, since $u \in \mathcal{B}^\beta$ and $u\varphi' \in H_{\nu_\beta}^\infty$. □

Theorem 2.3 The weighted composition operator $uC_\varphi : \mathcal{Z} \rightarrow \mathcal{B}^\beta$ is bounded if and only if $u \in \mathcal{B}^\beta$ and

$$\sup_{n \geq 0} (\log n) \|u\varphi'^n\|_{\nu_\beta} \asymp \sup_{z \in \mathbb{D}} (1 - |z|^2)^\beta |u(z)\varphi'(z)| \log \frac{2}{1 - |\varphi(z)|^2} < \infty.$$

Proof Suppose that the weighted composition operator $uC_\varphi : \mathcal{Z} \rightarrow \mathcal{B}^\beta$ is bounded. Then $u \in \mathcal{B}^\beta$ and $u\varphi' \in H_{\nu_\beta}^\infty$. Using [7, Theorem 3.1], for $a \in \mathbb{D}$ define the function

$$k_a(z) = \frac{h(\overline{\varphi(a)}z)}{\varphi(a)} \left(\log \frac{2}{1 - |\varphi(a)|^2} \right)^{-1},$$

where $h(z) = (z - 1) \left((1 + \log \frac{2}{1-z})^2 + 1 \right)$. It is proved that $\sup_{1/2 < |\varphi(a)| < 1} \|k_a\|_{\mathcal{Z}} < \infty$ and $k'_a(\varphi(a)) = \log \frac{2}{1 - |\varphi(a)|^2}$. Since the operator $uC_\varphi : \mathcal{Z} \rightarrow \mathcal{B}^\beta$ is bounded

$$\|uC_\varphi k_a\|_{\mathcal{B}^\beta} \geq (1 - |a|^2)^\beta |u(a)\varphi'(a)k'_a(\varphi(a))| - (1 - |a|^2)^\beta |u'(a)k_a(\varphi(a))|.$$

Thus, by Lemma 2.1 (ii), we have

$$\begin{aligned} \sup_{|\varphi(a)|>1/2} (1 - |a|^2)^\beta |u(a)\varphi'(a)k'_a(\varphi(a))| &\leq \sup_{|\varphi(a)|>1/2} \|uC_\varphi k_a\|_{\mathcal{B}^\beta} \\ &+ \|u\|_{\mathcal{B}^\beta} \sup_{|\varphi(a)|>1/2} \|k_a\|_{\mathcal{Z}} < \infty. \end{aligned}$$

On the other hand,

$$\begin{aligned} \sup_{|\varphi(a)|\leq 1/2} (1 - |a|^2)^\beta |u(a)\varphi'(a)k'_a(\varphi(a))| &= \sup_{|\varphi(a)|\leq 1/2} (1 - |a|^2)^\beta |u(a)\varphi'(a)| \log \frac{2}{1 - |\varphi(a)|^2} \\ &\leq \log \frac{8}{3} \sup_{|\varphi(a)|\leq 1/2} (1 - |a|^2)^\beta |u(a)\varphi'(a)| \\ &\leq \log \frac{8}{3} \sup_{a \in \mathbb{D}} (1 - |a|^2)^\beta |u(a)\varphi'(a)| < \infty. \end{aligned}$$

Therefore,

$$\begin{aligned} \sup_{a \in \mathbb{D}} (1 - |a|^2)^\beta |u(a)\varphi'(a)| \log \frac{2}{1 - |\varphi(a)|^2} &\leq \sup_{|\varphi(a)|>1/2} (1 - |a|^2)^\beta |u(a)\varphi'(a)k'_a(\varphi(a))| \\ &+ \sup_{|\varphi(a)|\leq 1/2} (1 - |a|^2)^\beta |u(a)\varphi'(a)k'_a(\varphi(a))| \\ &< \infty, \end{aligned}$$

and finally, Theorem 1.1 (i) and Lemma 1.2 (ii) complete the proof. The converse part is a consequence of Lemma 2.1 (ii). □

