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On the equivariant cohomology algebra for solenoidal actions

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Abstract: We prove, under certain conditions, that if a solenoidal group (i.e. 1-dimensional compact connected abelian group) acts effectively on a compact space then the fixed point set is nonempty and $H^*_G(X,\mathbb{Q})$ has a presentation similar to the presentation of $H^*(X,\mathbb{Q})$ as proven by Chang in the case of a circle group.

Key words: TNHZ, solenoid, c-symplectic, algebra presentation

1. Introduction

In the cohomology theory of transformation groups (based on the Borel construction), most of the results concern Lie group, especially abelian Lie group actions. Results for non-Lie group actions are fewer. The main reason for this is the complexity of determining the cohomology ring of classifying space for non-Lie groups and equivariant cohomology algebra of the space on which the non-Lie group acts. However, there is considerable information about compact non-Lie transformation groups in [10].

It is well known that locally compact groups can be "approximated" by Lie groups. This means if G is a locally compact group with finitely many components then G has arbitrarily small compact normal subgroup N such that G/N is a Lie group. This was proven by Yamabe [20]; see also the work of Montgomery and Zippin [17].

We say that G is an n-dimensional compact connected abelian group if G is the projective limit of n-dimensional tori and write $dimG = n$. One can say that if G is an n-dimensional compact connected abelian group then G has a totally disconnected closed subgroup N such that $G/N \simeq T^n$, an n torus. For details see [10], 8.17–8.24. We say G is a finite-dimensional compact connected abelian group if $dimG = n$ for some $n \in \mathbb{N}$. If $dimG = 1$, then G is called solenoid.

As a well-known example for a solenoid, let us choose a prime number p. Let set G_n be the circle group $T = \{z \in \mathbb{C} : |z| = 1\}$ and define $f_n^{n+1} : G_{n+1} \to G_n$, $f_n^{n+1}(z) = z^p$ for all $n \in \mathbb{N}$ and $z \in T$. The projective limit of the projective system $\{G_n, f_n^{n+1}\}$ is called the p-adic solenoid T_p. This projective limit would have the p-adic integers, Z_p, as a totally disconnected closed subgroup such that $T_p/Z_p \simeq T$. Solenoids are one of the prototypes of compact abelian groups that are connected, but not arc-wise connected.

If an n-dimensional compact connected abelian group G acts effectively on a Hausdorff space X (all actions are assumed to be continuous), then there is an induced, almost effective action of the n torus G/N on

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the orbit space X/N and $X \to X/N$ induces a homeomorphism $X^G \approx (X/N)^{G/N}$. (Here X^G denotes fixed point set of action G.)

The orbit space of the action of G/N on X/N is homeomorphic to the orbit space X/G. Moreover, the orbit space X/N inherits global and local cohomological properties from the space X. Thus, many questions about the cohomological properties of orbit spaces and fixed point set of actions of finite-dimensional compact connected abelian groups are reduced to questions about torus actions by comparing actions of G on X and G/N on X/N.

The study of such actions is motivated by a classic unresolved problem of topological transformation groups, known as the generalized Hilbert–Smith conjecture, which states that a locally compact effective transformation group on manifold is a Lie group. A well-known fact (see [11]) states that the Hilbert–Smith conjecture is equivalent the following conjecture:

Conjecture 1.1 *A p-adic group cannot act effectively on a connected finite dimensional manifold.*

The construction of effective p-adic spaces plays an important role in the study of the Hilbert–Smith conjecture. One way to obtain a compact space where a p-adic group acts effectively is to take the inverse limit of inverse systems of effective T-spaces with bonding maps that satisfy certain equivariance properties. This is because, if $\{X_\alpha, f_\alpha^\beta\}$ is an inverse system of topological spaces and $\{G_\alpha, \varphi_\alpha^\beta\}$ is an inverse system of topological groups, where each X_α is a G_α-space and each bonding map f_α^β is φ_α^β-equivariant, then $\varprojlim X_\alpha$ is a $\varprojlim G_\alpha$-space with the action given by

$$(g_\alpha)(x_\alpha) = (g_\alpha x_\alpha).$$

In this paper, under certain conditions, we try to determine the structure of equivariant cohomology algebra with rational coefficients for solenoidal actions on compact spaces.

