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On a generalization of Kelly's combinatorial lemma

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Abstract: Kelly's combinatorial lemma is a basic tool in the study of Ulam's reconstruction conjecture. A generalization in terms of a family of t-elements subsets of a v-element set was given by Pouzet. We consider a version of this generalization modulo a prime p. We give illustrations to graphs and tournaments.

Key words: Set, matrix, graph, tournament, isomorphism

1. Introduction

Kelly's combinatorial lemma [24] is the assertion that the number $s(F,G)$ of induced subgraphs of a given graph G, isomorphic to F, is determined by the deck of G, provided that $|V(F)| < |V(G)|$, namely $s(F,G) = \frac{1}{|V(G)|-|V(F)|}\sum_{x\in V(G)} s(F,G_{-x})$ (where G_{-x} is the graph induced by G on $V(G)\setminus\{x\}$).

In terms of a family $\mathcal{F}$ of t-element subsets of a v-element set, it simply says that $|\mathcal{F}| = \frac{1}{v-t}\sum_{x\in V(G)} |\mathcal{F}_{-x}|$ where $\mathcal{F}_{-x} := \mathcal{F}\cap [V(G)\setminus\{x\}]^t$.

For sets U, T, we put $U(T) := \{F : T\subseteq F \in U\}$ and $U_{\restriction K} := U\cap \mathfrak{P}(K)$ (where $\mathfrak{P}(K)$ is the set of subsets of K) so that $U_{\restriction K}(T) := \{F : T\subseteq F\subseteq K, F\in U\}$ and $e(U) := |U|$. Pouzet [31, 32] gave the following extension of this result.

Lemma 1.1 *(M.Pouzet [31]) Let t and r be integers, V be a set of size $v \geq t+r$ elements, and U and U' be sets of t-element subsets T of V. If for every subset K of $k = t+r$ elements of V, $e(U_{\restriction K}) = e(U'_{\restriction K})$, then for all finite subsets T' and K' of V, such that T' is contained in K' and $K'\setminus T'$ has at least $t+r$ elements, $e(U_{\restriction K'}(T')) = e(U'_{\restriction K'}(T'))$.*

In particular, if $|V| \geq 2t+r = t+k$, we have this particular version of the combinatorial lemma of Pouzet:

Lemma 1.2 *(M.Pouzet [31]) Let $v, t,$ and k be integers, $k\leq v$, V be a set of v elements with $t\leq min(k, v-k)$, and U and U' be sets of t-element subsets T of V. If for every k-element subset K of V, $e(U_{\restriction K}) = e(U'_{\restriction K})$, then $U = U'$.*

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Here we consider the case where $e(U_{\restriction K}) \equiv e(U'_{\restriction K})$ modulo a prime p for every k-element subset K of V; our main result, Theorem 1.3, is then a version, modulo a prime p, of the particular version of the combinatorial lemma of Pouzet.

Kelly's combinatorial lemma is a basic tool in the study of Ulam's reconstruction conjecture. Pouzet's combinatorial lemma has been used several times in reconstruction problems (see for example [1, 5, 6, 7, 11, 12]). Pouzet gave a proof of his lemma via a counting argument [32] and later by using linear algebra (related to incidence matrices) [31] (the paper was published earlier).

Let n, p be positive integers, the decomposition of $n = \sum_{i=0}^{n(p)} n_i p^i$ in the basis p is also denoted $[n_0, n_1, \ldots, n_{n(p)}]_p$ where $n_{n(p)} \neq 0$ if and only if $n \neq 0$.

Theorem 1.3 *Let p be a prime number. Let v, t, and k be nonnegative integers, $k \leq v$, $k = [k_0, k_1, \ldots, k_{k(p)}]_p$, $t = [t_0, t_1, \ldots, t_{t(p)}]_p$. Let V be a set of v elements with $t \leq min(k, v-k)$, and U and U' be sets of t-element subsets T of V. We assume that $e(U_{\restriction K}) \equiv e(U'_{\restriction K})$ modulo a prime p for every k-element subset K of V.*

1) If $k_i = t_i$ for all $i < t(p)$ and $k_{t(p)} \geq t_{t(p)}$, then $U = U'$.

2) If $t = t_{t(p)} p^{t(p)}$ and $k = \sum_{i=t(p)+1}^{k(p)} k_i p^i$, we have $U = U'$, or one of the sets U, U' is the set of all t-element subsets of V and the other is empty, or (whenever $p = 2$) for all t-element subsets T of V, $T \in U$ if and only if $T \notin U'$.

We prove Theorem 1.3 in Section 3. We use Wilson's theorem (Theorem 2.2) on incidence matrices.

In a reconstruction problem of graphs up to complementation [13], Wilson's theorem yielded the following result:

Theorem 1.4 *([13]) Let k be an integer, $2 \leq k \leq v-2$, $k \equiv 0$ (mod 4). Let G and G' be 2 graphs on the same set V of v vertices (possibly infinite). We assume that $e(G_{\restriction K})$ has the same parity as $e(G'_{\restriction K})$ for all k-element subsets K of V. Then $G' = G$ or $G' = \overline{G}$.*

Here we look for similar results whenever $e(G_{\restriction K}) \equiv e(G'_{\restriction K})$ modulo a prime p. As an illustration of Theorem 1.3, we obtain the following result.

Theorem 1.5 *Let G and G' be 2 graphs on the same set V of v vertices (possibly infinite). Let p be a prime number and k be an integer, $2 \leq k \leq v-2$. We assume that for all k-element subsets K of V, $e(G_{\restriction K}) \equiv e(G'_{\restriction K})$ (mod p).*

1) If $p \geq 3$, $k \not\equiv 0, 1$ (mod p), then $G' = G$.

2) If $p \geq 3$, $k \equiv 0$ (mod p), then $G' = G$, or one of the graphs G, G' is the complete graph and the other is the empty graph.

3) If $p = 2$, $k \equiv 2$ (mod 4), then $G' = G$.

We give other illustrations of Theorem 1.3, to graphs in section 4 and to tournaments in section 5.

2. Incidence matrices

We consider the matrix $W_{t\,k}$ defined as follows: Let V be a finite set, with v elements. Given nonnegative integers t, k with $t \leq k \leq v$, let $W_{t\,k}$ be the $\binom{v}{t}$ by $\binom{v}{k}$ matrix of 0's and 1's, the rows of which are indexed by the t-element subsets T of V, the columns are indexed by the k-element subsets K of V, and where the entry $W_{t\,k}(T,K)$ is 1 if $T \subseteq K$ and is 0 otherwise. The matrix transpose of $W_{t\,k}$ is denoted ${}^tW_{t\,k}$.

We say that a matrix D is a *diagonal form* for a matrix M when D is diagonal and there exist unimodular matrices (square integral matrices that have integral inverses) E and F such that $D = EMF$. We do not require that M and D are square; here "diagonal" just means that the (i,j) entry of D is 0 if $i \neq j$. A fundamental result, due to R.M. Wilson [36], is the following.

Theorem 2.1 *(R.M. Wilson [36]) For $t \leq min(k, v-k)$, $W_{t\,k}$ has as a diagonal form the $\binom{v}{t} \times \binom{v}{k}$ diagonal matrix with diagonal entries*

$$\binom{k-i}{t-i} \text{ with multiplicity } \binom{v}{i} - \binom{v}{i-1}, \quad i = 0, 1, \ldots, t.$$

In this statement and in Theorem 2.2, $\binom{v}{-1}$ should be interpreted as zero.

Denote $rank_{\mathbb{Q}}W_{t\,k}$ the rank of $W_{t\,k}$ over the field $\mathbb{Q}$ of rational numbers, resp. $rank_pW_{t\,k}$ the rank of $W_{t\,k}$ over the p-element field $\mathbb{F}_p$; similarly denote $Ker_{\mathbb{Q}}W_{t\,k}$, $Ker_pW_{t\,k}$ the corresponding kernels. Clearly from Theorem 2.1, $rank_{\mathbb{Q}}W_{t\,k} = \binom{v}{t}$. This yields Theorem 2.3 below due to D.H. Gottlieb [20] and independently W. Kantor [22]. On the other hand, from Theorem 2.1 follows $rank_pW_{t\,k}$, as given by Theorem 2.2.

