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On 4-dimensional almost para-complex pure-Walker manifolds

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Abstract: This paper is concerned with almost para-complex structures on Walker 4-manifolds. For these structures, we study some problems of Kähler manifolds. We also give an example of a flat almost para-complex manifold, which consists of a nonintegrable almost para-complex structure on Walker 4-manifolds.

Key words: Almost para-complex structure, pure metric, neutral metric, Walker metric, Kähler structure

1. Introduction

Let M_{2n} be a semi-Riemannian smooth manifold with the metric g, which is necessarily of neutral signature (n,n), and let $\Im_s^r(M_{2n})$ be the tensor field of M_{2n}, i.e. the field of all tensors of type (r,s) in M_{2n}.

An *almost para-complex structure* on M_{2n} is an affinor field φ on M_{2n}: $\varphi^2 = I$, and the 2 eigenbundles T^+M_{2n} and T^-M_{2n} corresponding to the two eigenvalues +1 and –1 have the same rank. The pair (M_{2n}, φ) is called an *almost para-complex manifold*.

Let (M_{2n}, φ) be an almost para-complex manifold with almost para-complex structure φ. If the Nijenhuis tensor of such a affinor field φ defined by

$$N_\varphi(X,Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + [X,Y]$$

is equivalent to the vanish, for any vector fields X, Y on M_{2n}, then the almost para-complex structure φ is integrable and it is said to be a para-complex structure.

1.1. Pure metrics

A *pure metric* with respect to the almost para-complex structure is a semi-Riemannian metric g such that

$$g(\varphi X, \varphi Y) = g(X,Y) \tag{1}$$

for any $X, Y \in \Im_0^1(M_{2n})$. For an almost para-complex manifold (M_{2n}, φ) with pure metric g, the triple (M_{2n}, φ, g) is called an almost para-complex manifold with pure metric g or almost para-complex pure metric manifold. If φ is integrable, then we say that (M_{2n}, φ, g) is a para-complex pure metric manifold. A similar geometry is generated if φ is an almost complex structure and acts as an antiisometry on the metric. Such metrics are known as B-metrics, Norden metrics, and anti-Hermitian metrics (see [1,2,4–10,16,17]).

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1.2. Holomorphic para-complex pure metric manifolds

Let the triple (M_{2n}, φ, g) be an almost para-complex manifolds with pure metric. A Tachibana operator applied to the pure metric g is given by [23]

$$(\Phi_\varphi g)(X, Y, Z) = g((\nabla_Y \varphi)Z, X) + g((\nabla_Z \varphi)X, Y) - g((\nabla_X \varphi)Y, Z), \tag{2}$$

for any $X, Y \in \Im_0^1(M_{2n})$. It is clear that a Tachibana operator Φ_φ is an operator from $\Im_2^0(M_{2n})$ to $\Im_3^0(M_{2n})$.

For an almost para-complex pure metric manifold (M_{2n}, φ, g), if $(\Phi_\varphi g)(X, Y, Z) = 0$ for any $X, Y, Z \in \Im_0^1(M_{2n})$, then a pure metric g is called a *holomorphic*. If the triple (M_{2n}, φ, g) is an almost para-complex pure metric manifold with holomorphic pure metric g, we say that (M_{2n}, φ, g) is a holomorphic para-complex pure metric manifold.

Now we give a theorem that we shall use later, which was proven in [17].

Theorem 1 *An almost para-complex pure metric manifold is a holomorphic para-complex pure metric manifold if and only if the almost para-complex structure is parallel with respect to the Levi-Civita connection* ∇ *of* g*, i.e. the condition* $\Phi_\varphi g = 0$ *is equivalent to* $\nabla\varphi = 0$.