2. Preliminaries

Throughout this paper X will be a compact space and we shall use sheaf cohomology with coefficients in a field k of characteristic 0.

We need to recall definitions on the notion of effectiveness.

Definition 2.1 (1) *Let G be a topological group and X a G space. If the ineffective kernel, $\bigcap_{x\in X} G_x$, is finite, then this action is called almost effective.*

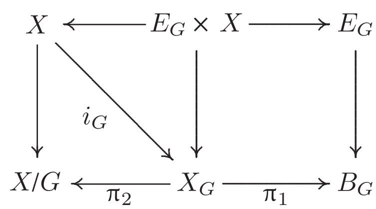
(2) *Let G be a compact connected Lie group and let X be a G space. The action of G on X is said to be cohomologically effective (with coefficients in k) if the restriction homomorphism*

$$H^*(X,k) \to H^*(X^K,k)$$

is not a monomorphism for any subcircle $K \subseteq G$.

Remark 2.2 *If G is a compact connected Lie group and X is a closed orientable manifold, then an action of G on X is cohomologically effective if and only if it is almost effective. More generally, this holds if X is a compact orientable cohomology manifold over $\mathbb{Q}$. (See* [5]*, Chapter 1, Corollary 4.6, and Chapter 5, Theorem 3.2.)*

For any topological group G, by introducing a suitable topology in the $(m+1)$-join $G^{(m+1)} = G*...*G$ and letting G act on it naturally, we obtain an m-universal G-bundle $(G^{(m+1)}, p, G^{(m+1)}/G, G)$ and a contractible space $E_G = \lim_m G^{(m+1)}$ by taking the direct limit, on which G acts freely and properly. We denote the quotient space by B_G, which is called a classifying space of G [14,15]. Thus, we have principal G-bundle $E_G \to B_G$, called the universal G-bundle. Let X be a G-space. A technique for studying G-actions is the construction of the so-called Borel space $E_G \times_G X = (E_G \times X)/G$ associated to the G-space X. (On $E_G \times X$, there is the diagonal action given by $g(e,x) = (ge, gx)$.) This leads to the following commutative diagram:



where π_1 is a fiber bundle mapping with fiber X and structure group G/K where K is the ineffective kernel of the G action on X, π_2 is a mapping such that $\pi_2^{-1}(x^*) = B_{G_x}$, where $x^* \in X/G$, and $x \in x^*$. The equivariant graded cohomology algebra of X with coefficient k is then defined by $H^*_G(X;k) = H^*(X_G;k)$.

X is said to be totally nonhomologous to zero (TNHZ) in $X_G \to B_G$ with respect to $H^*(-,k)$ if

$$i^*_G : H^*_G(X;k) \to H^*(X,k)$$

is surjective.

Definition 2.3 *Let X be a Poincare duality space of formal dimension $fd(X) = 2n$. (i.e. $H^i(X,k) = 0$ for $i > 2n$, $H^{2n}(X,k) \cong k$, $dim_k H^i(X,k) < \infty$, for all i, and for all $0 \le i \le 2n$ the cup product*

$$H^i(X,k) \times H^{2n-i}(X,k) \to H^{2n}(X,k) \cong k$$

is a nondegenerate bilinear form.) We say that X is cohomologically symplectic (c-symplectic for short) over k if there is a class $w \in H^2(X,k)$, which is called the c-symplectic class, such that $w^n \neq 0$.

Definition 2.4 *Let G be a compact connected Lie group and X a c-symplectic space. If G acts on X, then the action is said to be cohomologically Hamiltonian (c-Hamiltonian for short) if*

$$w \in im\{i^*_G : H^*_G(X,k) \to H^*(X,k)\}.$$

Remark 2.5 (1) *A closed symplectic manifold is c-symplectic (over $\mathbb{R}$) with $w = [w]$, the class of symplectic form.*

(2) *If X is a closed symplectic manifold, G is a compact connected Lie group, G is acting on X, and the action is symplectic, then the action is Hamiltonian if and only if it is c-Hamiltonian. Necessity follows from Frankel's theorem (see* [9]*). Sufficiency follows easily from the results and techniques of Atiyah-Bott (see* [3]*, Section 4; Audin* [4]*, Chapter 5, Proposition 3.1.1; and McDuff and Salamon* [13]*, Section 5.2).*

The next theorem is important for our main result.