Theorem 2.4 *If $1 < \alpha < 2$, then $uC_\varphi : \mathcal{Z}^\alpha \rightarrow \mathcal{B}^\beta$ is bounded if and only if $u \in \mathcal{B}^\beta$ and*

$$\sup_{n \geq 0} (n + 1)^{\alpha-1} \|u\varphi^n\|_{\nu_\beta} \asymp \sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^\beta}{(1 - |\varphi(z)|^2)^{\alpha-1}} |u(z)\varphi'(z)| < \infty.$$

Proof If $uC_\varphi : \mathcal{Z}^\alpha \rightarrow \mathcal{B}^\beta$ is bounded, then $u \in \mathcal{B}^\beta$ and $u\varphi' \in H_{\nu_\beta}^\infty$. By using the test function $f_{\varphi(a)}$, for $a \in \mathbb{D}$, we have

$$\begin{aligned} \|uC_\varphi f_{\varphi(a)}\|_{\mathcal{B}^\beta} &\geq (1 - |a|^2)^\beta |u(a)\varphi'(a)f'_{\varphi(a)}(\varphi(a))| - (1 - |a|^2)^\beta |u'(a)f_{\varphi(a)}(\varphi(a))| \\ &= (1 - |a|^2)^\beta |u(a)\varphi'(a)f'_{\varphi(a)}(\varphi(a))|. \end{aligned}$$

Since $\sup_{|\varphi(a)|>1/2} \|f_{\varphi(a)}\|_{\mathcal{Z}^\alpha} < \infty$,

$$\begin{aligned} \sup_{|\varphi(a)|>1/2} \frac{(1 - |a|^2)^\beta}{(1 - |\varphi(a)|^2)^{\alpha-1}} |u(a)\varphi'(a)| &= \sup_{|\varphi(a)|>1/2} (1 - |a|^2)^\beta |u(a)\varphi'(a)f'_{\varphi(a)}(\varphi(a))| |\varphi(a)| \\ &\leq \sup_{|\varphi(a)|>1/2} \|uC_\varphi f_{\varphi(a)}\|_{\mathcal{B}^\beta} \\ &\leq \|uC_\varphi\| \sup_{|\varphi(a)|>1/2} \|f_{\varphi(a)}\|_{\mathcal{Z}^\alpha} < \infty. \end{aligned}$$

On the other hand, since $u\varphi' \in H_{\nu_\beta}^\infty$, we have

$$\sup_{|\varphi(a)| \leq 1/2} \frac{(1 - |a|^2)^\beta}{(1 - |\varphi(a)|^2)^{\alpha-1}} |u(a)\varphi'(a)| < \infty.$$

Now, Theorem 1.1 (i) and Lemma 1.2 (i) imply that

$$\begin{aligned} \sup_{n \geq 0} (n + 1)^{\alpha-1} \|u\varphi'\varphi^n\|_{\nu_\beta} &\asymp \sup_{n \geq 0} \frac{\|u\varphi'\varphi^n\|_{\nu_\beta}}{\|z^n\|_{\nu_{\alpha-1}}} \\ &\asymp \sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^\beta}{(1 - |\varphi(z)|^2)^{\alpha-1}} |u(z)\varphi'(z)| < \infty. \end{aligned}$$

The converse part can be easily proved using Lemma 2.1. □

Theorem 2.5 *The weighted composition operator $uC_\varphi : \mathcal{Z}^2 \rightarrow \mathcal{B}^\beta$ is bounded if and only if*

$$(i) \sup_{n \geq 0} (\log n) \|u'\varphi^n\|_{\nu_\beta} \asymp \sup_{z \in \mathbb{D}} (1 - |z|^2)^\beta |u'(z)| \log \frac{2}{1 - |\varphi(z)|^2} < \infty,$$

$$(ii) \sup_{n \geq 0} (n + 1) \|u\varphi'\varphi^n\|_{\nu_\beta} \asymp \sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^\beta}{1 - |\varphi(z)|^2} |u(z)\varphi'(z)| < \infty.$$

Proof If $uC_\varphi : \mathcal{Z}^2 \rightarrow \mathcal{B}^\beta$ is assumed to be bounded, then (ii) can be proved in the same way as the proof of Theorem 2.4. Next, we prove (i).

