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General rotational surfaces in the 4-dimensional Minkowski space

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Abstract: General rotational surfaces as a source of examples of surfaces in the 4-dimensional Euclidean space were introduced by C. Moore. In this paper we consider the analogue of these surfaces in the Minkowski 4-space. On the basis of our invariant theory of spacelike surfaces we study general rotational surfaces with special invariants. We describe analytically the flat general rotational surfaces and the general rotational surfaces with flat normal connection. We classify completely the minimal general rotational surfaces and the general rotational surfaces consisting of parabolic points.

Key words: Surfaces in the 4-dimensional Minkowski space, general rotational surfaces, minimal surfaces, flat surfaces, surfaces with flat normal connection

1. Introduction

The local theory of spacelike surfaces in the 4-dimensional Minkowski space $\mathbb{R}^4_1$ was developed by the present authors in [6]. Our approach to this theory is based on the introduction of an invariant linear map of Weingarten type in the tangent plane at any point of the surface. This invariant map allowed us to introduce principal lines and a geometrically determined moving frame field at each point of the surface. Writing derivative formulas of the Frenet type for this frame field, we obtained 8 invariant functions, γ_1, γ_2, ν_1, ν_2, λ, μ, β_1, β_2, and proved a fundamental theorem of the Bonnet type, stating that these 8 invariants under some natural conditions determine the surface up to a motion in $\mathbb{R}^4_1$.

The basic geometric classes of surfaces in $\mathbb{R}^4_1$ are characterized by conditions on these invariant functions. For example, surfaces with flat normal connection are characterized by the condition $\nu_1 = \nu_2$, minimal surfaces are described by $\nu_1 + \nu_2 = 0$, and Chen surfaces are characterized by $\lambda = 0$.

Rotational surfaces are a basic source of examples of many geometric classes of surfaces. In [14], Moore introduced general rotational surfaces in the 4-dimensional Euclidean space $\mathbb{R}^4$ and described a special case of general rotational surfaces with constant Gauss curvature [15].

In the present paper we consider spacelike general rotational surfaces that are analogous to the general rotational surfaces of Moore. We apply the invariant theory of spacelike surfaces in $\mathbb{R}^4_1$ to the class of general rotational surfaces with plane meridians. Using the invariants of these surfaces, we describe analytically the flat general rotational surfaces (Theorem 3.2) and the general rotational surfaces with flat normal connection (Theorem 3.3). In Theorem 3.4 we give the complete classification of general rotational surfaces consisting of parabolic points. The classification of minimal general rotational surfaces is given in Theorem 3.5.

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2. Preliminaries

Let $\mathbb{R}^4_1$ be the 4-dimensional Minkowski space endowed with the metric $\langle , \rangle$ of signature $(3,1)$ and let $Oe_1e_2e_3e_4$ be a fixed orthonormal coordinate system, i.e. $e_1^2 = e_2^2 = e_3^2 = 1$, $e_4^2 = -1$, giving the orientation of $\mathbb{R}^4_1$. A surface $M^2 : z = z(u,v),\ (u,v) \in \mathcal{D}\ (\mathcal{D} \subset \mathbb{R}^2)$ in $\mathbb{R}^4_1$ is said to be *spacelike* if $\langle , \rangle$ induces a Riemannian metric g on M^2. Thus, at each point p of a spacelike surface M^2, we have the following decomposition:

$$\mathbb{R}^4_1 = T_pM^2 \oplus N_pM^2$$

with the property that the restriction of the metric $\langle , \rangle$ onto the tangent space T_pM^2 is of signature $(2,0)$ and the restriction of the metric $\langle , \rangle$ onto the normal space N_pM^2 is of signature $(1,1)$.

Denote by ∇' and ∇ the Levi-Civita connections on $\mathbb{R}^4_1$ and M^2, respectively. Let x and y be vector fields tangent to M and let ξ be a normal vector field. The formulas of Gauss and Weingarten give a decomposition of the vector fields $\nabla'_x y$ and $\nabla'_x \xi$ into a tangent and a normal component:

$$\begin{aligned}
\nabla'_x y &= \nabla_x y + \sigma(x,y),\\
\nabla'_x \xi &= -A_\xi x + D_x \xi,
\end{aligned}$$

which define the second fundamental tensor σ, the normal connection D, and the shape operator A_ξ with respect to ξ.

The mean curvature vector field H of M^2 is defined as $H = \dfrac{1}{2}\operatorname{tr}\sigma$, i.e. given a local orthonormal frame $\{x,y\}$ of the tangent bundle, $H = \dfrac{1}{2}(\sigma(x,x) + \sigma(y,y))$.

Let $M^2 : z = z(u,v),\ (u,v) \in \mathcal{D}\ (\mathcal{D} \subset \mathbb{R}^2)$ be a local parametrization on a spacelike surface in $\mathbb{R}^4_1$. The tangent space at an arbitrary point $p = z(u,v)$ of M^2 is $T_pM^2 = \operatorname{span}\{z_u, z_v\}$, where $\langle z_u, z_u\rangle > 0$, $\langle z_v, z_v\rangle > 0$. We use the standard denotations $E(u,v) = \langle z_u, z_u\rangle$, $F(u,v) = \langle z_u, z_v\rangle$, $G(u,v) = \langle z_v, z_v\rangle$ for the coefficients of the first fundamental form

$$I(\lambda,\mu) = E\lambda^2 + 2F\lambda\mu + G\mu^2, \quad \lambda,\mu \in \mathbb{R}.$$