Let (M_{2n}, φ, g) be an almost para-complex pure metric manifold. If $\nabla\varphi = 0$, where ∇ is the Levi-Civita connection of pure metric g, then the triple (M_{2n}, φ, g) is called a *Kähler para-complex pure metric manifold*. Therefore, from Theorem 1, the condition of being a holomorphic para-complex pure metric manifold of a manifold coincides with the condition of being a Kähler para-complex pure metric manifold of a manifold. From this, we can say that a holomorphic para-complex pure metric manifold is a Kähler para-complex pure metric manifold. In this paper, we shall use the term 'Kähler para-complex pure metric manifold' instead of 'holomorphic para-complex pure metric manifold'.

2. A pure metric on a neutral 4-manifold

In the present paper, we shall focus our attention on 4-dimensional almost para-complex pure metric manifolds of neutral signature $(++--)$. For the next step, it is appropriate to state a neutral metric g and the almost para-complex structure φ in terms of an orthonormal frame $\{e_i\}$ $(i = 1, ..., 4)$ of vertors, and its dual frame $\{e^j\}$ $(j = 1, ..., 4)$ of 1-forms. Actually, the metric g can be given by

$$g = (g(e_i, e_j)) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{3}$$

The almost para-complex structure φ can be written as

$$\varphi = (\varphi_i^j) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{4}$$

Furthermore, a simple form of pure metric g, which is pure with respect to φ in (4), can be written as

$$g=(g_{ij})=\begin{pmatrix}0&0&0&1\\0&0&-1&0\\0&-1&0&0\\1&0&0&0\end{pmatrix}. \tag{5}$$

3. Pure-Walker metrics

In this section, using (4) and (5), we study some properties of para-complex pure metric manifolds on a Walker 4-manifold.

3.1. Walker metrics

A triple (M_4,g,D) is said to be a 4-dimensional Walker manifold such that D is a 2-dimensional null plane and parallel distribution with respect to a neutral metric g. For such metrics a canonical form was obtained by Walker [22], showing the existence of suitable coordinates (x^1,x^2,x^3,x^4) where the metric is expressed as

$$g=(g_{ij})=\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&a&c\\0&1&c&b\end{pmatrix}, \tag{6}$$

where a, b, and c are some functions of the coordinates (x^1,x^2,x^3,x^4). The metric (6) is the most generic form of Walker metrics on the 4-dimensional Walker manifolds. Hereafter, we show by $\partial_i=\partial/\partial x^i \quad (i=1,2,3,4)$ the coordinate tangent vectors and we use subscript for partial derivatives, i.e. $h_i=\frac{\partial h}{\partial x^i}$, for any function h depending on (x^1,x^2,x^3,x^4). Such Walker metrics have been intensively investigated, e.g., [11,14,15,18,19].

In the present paper, we analyze a restricted form, as in [1,11,14], rather than the generic metric (6). The restricted Walker metric is the metric (6) with $c=0$, i.e.

$$g=(g_{ij})=\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&a&0\\0&1&0&b\end{pmatrix}. \tag{7}$$

3.2. Almost para-complex pure-Walker manifolds

Let (M_4,g) be a Walker 4-manifold with the Walker metric g, which is given in (7). If $\{e_1,e_2,e_3,e_4\}$ and $\{\partial_1,\partial_2,\partial_3,\partial_4\}$ are 2 orthonormal frames, the matrix $A=\left(A^i_j\right)$ of change of coordinates satisfies

$$g=A^Tg'A, \tag{8}$$

where the matrix A^T is the transpose matrix of the matrix A. In particular, $\det(A)=\mp 1$.

Substituting (3) and (7) into (8), one of the matrices A, which we will apply in the present analysis, is

$$A=\left(A^i_j\right)=\begin{pmatrix}\frac{1}{2}(1-a)&0&-\frac{1}{2}(1+a)&0\\0&\frac{1}{2}(1-b)&0&-\frac{1}{2}(1+b)\\1&0&1&0\\0&1&0&1\end{pmatrix}. \tag{9}$$

Furthermore, for affinors, the matrix $A = \left(A_j^i\right)$ of change of coordinates satisfies

$$\varphi = A^{-1}\varphi' A, \tag{10}$$

where the matrix A^{-1} is the inverse matrix of the matrix A.