For every nonzero $a \in \mathbb{D}$ define $k_a(z) = \log \frac{2}{1 - \bar{a}z}$. Then, $k_a \in \mathcal{Z}^2$ and $\sup_{1/2 < |a| < 1} \|k_a\|_{\mathcal{Z}^2} < \infty$. On the other hand,

$$\begin{aligned} \|uC_\varphi k_{\varphi(a)}\|_{\mathcal{B}^\beta} &\geq (1 - |a|^2)^\beta |u'(a)k_{\varphi(a)}(\varphi(a))| - (1 - |a|^2)^\beta |u(a)\varphi'(a)k'_{\varphi(a)}(\varphi(a))| \\ &= (1 - |a|^2)^\beta |u'(a)| \log \frac{2}{1 - |\varphi(a)|^2} - \frac{(1 - |a|^2)^\beta}{1 - |\varphi(a)|^2} |u(a)\varphi'(a)\varphi(a)|. \end{aligned}$$

Therefore,

$$\begin{aligned} \sup_{|\varphi(a)| > 1/2} (1 - |a|^2)^\beta |u'(a)| \log \frac{2}{1 - |\varphi(a)|^2} &\leq \sup_{|\varphi(a)| > 1/2} \|uC_\varphi k_{\varphi(a)}\|_{\mathcal{B}^\beta} \\ &\quad + \sup_{|\varphi(a)| > 1/2} \frac{(1 - |a|^2)^\beta}{1 - |\varphi(a)|^2} |u(a)\varphi'(a)| < \infty. \end{aligned}$$

On the other hand, we have

$$\sup_{|\varphi(a)| \leq 1/2} (1 - |a|^2)^\beta |u'(a)| \log \frac{2}{1 - |\varphi(a)|^2} < \infty,$$

which implies the desired result. Also, using Lemma 2.1, one can prove the converse part. □

Theorem 2.6 *If $\alpha > 2$, then $uC_\varphi : \mathcal{Z}^\alpha \rightarrow \mathcal{B}^\beta$ is bounded if and only if*

$$(i) \sup_{n \geq 0} (n + 1)^{\alpha-2} \|u' \varphi^n\|_{\nu_\beta} \asymp \sup_{z \in \mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha-2}} |u'(z)| < \infty,$$

$$(ii) \sup_{n \geq 0} (n + 1)^{\alpha-1} \|u \varphi^n \varphi'\|_{\nu_\beta} \asymp \sup_{z \in \mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha-1}} |u(z) \varphi'(z)| < \infty.$$

Proof Let $uC_\varphi : \mathcal{Z}^\alpha \rightarrow \mathcal{B}^\beta$ be bounded. Then (ii) holds by a similar method as in the proof of Theorem 2.4. Next, we prove (i).

For every $a, z \in \mathbb{D}$ define $t_a(z) = \frac{(1-|a|^2)^2}{(1-\bar{a}z)^\alpha}$. Then $t_a \in \mathcal{Z}^\alpha$, $\sup_{a \in \mathbb{D}} \|t_a\|_{\mathcal{Z}^\alpha} < \infty$, $t_a(a) = \frac{1}{(1-|a|^2)^{\alpha-2}}$, and $t'_a(a) = \frac{\alpha \bar{a}}{(1-|a|^2)^{\alpha-1}}$. Since the operator $uC_\varphi : \mathcal{Z}^\alpha \rightarrow \mathcal{B}^\beta$ is bounded,

$$\begin{aligned} \|uC_\varphi t_{\varphi(a)}\|_{\mathcal{B}^\beta} &\geq (1-|a|^2)^\beta |u'(a) t_{\varphi(a)}(\varphi(a))| - (1-|a|^2)^\beta |u(a) \varphi'(a) t'_{\varphi(a)}(\varphi(a))| \\ &= \frac{(1-|a|^2)^\beta}{(1-|\varphi(a)|^2)^{\alpha-2}} |u'(a)| - \frac{\alpha(1-|a|^2)^\beta}{(1-|\varphi(a)|^2)^{\alpha-1}} |u(a) \varphi'(a) \varphi(a)|. \end{aligned}$$