Theorem 2.2 *(R.M. Wilson [36]) For $t \leq min(k, v-k)$, the rank of $W_{t\,k}$ modulo a prime p is*

$$\sum \binom{v}{i} - \binom{v}{i-1}$$

where the sum is extended over those indices i, $0 \leq i \leq t$, such that p does not divide the binomial coefficient $\binom{k-i}{t-i}$.

This yields Theorem 2.3 below due to D.H. Gottlieb [20], and independently W. Kantor [22]. A simpler proof of Theorem 2.2 was obtained by P. Frankl [17]. Applications of Wilson's theorem and its version modulo p have been considered by various authors, notably Chung and Graham [10] and Dammak et al. [13].

Theorem 2.3 *(D.H. Gottlieb [20], W. Kantor [22]) For $t \leq min(k, v-k)$, $W_{t\,k}$ has full row rank over the field $\mathbb{Q}$ of rational numbers.*

It is clear that $t \leq min(k, v-k)$ implies $\binom{v}{t} \leq \binom{v}{k}$. Thus, from Theorem 2.3, we have the following result:

Corollary 2.4 *(W. Kantor [22]) For $t \leq min(k, v-k)$, $rank_{\mathbb{Q}}W_{t\,k} = \binom{v}{t}$ and thus $Ker_{\mathbb{Q}}({}^tW_{t\,k}) = \{0\}$.*

If $k := v - t$ then, up to a relabelling, $W_{t\,k}$ is the adjacency matrix $A_{t,v}$ of the *Kneser graph* $KG(t,v)$ [19], a graph whose vertices are the t-element subsets of V, 2 subsets forming an edge if they are disjoint. The eigenvalues of Kneser graphs are computed in [19] (Theorem 9.4.3, page 200), and thus an equivalent form of Theorem 2.3 is:

Theorem 2.5 *$A_{t,v}$ is nonsingular for $t \leq \frac{v}{2}$.*

We characterize values of t and k so that $dim\ Ker_p({}^tW_{t\ k}) \in \{0,1\}$ and give a basis of $Ker_p({}^tW_{t\ k})$ that appears in the following result.

Theorem 2.6 *Let p be a prime number. Let v, t, and k be nonnegative integers, $k \leq v$, $k = [k_0, k_1, \ldots, k_{k(p)}]_p$, $t = [t_0, t_1, \ldots, t_{t(p)}]_p$, $t \leq min(k, v-k)$. We have:*

1) $k_j = t_j$ for all $j < t(p)$ and $k_{t(p)} \geq t_{t(p)}$ if and only if $Ker_p({}^tW_{t\ k}) = \{0\}$.

2) $t = t_{t(p)}p^{t(p)}$ and $k = \sum_{i=t(p)+1}^{k(p)} k_i p^i$ if and only if $\dim Ker_p({}^tW_{t\ k}) = 1$ and $\{(1,1,\cdots,1)\}$ is a basis of $Ker_p({}^tW_{t\ k})$.

The proof of Theorem 2.6 uses Lucas's theorem. The notation $a \mid b$ (resp. $a \nmid b$) means a divides b (resp. a does not divide b).

Theorem 2.7 *(Lucas's theorem [29]) Let p be a prime number, t, k be positive integers, $t \leq k$, $t = [t_0, t_1, \ldots, t_{t(p)}]_p$ and $k = [k_0, k_1, \ldots, k_{k(p)}]_p$. Then*

$$\binom{k}{t} = \prod_{i=0}^{t(p)} \binom{k_i}{t_i}\ (mod\ p),\ \ where\ \binom{k_i}{t_i} = 0\ if\ t_i > k_i.$$

For an elementary proof of Theorem 2.7, see Fine [15]. As a consequence of Theorem 2.7, we have the following result, which is very useful in this paper.

Corollary 2.8 *Let p be a prime number, t, k be positive integers, $t \leq k$, $t = [t_0, t_1, \ldots, t_{t(p)}]_p$ and $k = [k_0, k_1, \ldots, k_{k(p)}]_p$. Then*

$$p|\tbinom{k}{t}\ \text{if and only if there is}\ i \in \{0,1,\ldots,t(p)\}\ \text{such that}\ t_i > k_i.$$

Proof of Theorem 2.6. 1) We prove that under the stated conditions $\binom{k-i}{t-i} \not\equiv 0\ (mod\ p)$ for every $i \in \{0,\ldots,t\}$. From Theorem 2.1 it follows that $Ker_p({}^tW_{t\ k}) = \{0\}$. Let $i \in \{0,\ldots,t\}$ then $i = [i_0, i_1, \ldots, i_{t(p)}]$ with $i_{t(p)} \leq t_{t(p)}$. Since $k_j = t_j$ for all $j < t(p)$, then $(t-i)_j = (k-i)_j$ for all $j < t(p)$. As $k_{t(p)} \geq t_{t(p)} \geq i_{t(p)}$ then $(k-i)_{t(p)} \geq (t-i)_{t(p)}$; thus, by Corollary 2.8, $p \nmid \binom{k-i}{t-i}$ for all $i \in \{0,1,\ldots,t\}$. Now from Theorem 2.2, $rank_p\ W_{tk} = \sum_{i=0}^{t} \binom{v}{i} - \binom{v}{i-1} = \binom{v}{t}$. Then $Ker_p({}^tW_{t\ k}) = \{0\}$.

Now we prove the converse implication. From Theorem 2.1, $Ker_p({}^tW_{t\ k}) = \{0\}$ implies $p \nmid \binom{k-i}{t-i}$ for all $i \in \{0,1,\ldots,t\}$, in particular $p \nmid \binom{k}{t}$. Then by Corollary 2.8, $k_j \geq t_j$ for all $j \leq t(p)$. We will prove that $k_j = t_j$ for all $j \leq t(p)-1$. By contradiction, let s be the least integer in $\{0,1,\ldots,t(p)-1\}$, such that $k_s > t_s$. We have $(t-(t_s+1)p^s)_s = p-1$, $(k-(t_s+1)p^s)_s = k_s - t_s - 1$ and $p-1 > k_s - t_s - 1$. From Corollary 2.8, $p \mid \binom{k-(t_s+1)p^s}{t-(t_s+1)p^s}$, which is impossible.

2) Set $n := t(p)$. We begin by the direct implication. Since $0 = k_n < t_n$ then, by Corollary 2.8, $p|\binom{k}{t}$. We will prove $p \nmid \binom{k-i}{t-i}$ for all $i = [i_0, i_1, \ldots, i_n] \in \{1,2,\ldots,t\}$.

Since $k_j = t_j = 0$ for all $j < n$, then $(t-i)_j = (k-i)_j$ for all $j < n$. From $t_n \geq i_n$, we have $(t-i)_n \in \{t_n - i_n, t_n - i_n - 1\}$. Note that $(k-i)_n \in \{p-i_n-1, p-i_n\}$ and $p-i_n-1 \geq t_n - i_n$; thus $(k-i)_n \geq (t-i)_n$.

Therefore, for all $j \le n$, $(k-i)_j \ge (t-i)_j$. Then, by Corollary 2.8, $p \nmid \binom{k-i}{t-i}$ for all $i \in \{1,2,\dots,t\}$. Now from Theorem 2.2, $rank_p\ W_{tk} = \sum_{i=1}^{t}\binom{v}{i} - \binom{v}{i-1} = \binom{v}{t} - 1$, and thus $\dim Ker_p({}^tW_{t\,k}) = 1$. Now $(1,1,\cdots,1)W_{t\,k} = (\binom{k}{t},\binom{k}{t},\cdots,\binom{k}{t})$. Since $p \mid \binom{k}{t}$, then $(1,1,\cdots,1)W_{t\,k} \equiv 0 \pmod p$. Then $\{(1,1,\cdots,1)\}$ is a basis of $Ker_p({}^tW_{t\,k})$.