The inverse of the matrix (9), A^{-1}, is given by

$$A^{-1} = \begin{pmatrix} 1 & 0 & \frac{1}{2}(1+a) & 0 \\ 0 & 1 & 0 & \frac{1}{2}(1+b) \\ -1 & 0 & \frac{1}{2}(1-a) & 0 \\ 0 & -1 & 0 & \frac{1}{2}(1-b) \end{pmatrix}. \tag{11}$$

Substituting (5) and (9) into (8), the pure-Walker metric g' is written as

$$g' = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & \frac{1}{2}(a-b) \\ 1 & 0 & \frac{1}{2}(a-b) & 0 \end{pmatrix} \tag{12}$$

with respect to the natural frame $\{\partial_1, \partial_2, \partial_3, \partial_4\}$.

Similarly, substituting (4), (9), and (11) into (10), the almost para-complex structure φ' is obtained as

$$\varphi' = \begin{pmatrix} 0 & -a & 0 & \frac{1}{2}(1-ab) \\ -b & 0 & \frac{1}{2}(1-ab) & 0 \\ 0 & 2 & 0 & b \\ 2 & 0 & a & 0 \end{pmatrix} \tag{13}$$

with respect to the natural frame $\{\partial_1, \partial_2, \partial_3, \partial_4\}$. Thus, the triple (M_4, φ', g') is an almost para-complex pure-Walker manifold.

3.3. Para-complex pure-Walker manifolds (*φ'-integrability)*

The almost para-complex structure φ' is integrable iff the torsion of φ' (Nijenhuis tensor) vanishes, i.e. the components

$$(N_{\varphi'})_{jk}^{i} = \varphi'_j{}^m \partial_m \varphi'_k{}^i - \varphi'_k{}^m \partial_m \varphi'_j{}^i - \varphi'_m{}^i \partial_j \varphi'_k{}^m + \varphi'_m{}^i \partial_k \varphi'_j{}^m \tag{14}$$

all vanish [12, p. 124].

From (13) and (14), we get the following integrability condition:

Theorem 2 *An almost para-complex pure-Walker manifold (M_4, φ', g') is a para-complex pure-Walker manifold (φ'-integrability) if and only if*

$$a_1 = b_2 = 0, \quad ba_2 - 2a_4 = 0, \quad ab_1 - 2b_3 = 0. \tag{15}$$

From this theorem, we easily see that if $a = b$ or $a = -b$, then φ' is integrable.

4. Kähler para-complex pure-Walker manifolds

In this part, we study some problems of Kähler manifolds on the almost para-complex pure-Walker manifold (M_4, φ', g').

4.1. Kähler pure-Walker metrics

Let (M_4, φ', g') be an almost para-complex pure-Walker manifold. If

$$(\Phi_{\phi'} g')_{kij} = \varphi'^{\,m}_{k} \partial_m g'_{ij} - \varphi'^{\,m}_{i} \partial_k g'_{mj} + g'_{mj}(\partial_i \varphi'^{\,m}_{k} - \partial_k \varphi'^{\,m}_{i}) + g'_{im} \partial_j \varphi'^{\,m}_{k} = 0, \tag{16}$$

then on account of Theorem 1, φ' is integrable and the triple (M_4, φ', g') is called a *Kähler para-complex pure-Walker manifold*.

Since $(\Phi_{\varphi'} g')_{kij} = (\Phi_{\varphi'} g')_{kji}$, we need only consider $(\Phi_{\varphi'} g')_{kij} \quad (i < j)$. Substituting (12) and (13) into (16), we find