Theorem 2.6 *(**Bredon et al.** [7, 5.1]**; Löwen** [12]). If N is a totally disconnected compact group and X is a locally compact N-space, then the orbit map $\pi : X \to X/N$ induces an isomorphism*

$$H_c^*(X/N, \mathbb{Q}) \simeq (H_c^*(X, \mathbb{Q}))^N$$

(H_c^* denotes sheaf cohomology with compact supports. For the details, the reader is referred to Bredon's monograph [6]*).*

Remark 2.7 *Let G be a finite-dimensional compact connected abelian group acting on a compact space X. Let N be a totally disconnected closed subgroup of G such that G/N is a torus. Since G is connected, its action (and hence that of N) on $H^*(X, \mathbb{Q})$ is trivial* (see [6, II.10.6 cf. II.11.11])*. Thus, Theorem 2.6 implies that*

$$H^*(X, \mathbb{Q}) \cong H^*(X/N, \mathbb{Q}).$$

3. Main result

Let $\mathfrak{L}$ denote the category of locally compact abelian groups, whose morphisms are continuous homomorphisms. For an object G of the category $\mathfrak{L}$, the group $Hom(G, T)$ of continuous homomorphisms from G to the circle group T endowed with the compact-open topology is an object of $\mathfrak{L}$. This group is called the character group of G and is denoted by $\hat{G}$.

The correspondence $G \mapsto \hat{G}$ defines a contravariant functor $\chi : \mathfrak{L} \to \mathfrak{L}$. The Pontryagin duality theorem states that

$$G \simeq \chi(\chi(G)) = \hat{\hat{G}}.$$

This means that χ is a contravariant category equivalence. (i.e. there are natural equivalences $q_1 : 1 \to \chi^2$ and $q_2 : \chi^2 \to 1$ where $1 : \mathfrak{L} \to \mathfrak{L}$ is identitiy functor in $\mathfrak{L}$). The next proposition is a well-known formal consequence of an equivalence of categories.

Proposition 3.1 *χ takes projective limits to direct limits.*

Proof Let $\{G, f_\alpha\}$ be the projective limit of the projective system $\{G_\alpha, f_\alpha^\beta\}$. Then χ induces morphisms $\hat{f}_\alpha : \hat{G}_\alpha \to \hat{G}$ where $\{\hat{G}_\alpha, \hat{f}_\alpha^\beta\}$ is a direct system in $\mathfrak{L}$ satisfying $\hat{f}_\beta \hat{f}_\alpha^\beta = \hat{f}_\alpha$ whenever $\alpha \leq \beta$. Let H be a locally compact abelian group and suppose $h_\alpha : \hat{G}_\alpha \to H$ are morphisms such that $h_\beta \hat{f}_\alpha^\beta = h_\alpha$ whenever $\alpha \leq \beta$. We apply χ and we see that $\{\hat{\hat{G}}, \hat{\hat{f}}_\alpha\}$ is projective limit of the projective system $\{\hat{\hat{G}}_\alpha, \hat{\hat{f}}_\alpha^\beta\}$ by Pontryagin's duality theorem. Therefore, there is a unique morphism $f : \hat{H} \to \hat{\hat{G}}$ such that $\hat{\hat{f}}_\alpha f = \hat{h}_\alpha$ for every α. Since χ is a contravariant category equivalence, there is a morphism $f_0 : \hat{G} \to H$ such that $\hat{f}_0 = f$ and f_0 is the unique morphism in $Hom(\hat{G}, H)$ such that $f_0 \hat{f}_\alpha = h_\alpha$ for every α (see [16, Prop. 10.1). Thus, $\{\hat{G}, \hat{f}_\alpha\}$ is the direct limit of the direct system $\{\hat{G}_\alpha, \hat{f}_\alpha^\beta\}$. □

($*$) *For the rest of this section G will be assumed to be a solenoid. X will be a compact G-space and TNHZ in $X_G \to B_G$ with respect to $H^*(-, \mathbb{Q})$.*