Since $I(\lambda,\mu)$ is positive definite, we set $W = \sqrt{EG - F^2}$. We choose a normal frame field $\{n_1, n_2\}$ such that $\langle n_1, n_1\rangle = 1$, $\langle n_2, n_2\rangle = -1$, and the quadruple $\{z_u, z_v, n_1, n_2\}$ is positively oriented in $\mathbb{R}^4_1$. Then we have the following derivative formulas:

$$\begin{aligned}
\nabla'_{z_u} z_u &= z_{uu} = \Gamma^1_{11}\, z_u + \Gamma^2_{11}\, z_v + c^1_{11}\, n_1 - c^2_{11}\, n_2,\\
\nabla'_{z_u} z_v &= z_{uv} = \Gamma^1_{12}\, z_u + \Gamma^2_{12}\, z_v + c^1_{12}\, n_1 - c^2_{12}\, n_2,\\
\nabla'_{z_v} z_v &= z_{vv} = \Gamma^1_{22}\, z_u + \Gamma^2_{22}\, z_v + c^1_{22}\, n_1 - c^2_{22}\, n_2,
\end{aligned}$$

where Γ^k_{ij} are the Christoffel symbols and the functions $c^k_{ij},\ i,j,k = 1,2$ are given by

$$\begin{array}{lll}
c^1_{11} = \langle z_{uu}, n_1\rangle; & c^1_{12} = \langle z_{uv}, n_1\rangle; & c^1_{22} = \langle z_{vv}, n_1\rangle;\\
c^2_{11} = \langle z_{uu}, n_2\rangle; & c^2_{12} = \langle z_{uv}, n_2\rangle; & c^2_{22} = \langle z_{vv}, n_2\rangle.
\end{array}$$

Obviously, the surface M^2 lies in a 2-plane if and only if M^2 is totally geodesic, i.e. $c_{ij}^k = 0,\ i,j,k = 1,2.$ Therefore, we assume that at least one of the coefficients c_{ij}^k is not zero.

The second fundamental form II of the surface M^2 at a point $p \in M^2$ is introduced by the following functions:

$$L = \frac{2}{W}\begin{vmatrix} c_{11}^1 & c_{12}^1 \\ c_{11}^2 & c_{12}^2 \end{vmatrix}; \quad M = \frac{1}{W}\begin{vmatrix} c_{11}^1 & c_{22}^1 \\ c_{11}^2 & c_{22}^2 \end{vmatrix}; \quad N = \frac{2}{W}\begin{vmatrix} c_{12}^1 & c_{22}^1 \\ c_{12}^2 & c_{22}^2 \end{vmatrix}.$$

Let $X = \lambda z_u + \mu z_v,\ (\lambda,\mu) \neq (0,0)$ be a tangent vector at a point $p \in M^2$. Then

$$II(\lambda,\mu) = L\lambda^2 + 2M\lambda\mu + N\mu^2, \quad \lambda,\mu \in \mathbb{R}.$$

The second fundamental form II is invariant up to the orientation of the tangent space or the normal space of the surface.

The condition $L = M = N = 0$ characterizes points at which the space $\{\sigma(x,y) : x,y \in T_pM^2\}$ is 1-dimensional. We call such points *flat points* of the surface [6]. These points are analogous to flat points in the theory of surfaces in $\mathbb{R}^3$ and $\mathbb{R}^4$ [2, 3]. In [6] we gave a local geometric description of spacelike surfaces consisting of flat points, proving that any spacelike surface consisting of flat points whose mean curvature vector at any point is a nonzero spacelike vector or timelike vector either lies in a hyperplane of $\mathbb{R}^4_1$ or is part of a developable ruled surface in $\mathbb{R}^4_1$.

We further consider surfaces free of flat points, i.e. $(L,M,N) \neq (0,0,0)$.

The second fundamental form II determines conjugate, asymptotic, and principal tangents at a point p of M^2 in the standard way. A line $c : u = u(q),\ v = v(q);\ q \in J \subset \mathbb{R}$ on M^2 is said to be an *asymptotic line*, respectively a *principal line*, if its tangent at any point is asymptotic, respectively principal. The surface M^2 is parameterized by principal lines if and only if $F = 0,\ M = 0$.

The second fundamental form II generates 2 invariant functions:

$$k = \frac{LN - M^2}{EG - F^2}, \qquad \varkappa = \frac{EN + GL - 2FM}{2(EG - F^2)}.$$

The functions k and $\varkappa$ are invariant under changes of the parameters of the surface and changes of the normal frame field. The sign of k is invariant under congruences and the sign of $\varkappa$ is invariant under motions in $\mathbb{R}^4_1$. However, the sign of $\varkappa$ changes under symmetries with respect to a hyperplane in $\mathbb{R}^4_1$. It turns out that the invariant $\varkappa$ is the curvature of the normal connection of the surface (see [6]). The number of asymptotic tangents at a point of M^2 is determined by the sign of the invariant k.

As in the theory of surfaces in $\mathbb{R}^3$ and $\mathbb{R}^4$, the invariant k divides the points of M^2 into the following types: *elliptic* ($k > 0$), *parabolic* ($k = 0$), and *hyperbolic* ($k < 0$).

Let H be the normal mean curvature vector field. Recall that a surface M^2 is said to be *minimal* if its mean curvature vector vanishes identically, i.e. $H = 0$. The minimal surfaces are characterized in terms of the invariants k and $\varkappa$ by the following equality [6]:

$$\varkappa^2 - k = 0.$$

It is interesting to note that the "*umbilical*" points, i.e. points at which the coefficients of the first and the second fundamental forms are proportional ($L = \rho E,\ M = \rho F,\ N = \rho G,\ \rho \neq 0$), are exactly the points at which the mean curvature vector H is zero. Thus, the spacelike surfaces consisting of "umbilical" points in $\mathbb{R}^4_1$ are exactly the minimal surfaces. If M^2 is a spacelike surface free of "umbilical" points ($H \neq 0$ at each point), then there exist exactly 2 principal tangents.