$$\begin{aligned}
(\Phi_{\varphi'} g')_{113} &= -a_1 + b_1,\ (\Phi_{\varphi'} g')_{123} = b_2,\ (\Phi_{\varphi'} g')_{124} = b_1,\ (\Phi_{\varphi'} g')_{133} = -aa_1 + 2b_3, \\
(\Phi_{\varphi'} g')_{134} &= -\frac{1}{2} b(a_2 - b_2) - a_4,\ (\Phi_{\varphi'} g')_{144} = bb_1,\ (\Phi_{\varphi'} g')_{213} = -a_2,\ (\Phi_{\varphi'} g')_{214} = -a_1, \\
(\Phi_{\varphi'} g')_{224} &= -a_2 + b_2,\ (\Phi_{\varphi'} g')_{233} = -aa_2,\ (\Phi_{\varphi'} g')_{234} = -\frac{1}{2} a(a_1 + b_1) - b_3, \\
(\Phi_{\varphi'} g')_{244} &= -2a_4 + bb_2,\ (\Phi_{\varphi'} g')_{311} = 2a_1,\ (\Phi_{\varphi'} g')_{312} = a_2,\ (\Phi_{\varphi'} g')_{313} = \frac{1}{2} a(a_1 + b_1), \\
(\Phi_{\varphi'} g')_{314} &= a_4,\ (\Phi_{\varphi'} g')_{323} = \frac{1}{2} a(a_2 + b_2),\ (\Phi_{\varphi'} g')_{324} = b_3,\ (\Phi_{\varphi'} g')_{333} = ab_3, \\
(\Phi_{\varphi'} g')_{334} &= \frac{1}{4}(1 - ab)(a_2 - b_2) + aa_4,\ (\Phi_{\varphi'} g')_{344} = bb_3,\ (\Phi_{\varphi'} g')_{412} = -b_1, \\
(\Phi_{\varphi'} g')_{413} &= -a_4,\ (\Phi_{\varphi'} g')_{414} = -\frac{1}{2} b(a_1 + b_1),\ (\Phi_{\varphi'} g')_{422} = -2b_2,\ (\Phi_{\varphi'} g')_{423} = -b_3, \\
(\Phi_{\phi'} g')_{424} &= -\frac{1}{2} b(a_2 + b_2),\ (\Phi_{\phi'} g')_{433} = -aa_4,\ (\Phi_{\phi'} g')_{434} = \frac{1}{4}(1 - ab)(a_1 - b_1) - bb_3, \\
(\Phi_{\varphi'} g')_{444} &= -ba_4.
\end{aligned} \tag{17}$$

From the above equations, we have:

Theorem 3 *An almost para-complex pure-Walker manifold (M_4, φ', g') is a Kähler para-complex pure-Walker manifold, i.e. $\Phi_{\varphi'} g' = 0$, if and only if a is only dependent on x^3 and b is only dependent on x^4. In fact, it holds that a and b satisfy the following PDEs:*

$$a_1 = a_2 = a_4 = b_1 = b_2 = b_3 = 0. \tag{18}$$

4.2. Quasi-Kähler para-complex pure-Walker metrics

The basic class of almost para-complex manifolds with pure metrics is the class of the quasi-Kähler manifolds with pure metrics, which is the basic class with nonintegrable almost para-complex structure [5,13]. A *quasi-Kähler manifold* is an almost para-complex pure metric manifold (M_{2n}, φ, g) such that

$$\underset{X,Y,Z}{\sigma} g((\nabla_X \varphi)Y, Z) = 0, \tag{19}$$

where σ is the cyclic sum over X, Y, Z.

From (2) and (19), we get

$$\underset{X,Y,Z}{\sigma} g((\nabla_X\varphi)Y,Z) = (\Phi_\varphi g)(X,Y,Z) + 2g((\nabla_X\varphi)Y,Z) = 0 \tag{20}$$

for all vector fields X, Y, and Z on M_{2n}.

Let (M_{2n},φ,g) be an almost para-complex pure metric manifold. The associated pure metric of the almost para-complex pure metric manifold is defined by $G(X,Y) = (g\circ\varphi)(X,Y)$ for any $X,Y,Z \in \Im^1_0(M_{2n})$. We can easily see that G is a new pure metric, which is also called the twin metric of g (see [3]). Then, on account of (20), an almost para-complex pure-Walker manifold (M_4,φ',g') is a *quasi-Kähler para-complex pure-Walker manifold* if

$$\Phi_k g'_{ij} + 2\nabla_k G_{ij} = 0, \tag{21}$$

where G is defined by $G_{ij} = \varphi'^{h}_{i} g'_{hj}$.