Now we prove the converse implication. Since $\{(1,1,\cdots,1)\}$ is a basis of $Ker_p({}^tW_{t\,k})$ and $(1,1,\cdots,1)W_{t\,k}$ $= (\binom{k}{t},\binom{k}{t},\cdots,\binom{k}{t})$, then $p \mid \binom{k}{t}$. Since $dim\ Ker_p({}^tW_{t\,k}) = 1$, then from Theorem 2.2, $p \nmid \binom{k-i}{t-i}$ for all $i \in \{1,2,\dots,t\}$.

First, let us prove that $t = t_n p^n$. Note that $t_n \neq 0$ since $t \neq 0$. Since $p|\binom{k}{t}$ then, from Corollary 2.8, there is an integer $j \in \{0,1,\dots,n\}$ such that $t_j > k_j$. Let $A := \{j < n\ :\ t_j \neq 0\}$. By contradiction, assume $A \neq \emptyset$.

Case 1. There is $j \in A$ such that $t_j > k_j$. We have $(t-p^n)_j = t_j$, $(k-p^n)_j = k_j$. Then from Corollary 2.8, we have $p \mid \binom{k-p^n}{t-p^n}$, which is impossible.

Case 2. For all $j \in A$, $t_j \le k_j$. Then $t_n > k_n$. We have $(t-p^j)_n = t_n$, $(k-p^j)_n = k_n$. Then, from Corollary 2.8, we have $p \mid \binom{k-p^j}{t-p^j}$, which is impossible.

From the above 2 cases, we deduce $t = t_n p^n$.

Secondly, since $p|\binom{k}{t}$, then by Corollary 2.8, $t_n > k_n$. Let us show that $k_n = 0$. By contradiction, if $k_n \neq 0$ then $(t-p^n)_n = t_n - 1 > k_n - 1 = (k-p^n)_n$. From Corollary 2.8, $p \mid \binom{k-p^n}{t-p^n}$, which is impossible. Let $s \in \{0,1,\dots,n-1\}$; let us show that $k_s = 0$. By contradiction, if $k_s \neq 0$ then $(t-p^s)_s = p-1$, $(k-p^s)_s = k_s - 1$, thus $(t-p^s)_s > (k-p^s)_s$ and so, from Corollary 2.8, $p \mid \binom{k-p^s}{t-p^s}$, which is impossible. □

3. Proof of Theorem 1.3.

Let $T_1, T_2, \cdots, T_{\binom{v}{t}}$ be an enumeration of the t-element subsets of V, let $K_1, K_2, \cdots, K_{\binom{v}{k}}$ be an enumeration of the k-element subsets of V, and let $W_{t\,k}$ be the matrix of the t-element subsets versus the k-element subsets.

Let w_U be the row matrix $(u_1, u_2, \cdots, u_{\binom{v}{t}})$ where $u_i = 1$ if $T_i \in U$, 0 otherwise. We have

$$w_U W_{t\,k} = (|\{T_i \in U : T_i \subseteq K_1\}|, \cdots, |\{T_i \in U : T_i \subseteq K_{\binom{v}{k}}\}|).$$

$$w_{U'} W_{t\,k} = (|\{T_i \in U' : T_i \subseteq K_1\}|, \cdots, |\{T_i \in U' : T_i \subseteq K_{\binom{v}{k}}\}|).$$

Since for all $j \in \{1,\dots,\binom{v}{k}\}$, $e(U_{\restriction K_j}) \equiv e(U'_{\restriction K_j}) \pmod p$, then $(w_U - w_{U'})W_{t\,k} = 0 \pmod p$, and so $w_U - w_{U'} \in Ker_p({}^tW_{t\,k})$.

1) Assume $k_i = t_i$ for all $i < t(p)$ and $k_{t(p)} \ge t_{t(p)}$. From 1) of Theorem 2.6, $w_U - w_{U'} = 0$, which gives $U = U'$.

2) Assume $t = t_{t(p)} p^{t(p)}$ and $k = \sum_{i=t(p)+1}^{k(p)} k_i p^i$. From 2) of Theorem 2.6, there is an integer $\lambda \in [0, p-1]$ such that $w_U - w_{U'} = \lambda(1,1,\cdots,1)$. It is clear that $\lambda \in \{0,1,-1\}$. If $\lambda = 0$ then $U = U'$. If $\lambda = 1$ and $p \ge 3$ then $U = \{T_1, T_2, \cdots, T_{\binom{v}{t}}\}$, $U' = \emptyset$. If $\lambda = 1$ and $p = 2$ then $U = \{T_1, T_2, \cdots, T_{\binom{v}{t}}\}$, $U' = \emptyset$, or $T \in U$ if

and only if $T \notin U'$. If $\lambda = -1$ and $p \geq 3$ then $U = \emptyset$, $U' = \{T_1, T_2, \cdots, T_{\binom{v}{t}}\}$. If $\lambda = -1$ and $p = 2$ then $U' = \{T_1, T_2, \cdots, T_{\binom{v}{t}}\}$, $U = \emptyset$, or $T \in U$ if and only if $T \notin U'$. □

4. Illustrations to graphs

Our notations and terminology follow [2]. A *digraph* $G = (V, E)$ or $G = (V(G), E(G))$ is formed by a finite set V of vertices and a set E of ordered pairs of distinct vertices, called *arcs* of G. The *order* (or *cardinal*) of G is the number of its vertices. If K is a subset of V, the *restriction* of G to K, also called the *induced subdigraph* of G on K, is the digraph $G_{\restriction K} := (K, K^2 \cap E)$. If $K = V \setminus \{x\}$, we denote this digraph by G_{-x}. Let $G = (V, E)$ and $G' = (V', E')$ be 2 digraphs. A one-to-one correspondence f from V onto V' is an *isomorphism from* G *onto* G' provided that for $x, y \in V$, $(x, y) \in E$ if and only if $(f(x), f(y)) \in E'$. The digraphs G and G' are then said to be *isomorphic*, which is denoted by $G \simeq G'$ if there is an isomorphism from one of them onto the other. A subset I of V is an *interval* [16, 21, 34] (or an *autonomous subset* [23] or a *clan* [14], or an *homogeneous subset* [18] or a *module* [35]) of G provided that for all $a, b \in I$ and $x \in V \setminus I$, $(a, x) \in E(G)$ if and only if $(b, x) \in E(G)$, and the same for (x, a) and (x, b). For example $\emptyset$, $\{x\}$ where $x \in V$, and V are intervals of G, called *trivial intervals*. A digraph is then said to be *indecomposable* [34] (or *primitive* [14]) if all its intervals are trivial; otherwise it is said to be *decomposable*.

We say that G is a *graph* (resp. *tournament*) when for all distinct vertices x, y of V, $(x, y) \in E$ if and only if $(y, x) \in E$ (resp. $(x, y) \in E$ if and only if $(y, x) \notin E$); we say that $\{x, y\}$ is an *edge* of the graph G if $(x, y) \in E$, thus E is identified with a subset of $[V]^2$, the set of pairs $\{x, y\}$ of distinct elements of V.

Let $G = (V, E)$ be a graph, the *complement* of G is the graph $\overline{G} := (V, [V]^2 \setminus E)$. We denote by $e(G) := |E(G)|$ the number of edges of G. The *degree* of a vertex x of G, denoted $d_G(x)$, is the number of edges that contain x. A 3-element subset T of V such that all pairs belong to $E(G)$ is a *triangle* of G. Let $T(G)$ be the set of *triangles* of G and let $t(G) :=| T(G) |$. A 3-element subset of V that is a triangle of G or of $\overline{G}$ is a 3-*homogeneous* subset of G. We set $H^{(3)}(G) := T(G) \cup T(\overline{G})$, the set of 3-homogeneous subsets of G, and $h^{(3)}(G) :=| H^{(3)}(G) |$.