Thus, using (ii), we have

$$\begin{aligned} \sup_{|\varphi(a)| > 1/2} \frac{(1-|a|^2)^\beta}{(1-|\varphi(a)|^2)^{\alpha-2}} |u'(a)| &\leq \sup_{|\varphi(a)| > 1/2} \|uC_\varphi t_{\varphi(a)}\|_{\mathcal{B}^\beta} \\ &\quad + \sup_{|\varphi(a)| > 1/2} \frac{\alpha(1-|a|^2)^\beta}{(1-|\varphi(a)|^2)^{\alpha-1}} |u(a) \varphi'(a) \varphi(a)| < \infty. \end{aligned}$$

Also,

$$\sup_{|\varphi(a)| \leq 1/2} \frac{(1-|a|^2)^\beta}{(1-|\varphi(a)|^2)^{\alpha-2}} |u'(a)| < \infty.$$

Consequently,

$$\sup_{z \in \mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha-2}} |u'(z)| < \infty,$$

and the rest of the proof holds by Theorem 1.1 (i) and Lemma 1.2 (i).

Conversely, if (i) and (ii) hold, then, using Lemma 2.1, one can see that the operator $uC_\varphi : \mathcal{Z}^\alpha \rightarrow \mathcal{B}^\beta$ is bounded. □

3. Essential norms

In this section we give estimates for the essential norm of weighted composition operator uC_φ mapping \mathcal{Z}^α into \mathcal{B}^β . Before stating the main results of this section, we note that if $D_\alpha : \mathcal{Z}^\alpha \rightarrow \mathcal{B}^\beta$ and $S_\alpha : \mathcal{B}^\alpha \rightarrow H^\infty_{\nu_\alpha}$ are the derivative operators, then D_α and S_α are linear isometries on $\tilde{\mathcal{Z}}^\alpha = \{f \in \mathcal{Z}^\alpha : f(0) = f'(0) = 0\}$ and $\tilde{\mathcal{B}}^\alpha = \{f \in \mathcal{B}^\alpha : f(0) = 0\}$, respectively, and

$$S_\beta uC_\varphi D_\alpha^{-1} S_\alpha^{-1} = u'C_\varphi D_\alpha^{-1} S_\alpha^{-1} + u\varphi' C_\varphi S_\alpha^{-1}.$$

Therefore,

$$\|uC_\varphi\|_{e, \tilde{\mathcal{Z}}^\alpha \rightarrow \mathcal{B}^\beta} \leq \|u'C_\varphi\|_{e, \tilde{\mathcal{Z}}^\alpha \rightarrow H^\infty_{\nu_\beta}} + \|u\varphi' C_\varphi\|_{e, \tilde{\mathcal{B}}^\alpha \rightarrow H^\infty_{\nu_\beta}}.$$

Using a similar method as in [5, Lemma 3.1], we can prove that $\|uC_\varphi\|_{e, \tilde{\mathcal{Z}}^\alpha \rightarrow \mathcal{B}^\beta} = \|uC_\varphi\|_{e, \mathcal{Z}^\alpha \rightarrow \mathcal{B}^\beta}$ and therefore,

$$\|uC_\varphi\|_{e, \mathcal{Z}^\alpha \rightarrow \mathcal{B}^\beta} \leq \|u'C_\varphi\|_{e, \mathcal{Z}^\alpha \rightarrow H_{\nu_\beta}^\infty} + \|u\varphi'C_\varphi\|_{e, \mathcal{B}^\alpha \rightarrow H_{\nu_\beta}^\infty}. \tag{3.1}$$

The next theorem gives essential norm estimates of a bounded weighted composition operator $uC_\varphi : \mathcal{Z}^\alpha \rightarrow H_\nu^\infty$ when ν is a radial and nonincreasing weight tending to zero at the boundary of \mathbb{D} .