Considering spacelike surfaces in $\mathbb{R}^4_1$ whose mean curvature vector at any point is a nonzero spacelike vector or timelike vector, on the base of the principal lines we introduced a geometrically determined orthonormal frame field $\{x, y, b, l\}$ at each point of such a surface [6]. The tangent vector fields x and y are collinear with the principal directions, and the normal vector field b is collinear with the mean curvature vector field H. Writing derivative formulas of the Frenet type for this frame field, we obtained 8 invariant functions, γ_1, γ_2, ν_1, ν_2, λ, μ, β_1, β_2, which determine the surface up to a rigid motion in $\mathbb{R}^4_1$.

The invariants γ_1, γ_2, ν_1, ν_2, λ, μ, β_1, and β_2 are determined by the geometric frame field $\{x, y, b, l\}$ as follows:

$$\nu_1 = \langle \nabla'_x x, b \rangle, \qquad \nu_2 = \langle \nabla'_y y, b \rangle, \qquad \lambda = \langle \nabla'_x y, b \rangle, \qquad \mu = \langle \nabla'_x y, l \rangle,$$

$$\gamma_1 = -y(\ln \sqrt{E}), \quad \gamma_2 = -x(\ln \sqrt{G}), \quad \beta_1 = \langle \nabla'_x b, l \rangle, \quad \beta_2 = \langle \nabla'_y b, l \rangle.$$

The invariants k and $\varkappa$ and the Gauss curvature K of M^2 are expressed by the functions $\nu_1, \nu_2, \lambda, \mu$ as follows:

$$k = -4\nu_1 \nu_2 \mu^2, \qquad \varkappa = (\nu_1 - \nu_2)\mu, \qquad K = \varepsilon(\nu_1 \nu_2 - \lambda^2 + \mu^2),$$

where $\varepsilon = sign\langle H, H \rangle$. The norm $\|H\|$ of the mean curvature vector is expressed as

$$\|H\| = \frac{|\nu_1 + \nu_2|}{2} = \frac{\sqrt{\varkappa^2 - k}}{2|\mu|}.$$

If M^2 is a spacelike surface whose mean curvature vector at any point is a nonzero spacelike vector or timelike vector, then M^2 is minimal if and only if $\nu_1 + \nu_2 = 0$.

The geometric meaning of the invariant λ is connected with the notion of Chen submanifolds. Let M be an n-dimensional submanifold of $(n+m)$-dimensional Riemannian manifold $\widetilde{M}$ and ξ be a normal vector field of M. Chen [1] defined the *allied vector field* $a(\xi)$ of ξ by the formula

$$a(\xi) = \frac{\|\xi\|}{n} \sum_{k=2}^{m} \{\mathrm{tr}(A_1 A_k)\} \xi_k,$$

where $\{\xi_1 = \dfrac{\xi}{\|\xi\|}, \xi_2, \dots, \xi_m\}$ is an orthonormal base of the normal space of M, and $A_i = A_{\xi_i}$, $i = 1, \dots, m$ is the shape operator with respect to ξ_i. The allied vector field $a(H)$ of the mean curvature vector field H is called the *allied mean curvature vector field* of M in $\widetilde{M}$. Chen defined the $\mathcal{A}$-submanifolds to be those submanifolds of $\widetilde{M}$ for which $a(H)$ vanishes identically [1]. In [7, 8], the $\mathcal{A}$-submanifolds are called *Chen submanifolds*. It is easy to see that minimal submanifolds, pseudoumbilical submanifolds, and hypersurfaces are Chen submanifolds. These Chen submanifolds are said to be trivial Chen-submanifolds. In [6] we showed that if M^2 is a spacelike surface in $\mathbb{R}^4_1$ with spacelike or timelike mean curvature vector field then the allied mean curvature vector field of M^2 is

$$a(H) = \frac{\sqrt{\varkappa^2 - k}}{2} \, \lambda \, l.$$

Hence, if M^2 is free of minimal points, then $a(H) = 0$ if and only if $\lambda = 0$. This gives the geometric meaning of the invariant λ: M^2 is a nontrivial Chen surface if and only if the invariant λ is zero.

3. Basic classes of general rotational surfaces

General rotational surfaces in the Euclidean 4-space $\mathbb{R}^4$ were introduced by Moore [14] as follows. Let $c : x(u) = \big(x^1(u), x^2(u), x^3(u), x^4(u)\big)$; $u \in J \subset \mathbb{R}$ be a smooth curve in $\mathbb{R}^4$, and let α, β be constants. A general rotation of the meridian curve c in $\mathbb{R}^4$ is defined by

$$X(u,v) = \big(X^1(u,v), X^2(u,v), X^3(u,v), X^4(u,v)\big),$$

where

$$\begin{aligned} X^1(u,v) &= x^1(u)\cos\alpha v - x^2(u)\sin\alpha v; & \qquad X^3(u,v) &= x^3(u)\cos\beta v - x^4(u)\sin\beta v;\\ X^2(u,v) &= x^1(u)\sin\alpha v + x^2(u)\cos\alpha v; & \qquad X^4(u,v) &= x^3(u)\sin\beta v + x^4(u)\cos\beta v. \end{aligned}$$

In the case of $\beta = 0$, $x^2(u) = 0$, the plane Oe_3e_4 is fixed and one gets the classical rotation about a fixed 2-dimensional axis.

In [13] we considered a special case of such surfaces, given by

$$\mathcal{M} : z(u,v) = \left(f(u)\cos\alpha v, f(u)\sin\alpha v, g(u)\cos\beta v, g(u)\sin\beta v\right), \tag{1}$$

where $u \in J \subset \mathbb{R}$, $v \in [0; 2\pi)$, $f(u)$ and $g(u)$ are smooth functions, satisfying $\alpha^2 f^2(u) + \beta^2 g^2(u) > 0$, $f'^{\,2}(u) + g'^{\,2}(u) > 0$, and α, β are positive constants. In the case of $\alpha \neq \beta$, each parametric curve $u = const$ is a curve in $\mathbb{R}^4$ with constant Frenet curvatures, and in the case of $\alpha = \beta$ each parametric curve $u = const$ is a circle. The parametric curves $v = const$ are plane curves with Frenet curvature $\dfrac{|g'f'' - f'g''|}{(\sqrt{f'^{\,2} + g'^{\,2}})^3}$. These curves are the meridians of $\mathcal{M}$.