Another proof of Theorem 1.4 using Theorem 1.3. Here $p = 2$, $t = 2 = [0, 1]_p$, and $k = [0, 0, k_2, \ldots]_p$. From 2) of Theorem 1.3, $U = U'$, or one of the sets U, U' is the set of all 2-element subsets of V and the other is empty, or for all 2-element subsets T of V, $T \in U$ if and only if $T \notin U'$. Thus $G' = G$ or $G' = \overline{G}$. □

Proof of Theorem 1.5. We may suppose V finite. We set $U := E(G)$, $U' := E(G')$. For all $K \subseteq V$ with $|K| = k$, we have: $\{\{x, y\} \subseteq K : \{x, y\} \in U\} = E(G_{\restriction K})$ and $\{\{x, y\} \subseteq K : \{x, y\} \in U'\} = E(G'_{\restriction K})$. Since $e(G_{\restriction K}) \equiv e(G'_{\restriction K}) \pmod p$, then $|\{\{x, y\} \subseteq K : \{x, y\} \in U\}| \equiv |\{\{x, y\} \subseteq K : \{x, y\} \in U'\}| \pmod p$.

1) $p \geq 3$, $t := 2 = [2]_p$ and $k_0 \geq 2$. From 1) of Theorem 1.3, $U = U'$; thus $G = G'$.

2) $p \geq 3$, $t := 2 = [2]_p$ and $k_0 = 0$. From 2) of Theorem 1.3, we have $U = U'$ or one of U, U' is the set of all 2-element subsets of V and the other is empty. Then $G = G'$ or one of the graphs G, G' is the complete graph and the other is the empty graph.

3) $p = 2$, $t = 2 = [0, 1]_p$, and $k = [0, 1, k_2, \ldots]_p$. From 1) of Theorem 1.3, we have $U = U'$; thus $G = G'$. □

The following result concerns graphs G and G' such that $h^{(3)}(G_{\restriction K}) \equiv h^{(3)}(G'_{\restriction K})$ modulo a prime p, for all k-element subsets K of V.

Theorem 4.1 *Let G and G' be 2 graphs on the same set V of v vertices (possibly infinite). Let p be a prime number and k be an integer, $3 \leq k \leq v-3$.*

1) If $h^{(3)}(G_{\restriction K}) = h^{(3)}(G'_{\restriction K})$ for all k-element subsets K of V then G and G' have the same 3-element homogeneous subsets.

2) Assume $p \geq 5$. If $k \not\equiv 1, 2 \pmod p$ and $h^{(3)}(G_{\restriction K}) \equiv h^{(3)}(G'_{\restriction K}) \pmod p$ for all k-element subsets K of V, then G and G' have the same 3-element homogeneous subsets.

3) If ($p = 2$ and $k \equiv 3 \pmod 4$) or ($p = 3$ and $3 \mid k$), and $h^{(3)}(G_{\restriction K}) \equiv h^{(3)}(G'_{\restriction K}) \pmod p$ for all k-element subsets K of V, then G and G' have the same 3-element homogeneous subsets.

Proof We may suppose V finite.

We have $H^{(3)}(G) = \{\{a,b,c\} : G_{\restriction\{a,b,c\}}$ is a 3-element homogeneous subset$\}$.

We set $U := H^{(3)}(G)$ and $U' := H^{(3)}(G')$. For all $K \subseteq V$ with $|K| = k$, we have: $\{T \subseteq K : T \in U\} = H^{(3)}_{G_{\restriction K}}$ and $\{T \subseteq K : T \in U'\} = H^{(3)}_{G'_{\restriction K}}$. Set $t :=| T |= 3$.

1) Since $h^{(3)}(G_{\restriction K}) = h^{(3)}(G'_{\restriction K})$ for all k-element subsets K of V then $|\{T \subseteq K : T \in U\}| = |\{T \subseteq K : T \in U'\}|$. From Lemma 1.2 it follows that $U = U'$; then G and G' have the same 3-element homogeneous subsets.

2) Since $h^{(3)}(G_{\restriction K}) \equiv h^{(3)}(G'_{\restriction K}) \pmod p$ for all k-element subsets K of V then $|\{T \subseteq K : T \in U\}| \equiv |\{T \subseteq K : T \in U'\}| \pmod p$.

Case 1. $k_0 \geq 3$. Then $p \geq 5$, $t := 3 = [3]_p$, and $t_0 = 3 \leq k_0$. From 1) of Theorem 1.3 we have $U = U'$; thus G and G' have the same 3-element homogeneous subsets.

Case 2. $k_0 = 0$. Then $p \geq 5$, $t := 3 = [3]_p$. By Ramsey's theorem [33], every graph with at least 6 vertices contains a 3-element homogeneous subset. Then U and U' are nonempty and so from 2) of Theorem 1.3, $U = U'$; thus G and G' have the same 3-element homogeneous subsets.

3) Since $h^{(3)}(G_{\restriction K}) \equiv h^{(3)}(G'_{\restriction K}) \pmod p$ for all k-element subsets K of V then $|\{T \subseteq K : T \in U\}| \equiv |\{T \subseteq K : T \in U'\}| \pmod p$.

Case 1. $p = 2$ and $k \equiv 3 \pmod 4$. Let $t := 3 = [1,1]_p$. In this case, $k = [1,1,k_2,\ldots]_p$; then from 1) of Theorem 1.3 we have $U = U'$; thus G and G' have the same 3-element homogeneous subsets.

Case 2. $p = 3$ and $3 \mid k$. Then $k = [0, k_1, \ldots, k_{k(p)}]_p$. Let $t := 3 = [0,1]_p$.

Case 2.1. $k_1 \in \{1,2\}$; then from 1) of Theorem 1.3 we have $U = U'$; thus G and G' have the same 3-element homogeneous subsets.

Case 2.2. $k_1 = 0$. By Ramsey's theorem [33], every graph with at least 6 vertices contains a 3-element homogeneous subset. Then U and U' are nonempty, and so from 2) of Theorem 1.3, $U = U'$; thus G and G' have the same 3-element homogeneous subsets. □

Let $G = (V,E)$ be a graph. From [34], every indecomposable graph of size 4 is isomorphic to $P_4 = (\{0,1,2,3\}, \{\{0,1\},\{1,2\},\{2,3\}\})$. Let $\mathcal{P}^{(4)}(G)$ be the set of subsets X of V such that the induced subgraph $G_{\restriction X}$ is isomorphic to P_4. We set $p^{(4)}(G) := |\mathcal{P}^{(4)}(G)|$. The following result concerns graphs G and G' such that $p^{(4)}(G_{\restriction K}) \equiv p^{(4)}(G'_{\restriction K})$ modulo a prime p, for all k-element subsets K of V.

Theorem 4.2 *Let G and G' be 2 graphs on the same set V of v vertices. Let p be a prime number and k be an integer, $4 \leq k \leq v-4$.*

1) If $p^{(4)}(G_{\restriction K}) = p^{(4)}(G'_{\restriction K})$ for all k-element subsets K of V then G and G' have the same indecomposable sets of size 4.

2) Assume $p^{(4)}(G_{\restriction K}) \equiv p^{(4)}(G'_{\restriction K}) \pmod p$ for all k-element subsets K of V.

a) If $p \geq 5$ and $k \not\equiv 1, 2, 3 \pmod p$, then G and G' have the same indecomposable sets of size 4.

b) If ($p = 2$, $4 \mid k$ and $8 \nmid k$) or ($p = 3$, $3 \mid k-1$ and $9 \nmid k-1$), then G and G' have the same indecomposable sets of size 4.

c) If $p = 2$ and $8 \mid k$, then G and G' have the same indecomposable sets of size 4, or for all 4-element subsets T of V, $G_{\restriction T}$ is indecomposable if and only if $G'_{\restriction T}$ is decomposable.