From (12) and (13), for the twin pure metric G, we obtain

$$G = (G_{ij}) = \begin{pmatrix} 2 & 0 & a & 0 \\ 0 & -2 & 0 & -b \\ a & 0 & \frac{1}{2}\left(a^2-1\right) & 0 \\ 0 & -b & 0 & -\frac{1}{2}\left(b^2-1\right) \end{pmatrix}. \tag{22}$$

Therefore, we can say that the twin pure metric G is not Walker.

Now, using a straightforward calculation, the inverse of the pure-Walker metric g' is given by

$$(g')^{-1} = \begin{pmatrix} 0 & \frac{1}{2}(a-b) & 0 & 1 \\ \frac{1}{2}(a-b) & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{23}$$

Furthermore, the Levi-Civita connection of a pure-Walker metric (12) is given by

$$\begin{aligned}
\nabla_{\partial_1}\partial_3 &= \frac{1}{4}(a_1-b_1)\partial_1, \nabla_{\partial_1}\partial_4 = -\frac{1}{4}(a_1-b_1)\partial_2, \nabla_{\partial_2}\partial_3 = \frac{1}{4}(a_2-b_2)\partial_1, \\
\nabla_{\partial_2}\partial_4 &= -\frac{1}{4}(a_2-b_2)\partial_2, \nabla_{\partial_3}\partial_1 = \frac{1}{4}(a_1-b_1)\partial_1, \nabla_{\partial_3}\partial_2 = \frac{1}{4}(a_2-b_2)\partial_1, \\
\nabla_{\partial_3}\partial_3 &= \frac{1}{2}(a_3-b_3)\partial_1, \nabla_{\partial_4}\partial_1 = -\frac{1}{4}(a_1-b_1)\partial_2, \nabla_{\partial_4}\partial_2 = -\frac{1}{4}(a_2-b_2)\partial_2, \\
\nabla_{\partial_3}\partial_4 &= -\frac{1}{8}(a-b)(a_2-b_2)\partial_1 - \frac{1}{8}(a-b)(a_1-b_1)\partial_2 + \frac{1}{4}(a_2-b_2)\partial_3 \\
&\quad -\frac{1}{4}(a_1-b_1)\partial_4, \\
\nabla_{\partial_4}\partial_3 &= -\frac{1}{8}(a-b)(a_2-b_2)\partial_1 - \frac{1}{8}(a-b)(a_1-b_1)\partial_2 + \frac{1}{4}(a_2-b_2)\partial_3 \\
&\quad -\frac{1}{4}(a_1-b_1)\partial_4, \\
\nabla_{\partial_4}\partial_4 &= -\frac{1}{2}(a_4-b_4)\partial_2.
\end{aligned} \tag{24}$$

For the covariant derivative ∇G of the twin pure metric G, put $(\nabla G)_{ijk} = \nabla_i G_{jk}$. Then, from (22) and (24), after some calculations, we obtain