The surfaces defined by (1) are general rotational surfaces in the sense of Moore with plane meridian curves. In [13] we found the invariants of these surfaces and completely classified the minimal superconformal general rotational surfaces in $\mathbb{R}^4$. The classification of the general rotational surfaces in $\mathbb{R}^4$ consisting of parabolic points is given in [4].

Similarly to the general rotations in $\mathbb{R}^4$, one can consider general rotational surfaces in the Minkowski 4-space $\mathbb{R}^4_1$ as follows. Let $c : x(u) = \big(x^1(u), x^2(u), x^3(u), x^4(u)\big)$; $u \in J \subset \mathbb{R}$ be a smooth spacelike or timelike curve in $\mathbb{R}^4_1$, and let α, β be constants. We consider the surface defined by

$$X(u,v) = \big(X^1(u,v), X^2(u,v), X^3(u,v), X^4(u,v)\big),$$

where

$$\begin{aligned} X^1(u,v) &= x^1(u)\cos\alpha v - x^2(u)\sin\alpha v; & \qquad X^3(u,v) &= x^3(u)\cosh\beta v + x^4(u)\sinh\beta v;\\ X^2(u,v) &= x^1(u)\sin\alpha v + x^2(u)\cos\alpha v; & \qquad X^4(u,v) &= x^3(u)\sinh\beta v + x^4(u)\cosh\beta v. \end{aligned}$$

In the case of $\beta = 0$, $x^2(u) = 0$ (or $x^1(u) = 0$) one gets the standard rotational surface of elliptic type in $\mathbb{R}^4_1$. A local classification of spacelike rotational surfaces of elliptic type, whose mean curvature vector field is either vanishing or lightlike, was obtained in [10].

In the case of $\alpha = 0$, $x^3(u) = 0$ we get the standard hyperbolic rotational surface of the first type, and in the case of $\alpha = 0$, $x^4(u) = 0$, we get the standard hyperbolic rotational surface of the second type. A local classification of spacelike rotational surfaces of hyperbolic type with either vanishing or lightlike mean curvature vector field is given in [9]. In [11] the timelike and spacelike hyperbolic rotational surfaces with nonzero constant mean curvature in the 3-dimensional de Sitter space $\mathbb{S}^3_1$ were classified. Similarly, a classification of the spacelike and timelike Weingarten rotational surfaces in $\mathbb{S}^3_1$ is given in [12]. In [5] we described the class of Chen spacelike rotational surfaces of hyperbolic or elliptic type in $\mathbb{R}^4_1$.

In the case of $\alpha > 0$ and $\beta > 0$, the surfaces defined above are analogous to the general rotational surfaces of Moore in $\mathbb{R}^4$.

In [6] we considered spacelike general rotational surfaces with plane meridians in the Minkowski space $\mathbb{R}^4_1$ and found their invariant functions. Here we shall describe and classify some basic geometric classes of such surfaces.

Let $\mathcal{M}_1$ be the surface parameterized by

$$\mathcal{M}_1 : z(u,v) = (f(u)\cos\alpha v, f(u)\sin\alpha v, g(u)\cosh\beta v, g(u)\sinh\beta v)\,, \tag{2}$$

where $u \in J \subset \mathbb{R}$, $v \in [0; 2\pi)$, $f(u)$ and $g(u)$ are smooth functions, satisfying $\alpha^2 f^2(u) - \beta^2 g^2(u) > 0$, $f'^{\,2}(u) + g'^{\,2}(u) > 0$, and α, β are positive constants.

The coefficients of the first fundamental form of $\mathcal{M}_1$ are $E = f'^{\,2}(u) + g'^{\,2}(u)$; $F = 0$; $G = \alpha^2 f^2(u) - \beta^2 g^2(u)$. $\mathcal{M}_1$ is a spacelike surface whose mean curvature vector at any point is a nonzero spacelike vector (see [6]). Moreover, $\mathcal{M}_1$ is parameterized by principal parameters (u,v).

The invariants k, $\varkappa$, and K of $\mathcal{M}_1$ are expressed by the functions $f(u)$, $g(u)$, and their derivatives, as follows:

$$k = \frac{4\alpha^2\beta^2(gf' - fg')^2(g'f'' - f'g'')(\alpha^2 fg' + \beta^2 gf')}{(f'^{\,2} + g'^{\,2})^3(\alpha^2 f^2 - \beta^2 g^2)^3}; \tag{3}$$

$$\varkappa = \frac{\alpha\beta(gf' - fg')[(\alpha^2 f^2 - \beta^2 g^2)(g'f'' - f'g'') + (f'^{\,2} + g'^{\,2})(\alpha^2 fg' + \beta^2 gf')]}{(f'^{\,2} + g'^{\,2})^2(\alpha^2 f^2 - \beta^2 g^2)^2}; \tag{4}$$

$$K = \frac{-(\alpha^2 f^2 - \beta^2 g^2)(\alpha^2 fg' + \beta^2 gf')(g'f'' - f'g'') + \alpha^2\beta^2(f'^{\,2} + g'^{\,2})(gf' - fg')^2}{(f'^{\,2} + g'^{\,2})^2(\alpha^2 f^2 - \beta^2 g^2)^2}\,. \tag{5}$$