Proof Let $U := \{T \subseteq V : |T| = 4,\ G_{\restriction T} \simeq P_4\} = \mathcal{P}^{(4)}(G)$, $U' := \{T \subseteq V : |T| = 4,\ G'_{\restriction T} \simeq P_4\} = \mathcal{P}^{(4)}(G')$. For all $K \subseteq V$, we have $\{T \subseteq K : T \in U\} = \mathcal{P}_4(G_{\restriction K})$ and $\{T \subseteq K : T \in U'\} = \mathcal{P}_4(G'_{\restriction K})$. Set $t := |T| = 4$.

1) Since $p^{(4)}(G_{\restriction K}) = p^{(4)}(G'_{\restriction K})$ then $|\{T \subseteq K : T \in U\}| = |\{T \subseteq K : T \in U'\}|$. From Lemma 1.2, $U = U'$; then G and G' have the same indecomposable sets of size 4.

2) We have $p^{(4)}(G_{\restriction K}) \equiv p^{(4)}(G'_{\restriction K}) \pmod p$ for all k-element subsets K of V; then $|\{T \subseteq K : T \in U\}| \equiv |\{T \subseteq K : T \in U'\}| \pmod p$.

a) Case 1. $k_0 \geq 4$. Then $p \geq 5$, $t = 4 = [4]_p$, and $t_0 = 4 \leq k_0$. From 1) of Theorem 1.3 we have $U = U'$; thus G and G' have the same indecomposable sets of size 4.

Case 2. $k_0 = 0$. Let $t := 4 = [4]_p$.

A graph H is *k-monomorphic* if $G_{\restriction X} \simeq G_{\restriction Y}$ for all k-element subsets X and Y of V. If a graph of order at least 6 is 4-monomorphic then it is 2-monomorphic and hence complete or empty. Since in every graph of order 6, there is a restriction of size 4 not isomorphic to P_4 then, from 2) of Theorem 1.3, $U = U'$; thus G and G' have the same indecomposable sets of size 4.

b) Case 1. $p = 2$, $4 \mid k$, and $8 \nmid k$. Then $t := 4 = [0, 0, 1]_p$ and $k = [0, 0, 1, k_3, \ldots, k_{k(p)}]_p$. From 1) of Theorem 1.3, we have $U = U'$; thus G and G' have the same indecomposable sets of size 4.

Case 2. $p = 3$, $3 \mid k-1$, and $9 \nmid k-1$. Then $t := 4 = [1, 1]_p$, $k = [1, k_1, \ldots, k_{k(p)}]_p$, and $t_1 = 1 \leq k_1$. From 1) of Theorem 1.3, $U = U'$, thus G and G' have the same indecomposable sets of size 4.

c) We have $p = 2$, $t := 4 = [0, 0, 1]_p$, and $k = [0, 0, 0, k_3, \ldots, k_{k(p)}]_p$. Since in every graph of order 6, there is a restriction of size 4 not isomorphic to P_4, then from 2) of Theorem 1.3, $U = U'$, or for all 4-element subsets T of V, $T \in U$ if and only if $T \notin U'$. Thus G and G' have the same indecomposable sets of size 4, or for all 4-element subsets T of V, $G_{\restriction T}$ is indecomposable if and only if $G'_{\restriction T}$ is decomposable. □

In a reconstruction problem of graphs up to complementation [13], Wilson's theorem yielded the following result:

Theorem 4.3 *([13]) Let G and G' be 2 graphs on the same set V of v vertices (possibly infinite). Let k be an integer, $5 \leq k \leq v-2$, $k \equiv 1 \pmod 4$. Then the following properties are equivalent:*

(i) $e(G_{\restriction K})$ has the same parity as $e(G'_{\restriction K})$ for all k-element subsets K of V; and $G_{\restriction K}$, $G'_{\restriction K}$ have the same 3-homogeneous subsets;

(ii) $G' = G$ or $G' = \overline{G}$.

Here, we just want to point out that we can obtain a similar result for $k \equiv 3 \pmod 4$, namely Theorem 4.4, using the same proof as that of Theorem 4.3.

The *boolean sum* $G \dot{+} G'$ of 2 graphs $G = (V, E)$ and $G' = (V, E')$ is the graph U on V whose edges are pairs e of vertices such that $e \in E$ if and only if $e \notin E'$.

Theorem 4.4 *Let G and G' be 2 graphs on the same set V of v vertices (possibly infinite). Let k be an integer, $3 \leq k \leq v-2$, $k \equiv 3 \pmod 4$. Then the following properties are equivalent:*

(i) $e(G_{\restriction K})$ has the same parity as $e(G'_{\restriction K})$ for all k-element subsets K of V; and $G_{\restriction K}$, $G'_{\restriction K}$ have the same 3-homogeneous subsets;

(ii) $G' = G$.

Proof It is exactly the same as that of Theorem 4.3 (see ([13]). The implication $(ii) \Rightarrow (i)$ is trivial. We prove $(i) \Rightarrow (ii)$. We may suppose V finite. We set $U := G \dot{+} G'$; let $T_1, T_2, \cdots, T_{\binom{v}{2}}$ be an enumeration of the 2-element subsets of V, and let $K_1, K_2, \cdots, K_{\binom{v}{k}}$ be an enumeration of the k-element subsets of V. Let w_U be the row matrix $(u_1, u_2, \cdots, u_{\binom{v}{2}})$ where $u_i = 1$ if T_i is an edge of U, 0 otherwise. We have $w_U W_{2\,k} = (e(U_{\restriction K_1}), e(U_{\restriction K_2}), \cdots, e(U_{\restriction K_{\binom{v}{k}}}))$. From the fact that $e(G_{\restriction K})$ has the same parity as $e(G'_{\restriction K})$ and $e(U_{\restriction K}) = e(G_{\restriction K}) + e(G'_{\restriction K}) - 2e(G_{\restriction K} \cap G'_{\restriction K})$ for all k-element subsets K, w_U belongs to $Ker_2({}^tW_{t\,k})$. According to Theorem 2.2, $rank_2 W_{2k} = \binom{v}{2} - v + 1$. Hence $\dim Ker_2({}^tW_{2\,k}) = v - 1$.

We give a similar claim as Claim 2.8 of [13]; the proof is identical.

Claim 4.5 *Let k be an integer such that $3 \leq k \leq v-2$, $k \equiv 3 \pmod 4$; then $Ker_2({}^tW_{2\,k})$ consists of complete bipartite graphs (including the empty graph).*

Proof Let us recall that a *star-graph* of v vertices consists of a vertex linked to all other vertices, those $v-1$ vertices forming an independent set. First we prove that each star-graph S belongs to $\mathbb{K} := Ker_2({}^tW_{2\,k})$. Let w_S be the row matrix $(s_1, s_2, \cdots, s_{\binom{v}{2}})$ where $s_i = 1$ if T_i is an edge of S, 0 otherwise. We have $w_S W_{2\,k} = (e(S_{\restriction K_1}), e(S_{\restriction K_2}), \cdots, e(S_{\restriction K_{\binom{v}{k}}}))$. For all $i \in \{1, \ldots, \binom{v}{k}\}$, $e(S_{\restriction K_i}) = k - 1$ if the center of the star-graph belongs to K_i, 0 otherwise. Since k is odd, each star-graph S belongs to $\mathbb{K}$. The vector space (over the 2-element field) generated by the star-graphs on V consists of all complete bipartite graphs; since $v \geq 3$, these are distinct from the complete graph (but include the empty graph). Moreover, its dimension is $v - 1$ (a basis being made of star-graphs). Since $\dim Ker_2({}^tW_{2\,k}) = v - 1$, then $\mathbb{K}$ consists of complete bipartite graphs as claimed. □

A *claw* is a star-graph on 4 vertices, that is a graph made of a vertex joined to 3 other vertices, with no edges between these 3 vertices. A graph is *claw-free* if no induced subgraph is a claw.