Theorem 3.1 [5, Theorem 3.3] *Let $0 < \alpha < \infty$, ν be a radial, nonincreasing weight tending to zero at the boundary of \mathbb{D} and the weighted composition operator $uC_\varphi : \mathcal{Z}^\alpha \rightarrow H_\nu^\infty$ be bounded.*

(i) *If $0 < \alpha < 2$, then uC_φ is a compact operator.*

(ii)

$$\|uC_\varphi\|_{e, \mathcal{Z}^2 \rightarrow H_\nu^\infty} \asymp \limsup_{n \rightarrow \infty} (\log n) \|u\varphi^n\|_\nu \asymp \limsup_{|\varphi(z)| \rightarrow 1} \nu(z) |u(z)| \log \frac{2}{1 - |\varphi(z)|^2}.$$

(iii) *If $\alpha > 2$, then*

$$\|uC_\varphi\|_{e, \mathcal{Z}^\alpha \rightarrow H_\nu^\infty} \asymp \limsup_{n \rightarrow \infty} (n + 1)^{\alpha-2} \|u\varphi^n\|_\nu \asymp \limsup_{|\varphi(z)| \rightarrow 1} \frac{\nu(z)}{(1 - |\varphi(z)|^2)^{\alpha-2}} |u(z)|.$$

The following lemma is an analogue of [7, Lemma 4.2].

Lemma 3.2 *Fix $0 < \alpha < 2$ and let (f_n) be a bounded sequence in \mathcal{Z}^α that converges to zero on compact subsets of \mathbb{D} . Then*

$$\limsup_{n \rightarrow \infty} \sup_{z \in \mathbb{D}} |f_n(z)| = 0.$$

Theorem 3.3 *Suppose that $uC_\varphi : \mathcal{Z}^\alpha \rightarrow \mathcal{B}^\beta$ is bounded.*

(i) *If $0 < \alpha < 1$, then*

$$\|uC_\varphi\|_{e, \mathcal{Z}^\alpha \rightarrow \mathcal{B}^\beta} = 0.$$

(ii) *If $\alpha = 1$, then*

$$\|uC_\varphi\|_{e, \mathcal{Z} \rightarrow \mathcal{B}^\beta} \asymp \limsup_{n \rightarrow \infty} (\log n) \|u\varphi'\varphi^n\|_{\nu_\beta}.$$

(iii) *If $1 < \alpha < 2$, then*

$$\|uC_\varphi\|_{e, \mathcal{Z}^\alpha \rightarrow \mathcal{B}^\beta} \asymp \limsup_{n \rightarrow \infty} (n + 1)^{\alpha-1} \|u\varphi'\varphi^n\|_{\nu_\beta}.$$

Proof Since the weighted composition operator $uC_\varphi : \mathcal{Z}^\alpha \rightarrow \mathcal{B}^\beta$ is bounded, $u \in \mathcal{B}^\beta$ and by Lemma 2.1 (i), (ii), and (iv) the operator $u'C_\varphi : \mathcal{Z}^\alpha \rightarrow H_{\nu_\beta}^\infty$ is bounded. Theorem 3.1 (i) implies that $\|u'C_\varphi\|_{e, \mathcal{Z}^\alpha \rightarrow H_{\nu_\beta}^\infty} = 0$.

On the other hand, by a similar method as in [5, Theorem 3.4], one can see that $u\varphi'C_\varphi$ is a compact operator from \mathcal{B}^α to $H_{\nu_\beta}^\infty$, for $0 < \alpha < 1$. Therefore, $\|u\varphi'C_\varphi\|_{e, \mathcal{B}^\alpha \rightarrow H_{\nu_\beta}^\infty} = 0$, and hence by applying (3.1) we get

$$\|uC_\varphi\|_{e, \mathcal{Z}^\alpha \rightarrow \mathcal{B}^\beta} = 0.$$

For $\alpha = 1$, by (3.1), Theorem 3.1 (i), and [20, Theorem 3.4] we have

$$\begin{aligned} \|uC_\varphi\|_{e, \mathcal{Z}^\alpha \rightarrow \mathcal{B}^\beta} &\leq \|u\varphi' C_\varphi\|_{e, \mathcal{B}^\alpha \rightarrow H_{\nu_\beta}^\infty} \\ &\asymp \limsup_{|\varphi(z)| \rightarrow 1} (1 - |z|^2)^\beta |u(z)\varphi'(z)| \log \frac{2}{1 - |\varphi(z)|^2} \\ &\asymp \limsup_{n \rightarrow \infty} (\log n) \|u\varphi' \varphi^n\|_{\nu_\beta}. \end{aligned}$$