The geometric invariant functions γ_1, γ_2, ν_1, ν_2, λ, μ, β_1, β_2 of $\mathcal{M}_1$ are:

$$\begin{array}{ll}
\gamma_1 = 0; & \gamma_2 = -\dfrac{\alpha^2 ff' - \beta^2 gg'}{\sqrt{f'^{\,2} + g'^{\,2}}(\alpha^2 f^2 - \beta^2 g^2)}; \\[2ex]
\nu_1 = \dfrac{g'f'' - f'g''}{(f'^{\,2} + g'^{\,2})^{\frac{3}{2}}}; & \nu_2 = -\dfrac{\alpha^2 fg' + \beta^2 gf'}{\sqrt{f'^{\,2} + g'^{\,2}}(\alpha^2 f^2 - \beta^2 g^2)}; \\[2ex]
\lambda = 0; & \mu = \dfrac{\alpha\beta(gf' - fg')}{\sqrt{f'^{\,2} + g'^{\,2}}(\alpha^2 f^2 - \beta^2 g^2)}; \\[2ex]
\beta_1 = 0; & \beta_2 = \dfrac{\alpha\beta(ff' + gg')}{\sqrt{f'^{\,2} + g'^{\,2}}(\alpha^2 f^2 - \beta^2 g^2)}.
\end{array} \tag{6}$$

In a similar way, we can consider the surface $\mathcal{M}_2$ parameterized by

$$\mathcal{M}_2 : z(u,v) = (f(u)\cos\alpha v, f(u)\sin\alpha v, g(u)\sinh\beta v, g(u)\cosh\beta v)\,, \tag{7}$$

where $u \in J$, $v \in [0;2\pi)$, $f(u)$ and $g(u)$ are smooth functions, satisfying $f'^{\,2}(u) - g'^{\,2}(u) > 0$, $\alpha^2 f^2(u) + \beta^2 g^2(u) > 0$, and α, β are positive constants.

The coefficients of the first fundamental form of $\mathcal{M}_2$ are $E = f'^{\,2}(u) - g'^{\,2}(u)$; $F = 0$; $G = \alpha^2 f^2(u) + \beta^2 g^2(u)$. $\mathcal{M}_2$ is a spacelike surface with timelike mean curvature vector field. The parameters (u,v) of $\mathcal{M}_2$ are principal.

The invariants k, $\varkappa$, and K of $\mathcal{M}_2$ are expressed similarly to the invariants of $\mathcal{M}_1$ [6]:

$$k = \frac{4\alpha^2\beta^2(gf' - fg')^2(g'f'' - f'g'')(\alpha^2 fg' + \beta^2 gf')}{(f'^{\,2} - g'^{\,2})^3(\alpha^2 f^2 + \beta^2 g^2)^3}; \tag{8}$$

$$\varkappa = \frac{\alpha\beta(gf' - fg')[(\alpha^2 f^2 + \beta^2 g^2)(g'f'' - f'g'') + (f'^{\,2} - g'^{\,2})(\alpha^2 fg' + \beta^2 gf')]}{(f'^{\,2} - g'^{\,2})^2(\alpha^2 f^2 + \beta^2 g^2)^2}; \tag{9}$$

$$K = \frac{(\alpha^2 f^2 + \beta^2 g^2)(\alpha^2 fg' + \beta^2 gf')(g'f'' - f'g'') - \alpha^2\beta^2(f'^{\,2} - g'^{\,2})(gf' - fg')^2}{(f'^{\,2} - g'^{\,2})^2(\alpha^2 f^2 + \beta^2 g^2)^2}\,. \tag{10}$$

The geometric invariant functions of $\mathcal{M}_2$ are given below:

$$\begin{array}{ll}
\gamma_1 = 0; & \gamma_2 = -\dfrac{\alpha^2 ff' + \beta^2 gg'}{\sqrt{f'^{\,2} - g'^{\,2}}(\alpha^2 f^2 + \beta^2 g^2)}; \\
\nu_1 = \dfrac{g'f'' - f'g''}{(f'^{\,2} - g'^{\,2})^{\frac{3}{2}}}; & \nu_2 = -\dfrac{\alpha^2 fg' + \beta^2 gf'}{\sqrt{f'^{\,2} - g'^{\,2}}(\alpha^2 f^2 + \beta^2 g^2)}; \\
\lambda = 0; & \mu = \dfrac{\alpha\beta(fg' - gf')}{\sqrt{f'^{\,2} - g'^{\,2}}(\alpha^2 f^2 + \beta^2 g^2)}; \\
\beta_1 = 0; & \beta_2 = \dfrac{\alpha\beta(gg' - ff')}{\sqrt{f'^{\,2} - g'^{\,2}}(\alpha^2 f^2 + \beta^2 g^2)}.
\end{array} \tag{11}$$

We shall call the general rotational surface $\mathcal{M}_1$, defined by (2), a *general rotational surface of first type*, and the general rotational surface $\mathcal{M}_2$, defined by (7), a *general rotational surface of second type.*

Note that the invariant λ of the general rotational surfaces of first or second type is zero. Hence, the following statement holds.

Theorem 3.1 *The general rotational surfaces of the first or second type, free of minimal points, are nontrivial Chen surfaces.*

In the following subsections we shall describe the classes of flat general rotational surfaces, general rotational surfaces with flat normal connection, general rotational surfaces consisting of parabolic points, and minimal general rotational surfaces.

3.1. Flat general rotational surfaces

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be general rotational surfaces of the first and second type, respectively. Recall that a surface is called *flat* if the Gauss curvature K is zero. Using equalities (5) and (10), we obtain the following:

Theorem 3.2 *(i) The general rotational surface of the first type is flat if and only if*

$$\alpha^2\beta^2(f'^{\,2}+g'^{\,2})(gf'-fg')^2 = (\alpha^2 f^2 - \beta^2 g^2)(\alpha^2 fg' + \beta^2 gf')(g'f''-f'g''). \tag{12}$$

(ii) The general rotational surface of the second type is flat if and only if

$$\alpha^2\beta^2(f'^{\,2}-g'^{\,2})(gf'-fg')^2 = (\alpha^2 f^2 + \beta^2 g^2)(\alpha^2 fg' + \beta^2 gf')(g'f''-f'g''). \tag{13}$$