Claim 4.6 *([13]) Let G and G' be 2 graphs on the same set and having the same 3-homogeneous subsets; then the boolean sum $U := G \dot{+} G'$ is claw-free.*

From Claim 4.5, U is a complete bipartite graph and, from Claim 4.6, U is claw-free. Since $v \geq 5$, it follows that U is the empty graph. Hence $G' = G$ as claimed. □

5. Illustrations to tournaments

Let $T=(V,E)$ be a tournament. For 2 distinct vertices x and y of T, $x \longrightarrow_T y$ (or simply $x \longrightarrow y$) means that $(x,y) \in E$. For $A \subseteq V$ and $y \in V$, $A \longrightarrow y$ means $x \longrightarrow y$ for all $x \in A$. The *degree* of a vertex x of T is $d_T(x) := |\{y \in V : x \longrightarrow y\}|$. We denote by T^* *the dual* of T that is $T^* = (V, E^*)$ with $(x,y) \in E^*$ if and only if $(y,x) \in E$. A *transitive* tournament or a *total order* or k*-chain* (denoted O_k) is a tournament of cardinality k, such that for $x,y,z \in V$, if $x \longrightarrow y$ and $y \longrightarrow z$, then $x \longrightarrow z$. If x and y are 2 distinct vertices of a total order, the notation $x < y$ means that $x \longrightarrow y$. The tournament $C_3 := \{\{0,1,2\}, \{(0,1),(1,2),(2,0)\}\}$ (resp. $C_4 := (\{0,1,2,3\}, \{(0,3),(0,1),(3,1),(1,2),(2,0),(2,3)\})$) is a 3*-cycle* (resp. 4*-cycle*) (see Figure 1). A *diamond* is a tournament on 4 vertices admitting only 1 interval of cardinality 3, which is a 3-cycle. Up to isomorphism, there are exactly 2 diamonds δ^+ and $\delta^- = (\delta^+)^*$, where δ^+ is the tournament defined on $\{0,1,2,3\}$ by $\delta^+_{\restriction\{0,1,2\}} = C_3$ and $\{0,1,2\} \rightarrow 3$. A tournament isomorphic to δ^+ (resp. isomorphic to δ^-) is said to be a *positive diamond* (resp. *negative diamond*) (see Figure 1). The *boolean sum* $U := T \dot{+} T'$ of 2 tournaments, $T=(V,E)$ and $T'=(V,E')$, is the graph U on V whose edges are pairs $\{x,y\}$ of vertices such that $(x,y) \in E$ if and only if $(x,y) \notin E'$.

Theorem 5.1 *Let $T=(V,E)$ and $T'=(V,E')$ be 2 tournaments on the same set V of v vertices (possibly infinite). Let p be a prime number and k be an integer, $2 \le k \le v-2$. Let $G := T \dot{+} T'$. We assume that for all k-element subsets K of V, $e(G_{\restriction K}) \equiv 0$ (mod p). Then*

1) $T' = T$ if ($p \ge 3$, $k \not\equiv 0,1$ (mod p)) or ($p=2$, $k \equiv 2$ (mod 4)).

2) $T' = T$ or $T' = T^*$ if ($p \ge 3$, $k \equiv 0$ (mod p)) or ($p=2$, $k \equiv 0$ (mod 4)).

Proof We may suppose V finite. The proof reduces to say when G is the empty graph or when G is either empty or full. We set $G' :=$ The empty graph. Then $e(G_{\restriction K}) \equiv e(G'_{\restriction K})$ (mod p).

1) Use respectively 1) of Theorem 1.5 and 3) of Theorem 1.5.

2) Use respectively 2) of Theorem 1.5 and Theorem 1.4. □

Let T be a tournament; we set $C^{(3)}(T) := \{\{a,b,c\} : T_{\restriction\{a,b,c\}}$ is a 3-cycle$\}$, and $c^{(3)}(T) := \mid C^{(3)}(T) \mid$. Let $T=(V,E)$ and $T'=(V,E')$ be 2 tournaments and let k be a nonnegative integer; T and T' are k*-hypomorphic* [8, 27] (resp. k*-hypomorphic* up to duality) if for every k-element subset K of V, the induced subtournaments $T'_{\restriction K}$ and $T_{\restriction K}$ are isomorphic (resp. $T'_{\restriction K}$ is isomorphic to $T_{\restriction K}$ or to $T^*_{\restriction K}$). We say that T and T' are $(\le k)$*-hypomorphic* if T and T' are h-hypomorphic for every $h \le k$. Similarly, we say that T and T' are $(\le k)$*-hypomorphic up to duality* if T and T' are h-hypomorphic up to duality for every $h \le k$. Clearly, 2 (≤ 3)-hypomorphic tournaments have the same diamonds. Furthermore, note that 2 (≤ 3)-hypomorphic tournaments have the same indecomposable structures and if a component in the tree decomposition is indecomposable, then the corresponding one is equal or dual [9].

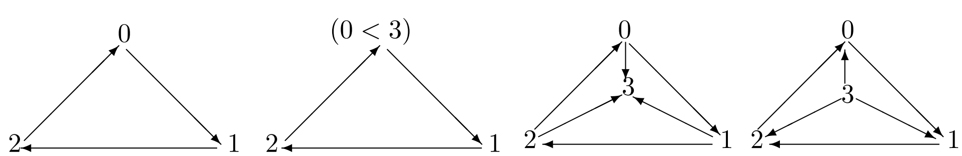

Figure 1. Cycle C_3, Cycle C_4, Positive Diamond, Negative Diamond.

Theorem 5.2 *Let T and T' be 2 tournaments on the same set V of v vertices. Let p be a prime number and k be an integer, $3 \leq k \leq v-3$.*

1) If $c^{(3)}(T_{\restriction K}) = c^{(3)}(T'_{\restriction K})$ for all k-element subsets K of V then T and T' are (≤ 3)-hypomorphic.

2) Assume $p \geq 5$. If $k \not\equiv 1, 2 \pmod p$, and $c^{(3)}(T_{\restriction K}) \equiv c^{(3)}(T'_{\restriction K}) \pmod p$ for all k-element subsets K of V, then T and T' are (≤ 3)-hypomorphic.

3) If ($p = 2$ and $k \equiv 3 \pmod 4$) or ($p = 3$ and $3 \mid k$), and $c^{(3)}(G_{\restriction K}) \equiv c^{(3)}(G'_{\restriction K}) \pmod p$ for all k-element subsets K of V, then T and T' are (≤ 3)-hypomorphic.

Proof Since every tournament of cardinality ≥ 4 has at least a restriction of cardinality 3 that is not a 3-cycle, then the proof is similar to that of Theorem 4.1. □

Let T be a tournament; we set $D_4^+(T) := \{\{a,b,c,d\} : T_{\restriction \{a,b,c,d\}} \simeq \delta^+\}$, $D_4^-(T) := \{\{a,b,c,d\} : T_{\restriction \{a,b,c,d\}} \simeq \delta^-\}$, $d_4^+(T) :=\mid D_4^+(T) \mid$, and $d_4^-(T) :=\mid D_4^-(T) \mid$.

It is well known that every subtournament of order 4 of a tournament is a diamond, a 4-chain, or a 4-cycle subtournament. We have $c^{(3)}(O_4) = 0$, $c^{(3)}(\delta^+) = c^{(3)}(\delta^-) = 1$, $c^{(3)}(C_4) = 2$, and $C_4 \simeq C_4^*$. The (≤ 4)-hypomorphy has been studied by G. Lopez and C. Rauzy [27, 28].

Theorem 5.3 *Let T and T' be 2 (≤ 3)-hypomorphic tournaments on the same set V of v vertices. Let p be a prime number and k be an integer, $4 \leq k \leq v-4$.*

1) If $d_4^+(T_{\restriction K}) = d_4^+(T'_{\restriction K})$ for all k-element subsets K of V then T' and T are (≤ 4)-hypomorphic.