$$\begin{aligned}
\nabla_1 G_{13} &= \frac{1}{2}(a_1+b_1), \nabla_1 G_{24} = -\frac{1}{2}(a_1+b_1), \nabla_1 G_{33} = \frac{1}{2}a(a_1+b_1),\\
\nabla_1 G_{44} &= -\frac{1}{2}b(a_1+b_1), \nabla_2 G_{24} = -\frac{1}{2}(a_2+b_2), \nabla_2 G_{33} = \frac{1}{2}a(a_2+b_2),\\
\nabla_2 G_{44} &= -\frac{1}{2}b(a_2+b_2), \nabla_3 G_{11} = -a_1+b_1, \nabla_3 G_{12} = \frac{1}{2}(-a_2+b_2),\\
\nabla_3 G_{13} &= \frac{1}{4}a(-a_1+b_1)+b_3, \nabla_3 G_{14} = -\frac{1}{4}b(a_2-b_2), \nabla_3 G_{23} = \frac{1}{4}a(-a_2+b_2),\\
\nabla_3 G_{24} &= \frac{1}{4}a(-a_1+b_1)-b_3, \nabla_3 G_{33} = ab_3, \nabla_3 G_{34} = \frac{1}{8}(1-ab)(a_2-b_2),\\
\nabla_3 G_{44} &= \frac{1}{4}(1-ab)(a_1-b_1)-bb_3, \nabla_4 G_{12} = \frac{1}{2}(-a_1+b_1),\\
\nabla_4 G_{13} &= \frac{1}{4}b(-a_2+b_2)+a_4, \nabla_4 G_{14} = \frac{1}{4}b(-a_1+b_1), \nabla_4 G_{22} = -a_2+b_2,\\
\nabla_4 G_{23} &= \frac{1}{4}a(-a_1+b_1), \nabla_4 G_{24} = \frac{1}{4}b(-a_2+b_2)-a_4, \nabla_4 G_{33} = \frac{1}{4}(1-ab)(a_2-b_2)+aa_4,\\
\nabla_4 G_{34} &= \frac{1}{4}b(-a_2+b_2), \nabla_4 G_{44} = -ba_4. \qquad (25)
\end{aligned}$$

From (17), (21), and (25), we get:

Theorem 4 *An almost para-complex pure-Walker manifold* (M_4, φ', g') *is a quasi-Kähler para-complex pure-Walker manifold, i.e.* $\Phi_k g'_{ij} + 2\nabla_k G_{ij} = 0$*, if and only if*

$$a_1 = a_2 = a_4 = b_1 = b_2 = b_3 = 0. \qquad (26)$$

4.3. Isotropic-Kähler structures on almost para-complex pure-Walker 4-manifolds

Let (M_4, φ', g') be an almost para-complex pure-Walker manifold. If $\nabla\varphi' = 0$, then (M_4, φ', g') becomes a Kähler para-complex pure-Walker manifold. If (M_4, φ', g') is a Kähler para-complex pure-Walker manifold, then $\|\nabla\varphi'\|^2$ vanishes. However, the inverse position is not always being true. That is, in general, the vanishing of the square norm $\|\nabla\varphi'\|^2$ does not always imply the Kähler condition $\nabla\varphi' = 0$.

An almost para-complex pure metric manifold satisfying the condition $\|\nabla\varphi'\|^2 = 0$ $(\nabla\varphi' \neq 0)$ is called an *isotropic-Kähler manifold*. Here, the square norm $\|\nabla\varphi'\|^2$ of $\nabla\varphi'$ is defined by

$$\|\nabla\varphi'\|^2 = g'^{ij} g'^{kl} g'_{ms} \nabla_i \varphi'^{\,m}_k \nabla_j \varphi'^{\,s}_l. \qquad (27)$$

Now, with long but straightforward calculations, we find the following condition:

Theorem 5 *An almost para-complex pure-Walker manifold* (M_4, φ', g') *is a isotropic-Kähler para-complex pure-Walker manifold, i.e.* $\|\nabla\varphi'\|^2 = 0$*, if and only if*

$$a_1 + b_1 = 0, \quad a_2 + b_2 = 0. \qquad (28)$$

4.4. Nearly-Kähler structures on almost para-complex pure-Walker 4-manifolds

Let (M_4, φ', g') be an almost para-complex pure-Walker manifold. If

$$(\nabla_X \varphi')\, Y + (\nabla_Y \varphi')\, X = 0, \quad \left(\nabla_i \varphi'^{\,k}_j + \nabla_j \varphi'^{\,k}_i = 0\right) \tag{29}$$

for all vector fields X, Y, and Z on M_{2n}, then the triple (M_4, φ', g') is called a *nearly-Kähler manifold* with pure metric [20,24].