Assume that the meridian curve is parameterized by $f = f(u);\ g = u$. Then equation (12) takes the form of

$$\frac{f''}{1+f'^{\,2}} = \frac{\alpha^2\beta^2(uf'-f)^2}{(\alpha^2 f^2 - \beta^2 u^2)(\alpha^2 f + \beta^2 uf')},$$

which is equivalent to

$$(\arctan f')' = \frac{\alpha^2\beta^2(uf'-f)^2}{(\alpha^2 f^2 - \beta^2 u^2)(\alpha^2 f + \beta^2 uf')}. \tag{14}$$

Similarly, equation (13) takes the form of

$$\frac{f''}{1-f'^{\,2}} = \frac{-\alpha^2\beta^2(uf'-f)^2}{(\alpha^2 f^2 + \beta^2 u^2)(\alpha^2 f + \beta^2 uf')},$$

which is equivalent to

$$\left(\ln\left|\frac{1+f'}{1-f'}\right|\right)' = \frac{-2\alpha^2\beta^2(uf'-f)^2}{(\alpha^2 f^2 + \beta^2 u^2)(\alpha^2 f + \beta^2 uf')}. \tag{15}$$

Equations (14) and (15) describe analytically the class of flat general rotational surfaces of the first and second type.

3.2. General rotational surfaces with flat normal connection

A surface is said to have *flat normal connection* if the curvature of the normal connection is zero. The curvature of the normal connection of the general rotational surface $\mathcal{M}_1$ (resp. $\mathcal{M}_2$) is given by formula (4) [resp. (9)]. Using these formulas, we obtain the next theorem.

Theorem 3.3 *(i) The general rotational surface of the first type has flat normal connection if and only if*

$$\frac{g'f''-f'g''}{f'^{\,2}+g'^{\,2}} = -\frac{\alpha^2 fg' + \beta^2 gf'}{\alpha^2 f^2 - \beta^2 g^2}. \tag{16}$$

(ii) The general rotational surface of the second type has flat normal connection if and only if

$$\frac{g'f''-f'g''}{f'^{\,2}-g'^{\,2}} = -\frac{\alpha^2 fg' + \beta^2 gf'}{\alpha^2 f^2 + \beta^2 g^2}. \tag{17}$$

If we assume that the meridian curve is parameterized by $f = f(u);\ g = u$, then equation (16) takes the form of

$$(\arctan f')' = -\frac{\alpha^2 f + \beta^2 u f'}{\alpha^2 f^2 - \beta^2 u^2}. \tag{18}$$

Similarly, equation (17) takes the form of

$$\left(\ln\left|\frac{1+f'}{1-f'}\right|\right)' = \frac{2(\alpha^2 f + \beta^2 u f')}{\alpha^2 f^2 + \beta^2 u^2}. \tag{19}$$

The class of general rotational surfaces with flat normal connection is described analytically by equations (18) and (19).

Example 1 Let $f(u) = a\cos u,\ g(u) = a\sin u,\ a = const\ (a \neq 0)$. A direct computation shows that equation (16) is fulfilled. Hence, the surface parameterized by

$$z(u,v) = (a\cos u\cos\alpha v, a\cos u\sin\alpha v, a\sin u\cosh\beta v, a\sin u\sinh\beta v)$$

is a spacelike general rotational surface of the first type with flat normal connection.

In the special case when $a = 1,\ \alpha = \beta = 1$, we obtain a spacelike surface lying on the de Sitter space $S^3_1 = \{x \in \mathbb{R}^4_1; \langle x, x\rangle = 1\}$.

Example 2 If we choose $f(u) = a\sinh u,\ g(u) = a\cosh u,\ a = const\ (a \neq 0)$, by a direct computation we obtain that equation (17) is fulfilled. Hence, the surface parameterized by

$$z(u,v) = (a\sinh u\cos\alpha v, a\sinh u\sin\alpha v, a\cosh u\sinh\beta v, a\cosh u\cosh\beta v)$$

is a spacelike general rotational surface of the second type with flat normal connection.

In the special case when $a = 1,\ \alpha = \beta = 1$, we obtain a spacelike surface lying on the unit hyperbolic sphere $H^3_1 = \{x \in \mathbb{R}^4_1; \langle x, x\rangle = -1\}$.

3.3. General rotational surfaces consisting of parabolic points

Recall that surfaces consisting of parabolic points are characterized by the condition $k = 0$. The next theorem classifies the general rotational surfaces of the first and second type with $k = 0$.

Theorem 3.4 *A general rotational surface of first or second type consists of parabolic points if and only if it is one of the following:*

(i) a developable ruled surface in $\mathbb{R}^4_1$*;*

(ii) a nondevelopable ruled surface in $\mathbb{R}^4_1$*;*

(iii) a nonruled surface in $\mathbb{R}^4_1$ *whose meridian curve is given by* $g = c\, f^{-\frac{\beta^2}{\alpha^2}}$*, where* $c = const \neq 0$*.*

Proof Consider a general rotational surface of first or second type. Equality (3) [or (8)] implies that $k = 0$ if and only if

$$(gf' - fg')(g'f'' - f'g'')(\alpha^2 fg' + \beta^2 gf') = 0. \tag{20}$$

It follows from equality (20) that the invariant k is zero in the following 3 cases:

1. $gf' - fg' = 0$, i.e. $g = af$, where $a = const \neq 0$. In this case $k = \varkappa = K = 0$, and by a result in [6] the corresponding general rotational surface (of first or second type) is a developable ruled surface in $\mathbb{R}^4_1$.

2. $g'f'' - f'g'' = 0$, i.e. $g = af + b$, where $a = const \neq 0, b = const \neq 0$. Hence, the meridians are straight lines. It can easily be seen that in this case $\varkappa \neq 0$. Consequently, the corresponding general rotational surface is a nondevelopable ruled surface in $\mathbb{R}^4_1$.