2) Assume $d_4^+(T_{\restriction K}) \equiv d_4^+(T'_{\restriction K}) \pmod p$ for all k-element subsets K of V.

a) If $p \geq 5$ and $k \not\equiv 1, 2, 3 \pmod p$, then T' and T are (≤ 4)-hypomorphic.

b) If ($p = 3$, $3 \mid k-1$ and $9 \nmid k-1$) or ($p = 2$, $4 \mid k$ and $8 \nmid k$), then T' and T are (≤ 4)-hypomorphic.

c) If $p = 2$ and $8 \mid k$, then T' and T are (≤ 4)-hypomorphic.

Proof Let $U^+ := \{S \subseteq V,\ T_{\restriction S} \simeq \delta^+\} = D_4^+(T)$, $U'^+ := D_4^+(T')$, $U^- := D_4^-(T)$, and $U'^- := D_4^-(T')$.

Claim 5.4 *If T and T' are (≤ 3)-hypomorphic and $U^+ = U'^+$, then $U^- = U'^-$; T and T' are (≤ 4)-hypomorphic.*

Proof Let $S \in U^-$, $T_{\restriction S} \simeq \delta^-$. Since T and T' are (≤ 3)-hypomorphic, then $T'_{\restriction S} \simeq \delta^+$ or $T'_{\restriction S} \simeq \delta^-$. We have $\{S \subseteq V,\ T'_{\restriction S} \simeq \delta^+\} = \{S \subseteq V,\ T_{\restriction S} \simeq \delta^+\}$; then $T'_{\restriction S} \simeq \delta^-$, $S \in U'^-$ and $U^- = U'^-$. Therefore, for $X \subset V$, if $T_{\restriction X}$ is a diamond then $T'_{\restriction X} \simeq T_{\restriction X}$.

Now we prove that T and T' are 4-hypomorphic. Let $X \subset V$ such that $|X| = 4$. If $T_{\restriction X} \simeq C_4$, then $c^{(3)}(T_{\restriction X}) = 2$. Since T and T' are (≤ 3)-hypomorphic then $c^{(3)}(T'_{\restriction X}) = 2$; thus $T'_{\restriction X} \simeq T_{\restriction X} \simeq C_4$. The same, if $T_{\restriction X} \simeq O_4$ then $T'_{\restriction X} \simeq T_{\restriction X} \simeq O_4$. Therefore, T' and T are (≤ 4)-hypomorphic. □

From Claim 5.4, it is sufficient to prove that $U^+ = U'^+$.

For all $K \subseteq V$ with $|K| = k$, we have $\{S \subseteq K : S \in U^+\} = D_4^+(T_{\restriction K})$ and $\{S \subseteq K : S \in U'^+\} = D_4^+(T'_{\restriction K})$.

1) Since $d_4^+(T_{\restriction K}) = d_4^+(T'_{\restriction K})$ then $|\{S \subseteq K : S \in U^+\}| = |\{S \subseteq K : S \in U'^+\}|$. From Lemma 1.2, we have $U^+ = U'^+$.

2) We have $d_4^+(T_{\restriction K}) \equiv d_4^+(T'_{\restriction K}) \pmod p$ for all k-element subsets K of V; then $|\{S \subseteq K : S \in U^+\}| \equiv |\{S \subseteq K : S \in U'^+\}| \pmod p$.

a) Case 1. $k_0 \geq 4$. Then $p \geq 5$, $t := 4 = [4]_p$, $k = [k_0, \dots]_p$, and $t_0 = 4 \leq k_0$. From 1) of Theorem 1.3 we have $U^+ = U'^+$.

Case 2. $k_0 = 0$. Then $p \geq 5$, $t := 4 = [4]_p$, and $k = [0, k_1, \dots]_p$. Since every tournament of cardinality ≥ 5 has at least a restriction of cardinality 4 that is not a diamond, then from 2) of Theorem 1.3, $U^+ = U'^+$.

b) Case 1. $p = 3$, $3 \mid k-1$ and $9 \nmid k-1$. Then $t := 4 = [1,1]_p$, $k = [1, k_1, \dots, k_{k(p)}]_p$ and $t_1 = 1 \leq k_1$. From 1) of Theorem 1.3 we have $U^+ = U'^+$.

Case 2. $p = 2$, $4 \mid k$ and $8 \nmid k$. Then $t := 4 = [0,0,1]_p$ and $k = [0,0,1,k_3, \dots, k_{k(p)}]_p$.

From 1) of Theorem 1.3 we have $U^+ = U'^+$.

c) We have $p = 2$, $t := 4 = [0,0,1]_p$, $k = [0,0,0,k_3, \dots, k_{k(p)}]_p$. Since every tournament of cardinality ≥ 5 has at least a restriction of cardinality 4 that is not a diamond, and the fact that T and T' are 3-hypomorphic, then from 2) of Theorem 1.3, $U^+ = U'^+$; thus T' and T are (≤ 5)-hypomorphic, or for all 4-element subsets S of V, $T_{\restriction S}$ is isomorphic to δ^+ if and only if $T'_{\restriction S}$ is isomorphic to δ^-. □

In fact, in Theorem 5.3, the conclusion is that T' and T are (≤ 5)-hypomorphic; this follows from Lemma 5.5 below.

Lemma 5.5 *([5]) Let T and T' be 2 (≤ 4)-hypomorphic tournaments on at least 5 vertices. Then, T and T' are (≤ 5)-hypomorphic.*

Comment. Let T and T' be 2 (≤ 3)-hypomorphic tournaments on the same set V of v vertices. Let U (respectively U') be the set of positive diamonds of T (respectively of T'). Then 2) of Theorem 1.3 with $U \neq U'$ cannot occur. Indeed, from 2) of Theorem 1.3, it follows that if $U \neq U'$ then for every 4-element subset X of V, $T_{\restriction X}$ is a positive diamond if and only if $T'_{\restriction X}$ is not a positive diamond. This implies that for every 4-element subset Y of V such that $T'_{\restriction Y}$ is not a diamond, $T_{\restriction Y}$ is a positive diamond. Since there are such Y (a 5-element tournament has 0 or 2 diamonds, see H. Bouchaala [4]), this contradicts the 3-hypomorphy.

Let m be an integer, $m \geq 1$, $S = (\{0,1,\dots,m-1\}, A)$ be a digraph and for $i < m$ a digraph $G_i = (V_i, A_i)$ such that the V_i's are nonempty and pairwise disjoint. The *lexicographic sum over S of the G_i*'s or simply the S-*sum* of the G_i's is the digraph denoted by $S(G_0, G_1, \dots, G_{m-1})$ and defined on the union of the V_i's as follows: given $x \in V_i$ and $y \in V_j$, where $i, j \in \{0,1,\dots,m-1\}$, (x,y) is an arc of $S(G_0, G_1, \dots, G_{m-1})$ if either $i = j$ and $(x,y) \in A_i$ or $i \neq j$ and $(i,j) \in A$: this digraph replaces each vertex i of S by G_i. We say that the vertex i of S is *dilated by* G_i.

We define, for each integer $h \geq 0$, the tournament T_{2h+1} (see Figure 2) on $\{0, \dots, 2h\}$ as follows. For $i, j \in \{0, \dots, 2h\}$, $i \longrightarrow j$ if there exists $k \in \{1, \dots, h\}$ such that $j = i + k$ modulo $2h+1$. A tournament T is said to be an element of $D(T_{2h+1})$ if T is obtained by dilating each vertex of T_{2h+1} by a finite chain p_i, and then $T = T_{2h+1}(p_0, p_1, \dots, p_{2h})$. We recall that T_{2h+1} is indecomposable and $D(T_{2h+1})$ is the class of finite tournaments without a diamond [27]; this class was obtained previously by Moon [30].

We define the tournament $\beta_6^+ := T_3(p_0, p_1, p_2)$ with $p_0 = (0 < 1 < 2)$, $p_1 = (3 < 4)$, and $|p_2| = 1$ (see Figure 3). We set $\beta_6^- := (\beta_6^+)^*$.