This proves the upper bound in (ii). Let (z_n) be a sequence in \mathbb{D} such that $|\varphi(z_n)| > 1/2$ and $|\varphi(z_n)| \rightarrow 1$ as $n \rightarrow \infty$. Define the test function

$$f_n(z) = \frac{h(\overline{\varphi(z_n)}z)}{\varphi(z_n)} \left(\log \frac{2}{1 - |\varphi(z_n)|^2} \right)^{-1},$$

for all $z \in \mathbb{D}$, where $h(z) = (z - 1) \left((1 + \log \frac{2}{1-z})^2 + 1 \right)$. One can see that (f_n) is a bounded sequence in \mathcal{Z} that converges to zero uniformly on compact subsets of \mathbb{D} and $f'_n(\varphi(z_n)) = \log \frac{2}{1 - |\varphi(z_n)|^2}$. If $c = \sup_{n \geq 1} \|f_n\|_{\mathcal{Z}}$, then

$$\begin{aligned} c \|uC_\varphi\|_{e, \mathcal{Z} \rightarrow \mathcal{B}^\beta} &\geq \limsup_{n \rightarrow \infty} \|uC_\varphi f_n\|_{\mathcal{B}^\beta} \\ &\geq \limsup_{n \rightarrow \infty} (1 - |z_n|^2)^\beta |u(z_n)\varphi'(z_n)f'_n(\varphi(z_n))| \\ &\quad - \limsup_{n \rightarrow \infty} (1 - |z_n|^2)^\beta |u'(z_n)f_n(\varphi(z_n))| \\ &= \limsup_{n \rightarrow \infty} (1 - |z_n|^2)^\beta |u(z_n)\varphi'(z_n)| \log \frac{2}{1 - |\varphi(z_n)|^2} \\ &\quad - \limsup_{n \rightarrow \infty} (1 - |z_n|^2)^\beta |u'(z_n)f_n(\varphi(z_n))|. \end{aligned}$$

By Lemma 3.2, we have

$$\limsup_{n \rightarrow \infty} (1 - |z_n|^2)^\beta |u'(z_n)f_n(\varphi(z_n))| = 0.$$

Thus,

$$\begin{aligned} c \|uC_\varphi\|_{e, \mathcal{Z} \rightarrow \mathcal{B}^\beta} &\geq \limsup_{n \rightarrow \infty} (1 - |z_n|^2)^\beta |u(z_n)\varphi'(z_n)| \log \frac{2}{1 - |\varphi(z_n)|^2} \\ &\asymp \limsup_{n \rightarrow \infty} (\log n) \|u\varphi' \varphi^n\|_{\nu_\beta}. \end{aligned}$$

This completes the proof of (ii). Similarly, one can prove (iii) by using the test function $f_n = f_{\varphi(z_n)}$. □

Theorem 3.4 *Let the weighted composition operator $uC_\varphi : \mathcal{Z}^2 \rightarrow \mathcal{B}^\beta$ be bounded. Then*

$$\|uC_\varphi\|_{e, \mathcal{Z}^2 \rightarrow \mathcal{B}^\beta} \asymp \max \left\{ \limsup_{n \rightarrow \infty} (\log n) \|u'\varphi^n\|_{\nu_\beta}, \limsup_{n \rightarrow \infty} (n + 1) \|u\varphi' \varphi^n\|_{\nu_\beta} \right\}.$$

Proof Using [20, Theorem 3.2] and Theorem 3.1 (ii), by (3.1) we have

$$\|uC_\varphi\|_{e, \mathcal{Z}^2 \rightarrow \mathcal{B}^\beta} \leq 2 \max \left\{ \limsup_{n \rightarrow \infty} (\log n) \|u'\varphi^n\|_{\nu_\beta}, \limsup_{n \rightarrow \infty} (n+1) \|u\varphi'\varphi^n\|_{\nu_\beta} \right\}.$$