Therefore, from (29), we have:

Theorem 6 *A triple (M_4, φ', g') is a nearly-Kähler para-complex pure-Walker manifold if and only if*

$$a_1 = a_2 = a_4 = b_1 = b_2 = b_3 = 0. \tag{30}$$

5. On flat nonintegrable almost para-complex pure-Walker manifold

In this section, we give an example of a 4-dimensional flat almost para-complex pure-Walker manifold whose almost para-complex structure is not integrable. Examples of 4-dimensional almost Hermitian manifolds whose almost complex structures are not integrable are given in [21]. Additionally, examples of flat almost Norden manifolds with nonintegrable almost complex structures are given in [1].

Let (M_4, φ', g') be an almost para-complex pure-Walker manifold with pure-Walker metric g' in (12) and almost para-complex structure φ' in (13). We see that g' depends on the difference $a - b$. If we put

$$a(x^1, x^2, x^3, x^4) - b(x^1, x^2, x^3, x^4) = \alpha(x^1, x^2, x^3, x^4), \tag{31}$$

then the nonzero components R'_{ijkl} of the Riemannian curvature tensor R' with respect to pure-Walker metric g' are given by

$$\begin{aligned}
R'_{1314} &= -\frac{1}{2}\alpha_{11}, R'_{1324} = R'_{1423} = -\frac{1}{2}\alpha_{12}, R'_{1334} = -\frac{1}{2}\alpha_{13} + \frac{1}{4}\alpha_1\alpha_2,\\
R'_{1434} &= \frac{1}{2}\alpha_{14} - \frac{1}{4}(\alpha_1)^2, R'_{2324} = -\frac{1}{2}\alpha_{22}, R'_{2334} = -\frac{1}{2}\alpha_{23} + \frac{1}{4}(\alpha_2)^2,\\
R'_{2434} &= \frac{1}{2}\alpha_{24} - \frac{1}{4}\alpha_1\alpha_2, R'_{3434} = \alpha_{34} - \frac{1}{2}\alpha\alpha_1\alpha_2.
\end{aligned} \tag{32}$$

For seeking a flat metric, we suppose that

$$a(x^1, x^2, x^3, x^4) - b(x^1, x^2, x^3, x^4) = 2\alpha(x^3, x^4). \tag{33}$$

In this case, the nonzero component of the curvature tensor R' is

$$R'_{3434} = \alpha_{34}. \tag{34}$$

We see that the pure-Walker metric g' is still not flat. If $\alpha(x^3, x^4) = \beta\left(x^3\right) + \gamma\left(x^4\right)$, then the pure-Walker metric g' is flat.

Corollary 1 *The pure-Walker metric*

$$g' = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & \frac{1}{2}\left(\alpha(x^3)+\beta(x^4)\right) \\ 1 & 0 & \frac{1}{2}\left(\alpha(x^3)+\beta(x^4)\right) & 0 \end{pmatrix} \tag{35}$$

is flat.

From (15), for almost para-complex structure φ', we have:

Corollary 2 *If a and b satisfy*

$$a(x^1,x^2,x^3,x^4) - b(x^1,x^2,x^3,x^4) = 2\left(\beta\left(x^3\right)+\gamma\left(x^4\right)\right), \tag{36}$$

then φ' is not integrable in general. Moreover, φ' is integrable if and only if $a_1 = a_2 = a_4 = 0, \quad a_3 - 2\beta_3 = 0$.

Therefore, we have:

Theorem 7 *Let (M_4, φ', g') be an almost para-complex pure-Walker manifold endowed with the metric g' as in (35) and an almost para-complex structure φ' as in (13). If $a(x^1,x^2,x^3,x^4) - b(x^1,x^2,x^3,x^4) = 2\left(\beta\left(x^3\right)+\gamma\left(x^4\right)\right)$ as in (36), then the almost para-complex pure-Walker manifold (M_4, φ', g') admits a flat pure-Walker metric g' and nonintegrable almost para-complex structure φ'.*

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