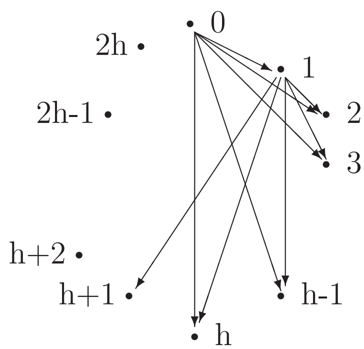

Figure 2. Circular tournament T_{2h+1}.

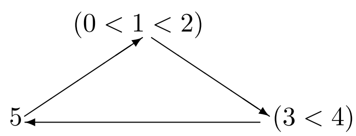

Figure 3. β_6^+.

For a tournament $T=(V,E)$, we set $B_6^+(T) := \{S \subseteq V : T_{\restriction S} \simeq \beta_6^+\}$, $B_6^-(T) := \{S \subseteq V : T_{\restriction S} \simeq \beta_6^-\}$, $b_6^+(T) :=\mid B_6^+(T) \mid$, and $b_6^-(T) :=\mid B_6^-(T) \mid$.

Two tournaments T and T' on the same vertex set V are *hereditarily isomorphic* if for all $X \subseteq V$, $T_{\restriction X}$ and $T'_{\restriction X}$ are isomorphic [3].

Let $G=(V,E)$ and $G'=(V,E')$ be 2 (≤ 2)-hypomorphic digraphs. Denote $D_{G,G'}$ the binary relation on V such that: for $x \in V$, $xD_{G,G'}x$; and for $x \neq y \in V$, $xD_{G,G'}y$ if there exists a sequence $x_0=x,...,x_n=y$ of elements of V satisfying $(x_i,x_{i+1}) \in E$ if and only if $(x_i,x_{i+1}) \notin E'$, for all i, $0 \leq i \leq n-1$. The relation $D_{G,G'}$ is an equivalence relation called *the difference relation*; its classes are called *difference classes.*

Using difference classes, G. Lopez [25, 26] showed that if T and T' are (≤ 6)-hypomorphic then T and T' are isomorphic. One may deduce the next corollary.

Corollary 5.6 *([25, 26]) Let T and T' be 2 tournaments. We have the following properties:*

1) If T and T' are (≤ 6)-hypomorphic then T and T' are hereditarily isomorphic.

2) If for each equivalence class C of $D_{T,T'}$, C is an interval of T and T', and $T'_{\restriction C}$, $T_{\restriction C}$ are (≤ 6)-hypomorphic, then T and T' are hereditarily isomorphic.

Lemma 5.7 *[27] Given 2 (≤ 4)-hypomorphic tournaments T and T', and C an equivalence class of $D_{T,T'}$, then:*

1) C is an interval of T' and T.

2) Every 3-cycle in $T_{\restriction C}$ is reversed in $T'_{\restriction C}$.

3) There exists an integer $h \geq 0$ such that $T_{\restriction C} = T_{2h+1}(p_0,p_1,\dots,p_{2h})$ and $T'_{\restriction C} = T^*_{2h+1}(p'_0,p'_1,\dots,p'_{2h})$ with p_i, p'_i as chains on the same basis, for all $i \in \{0,1,\dots,2h\}$.

Theorem 5.8 *Let T and T' be 2 (≤ 4)-hypomorphic tournaments on the same set V of v vertices. Let p be a prime number and $k=[k_0,k_1,\dots,k_{k(p)}]_p$ be an integer, $6 \leq k \leq v-6$.*

1) If $b_6^+(T_{\restriction K}) = b_6^+(T'_{\restriction K})$ *for all* k*-element subsets* K *of* V *then* T' *and* T *are* (≤ 6)*-hypomorphic and thus hereditarily isomorphic.*

2) Assume $b_6^+(T_{\restriction K}) \equiv b_6^+(T'_{\restriction K})$ *(mod* p*) for all* k*-element subsets* K *of* V*.*

a) If $p \geq 7$*, and* $k_0 \geq 6$ *or* $k_0 = 0$*, then* T' *and* T *are (*≤ 6*)-hypomorphic and thus hereditarily isomorphic.*

b) If ($p = 5$*,* $k_0 = 1$*, and* $k_1 \neq 0$*) or (*$p = 3$*,* $k_0 = 0$*, and* $k_1 = 2$*) or (*$p = 3$ *and* $k_0 = k_1 = 0$*) or (*$p = 2$*,* $k_0 = 0$*, and* $k_1 = k_2 = 1$*), then* T' *and* T *are* (≤ 6)*-hypomorphic and thus hereditarily isomorphic.*

Proof From Lemma 5.5, T and T' are (≤ 5)-hypomorphic. Let $U^+ := \{S \subseteq V,\ T_{\restriction S} \simeq \beta_6^+\} = B_6^+(T)$, $U'^+ := B_6^+(T')$, $U^- := \{S \subseteq V,\ T_{\restriction S} \simeq \beta_6^-\} = B_6^-(T)$, $U'^- := B_6^-(T')$.

Every tournament of cardinality ≥ 7 has at least a restriction of cardinality 6 that is neither isomorphic to β_6^+ nor to β_6^-. Then, for all cases, similarly to the proof of Theorem 5.3, we have $U^+ = U'^+$.

Let C be an equivalence class of $D_{T,T'}$, $S \in U^-$, $T_{\restriction S} \simeq \beta_6^-$. Since T and T' are (≤ 3)-hypomorphic, then $T'_{\restriction S} \simeq \beta_6^+$ or $T'_{\restriction S} \simeq \beta_6^-$. We have $\{S \subseteq V,\ T'_{\restriction S} \simeq \beta_6^+\} = \{S \subseteq V,\ T_{\restriction S} \simeq \beta_6^+\}$; then $T'_{\restriction S} \simeq \beta_6^-$, $S \in U'^-$, and $U^- = U'^-$. Let $X \subseteq C$ such that $|X| = 6$; if $T_X \simeq \beta_6^+$ then, from 2) of Lemma 5.7, $T'_X \simeq \beta_6^-$, which is impossible, and so T_C and T'_C do not have a restriction of cardinality 6 isomorphic to β_6^+ and β_6^-. From Lemma 5.9 below, $T_{\restriction C}$ and $T'_{\restriction C}$ are (≤ 6)-hypomorphic.

Lemma 5.9 *([3]) Let* T *and* T' *be 2* (≤ 5)*-hypomorphic tournaments defined on a vertex set* V *such that for all* $X \subseteq V$*, if* $T_{\restriction X}$ *is isomorphic to* β_6^+ *or to* β_6^-*, then* $T'_{\restriction X}$ *is isomorphic to* $T_{\restriction X}$*. Then* T *and* T' *are* (≤ 6)*-hypomorphic.*

From 1) of Lemma 5.7, C is an interval of T' and T. Then, from 2) of Corollary 5.6, T and T' are hereditarily isomorphic. □

From Theorem 5.2, Theorem 5.3, and Theorem 5.8, we deduce the following result.

Corollary 5.10 *Let* T *and* T' *be 2 tournaments on the same set* V *of* v *vertices. Let* p *be a prime number and* $k = [k_0, k_1, \ldots, k_{k(p)}]_p$ *be an integer,* $6 \leq k \leq v - 6$*.*

1) If $c^{(3)}(T_{\restriction K}) = c^{(3)}(T'_{\restriction K})$*,* $d_4^+(T_{\restriction K}) = d_4^+(T'_{\restriction K})$*, and* $b_6^+(T_{\restriction K}) = b_6^+(T'_{\restriction K})$ *for all* k*-element subsets* K *of* V *then* T' *and* T *are hereditarily isomorphic.*

2) Assume $c^{(3)}(T_{\restriction K}) \equiv c^{(3)}(T'_{\restriction K})$*,* $d_4^+(T_{\restriction K}) \equiv d_4^+(T'_{\restriction K})$*, and* $b_6^+(T_{\restriction K}) \equiv b_6^+(T'_{\restriction K})$ *(mod* p*) for all* k*-element subsets* K *of* V*.*

If $p \geq 7$*, and* $k_0 \geq 6$ *or* $k_0 = 0$*, then* T' *and* T *are hereditarily isomorphic.*

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