3. $\alpha^2 f g' + \beta^2 g f' = 0$, i.e. $\alpha^2 \dfrac{g'}{g} + \beta^2 \dfrac{f'}{f} = 0$. Integrating the last equality we obtain $g = c\, f^{-\frac{\beta^2}{\alpha^2}}$, where $c = const \neq 0$. In this case the meridians are not straight lines. The invariants $\varkappa$ and K are nonzero, and, hence, the corresponding general rotational surface is a nonruled surface in $\mathbb{R}^4_1$. □

3.4. Minimal general rotational surfaces

In this subsection we shall find all minimal general rotational surfaces of first and second type. Recall that a surface is minimal if and only if $\nu_1 + \nu_2 = 0$. Hence, using (6) we get that the general rotational surface of the first type is minimal if and only if the functions $f(u)$ and $g(u)$ satisfy the following equality:

$$\frac{g'f'' - f'g''}{f'^{\,2} + g'^{\,2}} = \frac{\alpha^2 f g' + \beta^2 g f'}{\alpha^2 f^2 - \beta^2 g^2}. \tag{21}$$

Similarly, from (11) it follows that the general rotational surface of the second type is minimal if and only if

$$\frac{g'f'' - f'g''}{f'^{\,2} - g'^{\,2}} = \frac{\alpha^2 f g' + \beta^2 g f'}{\alpha^2 f^2 + \beta^2 g^2}. \tag{22}$$

We shall find the solutions of equalities (21) and (22). In such a way we shall describe the class of minimal general rotational surfaces of first and second type.

Theorem 3.5 *(i) The general rotational surface of the first type is minimal if and only if the meridian curve is given by the formula*

$$g = \frac{\sqrt{A}}{\beta} \sin\left(\varepsilon \frac{\beta}{\alpha} \ln\left|\alpha f + \sqrt{\alpha^2 f^2 - A}\right| + C\right), \qquad C = const, \;\; A = const > 0.$$

(ii) The general rotational surface of the second type is minimal if and only if the meridian curve is given by the formula

$$g = \frac{\sqrt{A}}{\beta} \sin\left(\varepsilon \frac{\beta}{\alpha} \ln\left|\alpha f + \sqrt{\alpha^2 f^2 + A}\right| + C\right), \qquad C = const, \;\; A = const > 0.$$

Proof First, we shall simplify equalities (21) and (22). Using that the connection ∇' of $\mathbb{R}^4_1$ is flat, from $R'(x,y,x) = 0$ and $R'(x,y,y) = 0$ we obtain that the invariants of each spacelike surface in $\mathbb{R}^4_1$ satisfy the following equalities (see [6]):

$$\begin{aligned} &2\mu\,\gamma_2 + \nu_1\,\beta_2 - \lambda\,\beta_1 = x(\mu), \\ &2\lambda\,\gamma_1 - \mu\,\beta_2 + (\nu_1 - \nu_2)\,\gamma_2 = -x(\nu_2) + y(\lambda). \end{aligned} \tag{23}$$

In the case that the surface is a minimal general rotational surface of first or second type, we have $\lambda = 0$, $\gamma_1 = 0$, $\beta_1 = 0$, and $\nu_2 = -\nu_1$. Hence, from (23) it follows that

$$\begin{aligned} 2\mu\,\gamma_2 + \nu_1\,\beta_2 &= x(\mu), \\ 2\nu_1\,\gamma_2 - \mu\,\beta_2 &= x(\nu_1), \end{aligned}$$

which implies $\gamma_2 = \dfrac{1}{4}\,x\left(\ln(\mu^2+\nu_1^2)\right)$. On the other hand, $\gamma_2 = -x\left(\ln\sqrt{G}\right)$. Hence, we get $\dfrac{1}{4}\,x\left(\ln(\mu^2+\nu_1^2)\right) + x\left(\ln\sqrt{G}\right)) = 0$, which implies

$$x\left(G^2(\mu^2+\nu_1^2)\right) = 0.$$

Now, using that μ, ν_1, and G are functions depending only on the parameter u, we obtain

$$G^2(\mu^2+\nu_1^2) = c^2, \tag{24}$$

where c is a constant.

Now let $\mathcal{M}_1$ be a minimal general rotational surface of the first type. Then $G = \alpha^2 f^2 - \beta^2 g^2$, and using (6) and (24) we obtain

$$\frac{\alpha^2\beta^2(gf'-fg')^2 + (\alpha^2 fg' + \beta^2 gf')^2}{f'^{\,2}+g'^{\,2}} = c^2,$$

which is equivalent to

$$\frac{\alpha^2 f^2 g'^{\,2} + \beta^2 g^2 f'^{\,2}}{f'^{\,2}+g'^{\,2}} = \frac{c^2}{\alpha^2+\beta^2}. \tag{25}$$

Equality (25) can also be obtained from (21) by a direct but very long computation.

Without loss of generality we assume that $f'^{\,2} + g'^{\,2} = 1$. From (25) we then get

$$\alpha^2 f^2 g'^{\,2} + \beta^2 g^2 f'^{\,2} = \frac{c^2}{\alpha^2+\beta^2}. \tag{26}$$

Denote $A = \dfrac{c^2}{\alpha^2+\beta^2}$. Now, using that $g'^{\,2} = 1 - f'^{\,2}$, from (26) it follows that

$$f'^{\,2} = \frac{\alpha^2 f^2 - A}{\alpha^2 f^2 - \beta^2 g^2}; \qquad g'^{\,2} = \frac{A - \beta^2 g^2}{\alpha^2 f^2 - \beta^2 g^2}. \tag{27}$$

Note that the constant A satisfies $\beta^2 g^2 < A < \alpha^2 f^2$, since $\alpha^2 f^2 - \beta^2 g^2 > 0$.