By a similar argument as in the proof of Theorem 3.3, for some constant $c_1 > 0$, we have

$$c_1 \|uC_\varphi\|_{e, \mathcal{Z}^2 \rightarrow \mathcal{B}^\beta} \geq \limsup_{n \rightarrow \infty} (n+1) \|u\varphi'\varphi^n\|_{\nu_\beta}. \tag{3.2}$$

Let (z_n) be a sequence in \mathbb{D} such that $|\varphi(z_n)| > 1/2$ and $|\varphi(z_n)| \rightarrow 1$ as $n \rightarrow \infty$. Define the function

$$f_n(z) = \left(1 + \left(\log \frac{2}{1 - \overline{\varphi(z_n)}z} \right)^2 \right) \left(\log \frac{2}{1 - |\varphi(z_n)|^2} \right)^{-1},$$

for all $z \in \mathbb{D}$. Then (f_n) is a bounded sequence in \mathcal{Z}^2 that converges to zero uniformly on compact subsets of \mathbb{D} and $f'_n(\varphi(z_n)) = \frac{2\overline{\varphi(z_n)}}{1 - |\varphi(z_n)|^2}$. If $c_2 = \sup_{n \geq 1} \|f_n\|_{\mathcal{Z}^2}$, then

$$\begin{aligned} c_2 \|uC_\varphi\|_{e, \mathcal{Z}^2 \rightarrow \mathcal{B}^\beta} &\geq \limsup_{n \rightarrow \infty} \|uC_\varphi f_n\|_{\mathcal{B}^\beta} \\ &\geq \limsup_{n \rightarrow \infty} (1 - |z_n|^2)^\beta |u'(z_n) f_n(\varphi(z_n))| \\ &\quad - \limsup_{n \rightarrow \infty} (1 - |z_n|^2)^\beta |u(z_n) \varphi'(z_n) f'_n(\varphi(z_n))|. \end{aligned}$$

On the other hand, (3.2) implies that

$$\begin{aligned} \limsup_{n \rightarrow \infty} (1 - |z_n|^2)^\beta |u(z_n) \varphi'(z_n) f'_n(\varphi(z_n))| &= 2 \limsup_{n \rightarrow \infty} \frac{(1 - |z_n|^2)^\beta |\varphi(z_n)|}{1 - |\varphi(z_n)|^2} |u(z_n) \varphi'(z_n)| \\ &\leq 2 \limsup_{n \rightarrow \infty} (n+1) \|u\varphi'\varphi^n\|_{\nu_\beta} \\ &\leq 2c_1 \|uC_\varphi\|_{e, \mathcal{Z}^2 \rightarrow \mathcal{B}^\beta}. \end{aligned}$$

Thus,

$$\begin{aligned} (c_2 + 2c_1) \|uC_\varphi\|_{e, \mathcal{Z}^2 \rightarrow \mathcal{B}^\beta} &\geq \limsup_{n \rightarrow \infty} (1 - |z_n|^2)^\beta |u'(z_n) f_n(\varphi(z_n))| \\ &\geq \limsup_{n \rightarrow \infty} (1 - |z_n|^2)^\beta |u'(z_n)| \log \frac{2}{1 - |\varphi(z_n)|^2} \\ &\asymp \limsup_{n \rightarrow \infty} (\log n) \|u'\varphi^n\|_{\nu_\beta}. \end{aligned}$$

□

Proof of the next theorem is similar to that of the previous theorem, by using the test function $g_n(z) = \frac{(1 - |\varphi(z_n)|^2)^2}{(1 - \varphi(z_n)z)^\alpha}$.

Theorem 3.5 *Let $\alpha > 2$ and the weighted composition operator $uC_\varphi : \mathcal{Z}^\alpha \rightarrow \mathcal{B}^\beta$ be bounded. Then*

$$\|uC_\varphi\|_{e, \mathcal{Z}^2 \rightarrow \mathcal{B}^\beta} \asymp \max \left\{ \limsup_{n \rightarrow \infty} (n+1)^{\alpha-2} \|u'\varphi^n\|_{\nu_\beta}, \limsup_{n \rightarrow \infty} (n+1)^{\alpha-1} \|u\varphi'\varphi^n\|_{\nu_\beta} \right\}.$$

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