Equalities (27) imply $(A - \beta^2 g^2) f'^{\,2} = (\alpha^2 f^2 - A) g'^{\,2}$, i.e.

$$\frac{f'}{\sqrt{\alpha^2 f^2 - A}} = \varepsilon \frac{g'}{\sqrt{A - \beta^2 g^2}}, \qquad \varepsilon = \pm 1.$$

Integrating the last equality, we obtain

$$\int \frac{df}{\sqrt{\alpha^2 f^2 - A}} = \varepsilon \int \frac{dg}{\sqrt{A - \beta^2 g^2}}.$$

Calculating the integrals, we get

$$\arcsin \frac{\beta g}{\sqrt{A}} = \varepsilon \frac{\beta}{\alpha} \ln \left| \alpha f + \sqrt{\alpha^2 f^2 - A} \right| + C, \qquad C = const.$$

Consequently, in the case of a minimal general rotational surface of the first type, the meridian curve is given by the following formula:

$$g = \frac{\sqrt{A}}{\beta} \sin \left(\varepsilon \frac{\beta}{\alpha} \ln \left| \alpha f + \sqrt{\alpha^2 f^2 - A} \right| + C \right). \tag{28}$$

Conversely, if the meridian curve is defined by formula (28), by a straightforward computation we obtain that equality (21) is fulfilled and, hence, the general rotational surface of the first type is minimal.

In the case of a minimal general rotational surface of the second type we have $G = \alpha^2 f^2 + \beta^2 g^2$. Then (11) and (24) imply

$$\frac{\alpha^2 f^2 g'^{\,2} + \beta^2 g^2 f'^{\,2}}{f'^{\,2} - g'^{\,2}} = \frac{c^2}{\alpha^2 + \beta^2}. \tag{29}$$

Without loss of generality we assume that $f'^{\,2} - g'^{\,2} = 1$. From (29) we then get

$$\alpha^2 f^2 g'^{\,2} + \beta^2 g^2 f'^{\,2} = A, \tag{30}$$

where $A = \dfrac{c^2}{\alpha^2 + \beta^2}$. Using that $g'^{\,2} = f'^{\,2} - 1$, from (30) we obtain

$$f'^{\,2} = \frac{A + \alpha^2 f^2}{\alpha^2 f^2 + \beta^2 g^2}; \qquad g'^{\,2} = \frac{A - \beta^2 g^2}{\alpha^2 f^2 + \beta^2 g^2}. \tag{31}$$

Note that in this case $A > \beta^2 g^2$, since $g'^{\,2} > 0$.

Equalities (31) imply

$$\frac{f'}{\sqrt{A + \alpha^2 f^2}} = \varepsilon \frac{g'}{\sqrt{A - \beta^2 g^2}}, \qquad \varepsilon = \pm 1.$$

After integration we obtain

$$\arcsin \frac{\beta g}{\sqrt{A}} = \varepsilon \frac{\beta}{\alpha} \ln \left| \alpha f + \sqrt{A + \alpha^2 f^2} \right| + C, \qquad C = const.$$

Consequently, in the case of a minimal general rotational surface of the second type, the meridian curve is given by

$$g = \frac{\sqrt{A}}{\beta} \sin \left(\varepsilon \frac{\beta}{\alpha} \ln \left| \alpha f + \sqrt{A + \alpha^2 f^2} \right| + C \right). \tag{32}$$

A direct computation shows that if the meridian curve is given by formula (32), then equality (22) is satisfied. Hence, the general rotational surface of the second type is minimal. □

Finally, it should be mentioned that the classes of minimal general rotational surfaces and general rotational surfaces consisting of parabolic points are found explicitly. The classes of flat general rotational surfaces and general rotational surfaces with flat normal connection are described analytically by ordinary differential equations. An open question is to find them explicitly.

References

[1] Chen BY. Geometry of Submanifolds. New York, NY, USA: Marcel Dekker, 1973.

[2] Ganchev G, Milousheva V. On the theory of surfaces in the four-dimensional Euclidean space. Kodai Math J 2008; 31: 183–198.

[3] Ganchev G, Milousheva, V. Invariants and Bonnet-type theorem for surfaces in $\mathbb{R}^4$. Cent Eur J Math 2010; 8: 993–1008.

[4] Ganchev G, Milousheva V. Invariants of lines on surfaces in $\mathbb{R}^4$. C R Acad Bulgare Sci 2010; 63: 835–842.

[5] Ganchev G, Milousheva V. Chen rotational surfaces of hyperbolic or elliptic type in the four-dimensional Minkowski space. C R Acad Bulgare Sci 2011; 64: 641–652.

[6] Ganchev G, Milousheva V. An invariant theory of spacelike surfaces in the four-dimensional Minkowski space. Mediterr J Math 2012; 9: 267–294.

[7] Gheysens L, Verheyen P, Verstraelen L. Sur les surfaces $\mathcal{A}$ ou les surfaces de Chen. C R Acad Sci Paris 1981; 292: 913–916 (in French).

[8] Gheysens L, Verheyen P, Verstraelen L. Characterization and examples of Chen submanifolds. J Geom 1983; 20: 47–62.

[9] Haesen S, Ortega M. Boost invariant marginally trapped surfaces in Minkowski 4-space. Classical Quant Grav 2007; 24: 5441–5452.

[10] Haesen S, Ortega M. Marginally trapped surfaces in Minkowski 4-space invariant under a rotational subgroup of the Lorentz group. Gen Relat Gravit 2009; 41: 1819–1834.

[11] Liu H, Liu G. Hyperbolic rotation surfaces of constant mean curvature in 3-de Sitter space. Bull Belg Math Soc Simon Stevin 2000; 7: 455–466.

[12] Liu H, Liu G. Weingarten rotation surfaces in 3-dimensional de Sitter space. J Geom 2004; 79: 156–168.

[13] Milousheva V. General rotational surfaces in $\mathbb{R}^4$ with meridians lying in two-dimensional planes. C R Acad Bulgare Sci 2010; 63: 339–348.

[14] Moore C. Surfaces of rotation in a space of four dimensions. Ann Math 1919; 21: 81–93.

[15] Moore C. Rotation surfaces of constant curvature in space of four dimensions. Bull Amer Math Soc 1920; 26: 454–460.