Theorem 3.2 *Let $N \subseteq G$ be a totally disconnected closed subgroup such that G/N is the circle group. If $X^G = \emptyset$, then X/N is $TNHZ$ in $(X/N)_{G/N} \to B_{G/N}$ with respect to $H^*(-, \mathbb{Q})$.*

Proof Since $X \to X/N$ induces a homeomorphism,

$$X^G \approx (X/N)^{G/N},$$

we have

$$(X/N)^{G/N} = \emptyset$$

where $(X/N)^{G/N}$ is the fixed point set of induced action of the circle group G/N on the orbit space X/N. Thus, all isotropy subgroups, $(G/N)_{Nx}$, of G/N are finite. It is obvious that any finite subgroups of the circle group are the groups of the nth roots of unity for some n. For the isotropy subgroups, G_x of G is explicitly discussed in ([10, Prop.10.31ff]) and we have

$$(G/N)_{Nx} = NG_x/N \simeq G_x/(G_x \cap N).$$

In particular, it follows that

$$G_x = \varprojlim_{N\in\mathcal{N}}(G/N)_{Nx}$$

where $\mathcal{N}$ is a filter basis of compact normal subgroups of G such that G/N is a circle for $N \in \mathcal{N}$ and such that $\bigcap \mathcal{N} = 1$. (For $M \subseteq N$ in $\mathcal{N}$, let $f_N^M : G/M \to G/N$ denote the natural homomorphism given by $f_N^M(gM) = gN$. Then restriction of f_N^M to the $(G/M)_{Mx}$ gives a homomorphism from $(G/M)_{Mx}$ into the $(G/N)_{Nx}$. This restricted homomorphism constitutes a projective system.) This implies that all isotropy subgroups of G are projective limits of finite cyclic groups.

For the cohomology of the universal classifying space B_{G_x} with integer coefficient, since G_x is the projective limit of a projective system of finite cyclic groups, we have

$$H^r(B_{G_x}, \mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & r = 0, \\ \hat{G}_x & r = 2, \\ 0 & r \neq 0 \text{ or } 2. \end{cases}$$

(See [7, remarks for Theorem 1].) By Proposition 3.1, we have that $H^2(B_{G_x}, \mathbb{Z}) = \hat{G}_x$ is the direct limit of the direct system $\{(\hat{G/N})_{Nx}, \hat{f}_N^M\}$. Since all $(G/N)_{Nx}$ are finite cyclic groups,

$$(\hat{G/N})_{Nx} = (G/N)_{Nx} \text{ for all } N \in \mathcal{N}.$$

Thus, $H^2(B_{G_x}, \mathbb{Q}) = \hat{G}_x \otimes_{\mathbb{Z}} \mathbb{Q}$ is the direct limit of the direct system

$$\{(\hat{G/N})_{Nx} \otimes_{\mathbb{Z}} \mathbb{Q},\ \hat{f}_N^M \otimes 1_{\mathbb{Q}}\}.$$

It is a well-known fact that the tensor product of finite abelian groups and rationals over $\mathbb{Z}$ is 0. Therefore, we have

$$(\hat{G/N})_{Nx} \otimes_{\mathbb{Z}} \mathbb{Q} = 0.$$

It follows that

$$H^2(B_{G_x}, \mathbb{Q}) = 0.$$

Thus, for $r \geq 1$, $H^r(B_{G_x}, \mathbb{Q})$ are trivial for all $x \in X$. By the Vietoris–Begle mapping theorem (see [19]) the orbit projection $\pi_2 : X_G \to X/G$ thus induces an isomorphism:

$$\pi_2^* : H^*(X/G, \mathbb{Q}) \cong H_G^*(X, \mathbb{Q}).$$

Similarly, we have

$$H^*(X/G, \mathbb{Q}) \cong H_{G/N}^*(X/N, \mathbb{Q})$$

by considering the orbit projection

$$(X/N)_{G/N} \to (X/N)/(G/N) \approx X/G.$$

On the other hand,

$$H^*(X, \mathbb{Q}) \cong H^*(X/N, \mathbb{Q})$$

by Remark 2.7. From the commutative diagram,

$$\begin{array}{ccc} H_G^*(X, \mathbb{Q}) & \xrightarrow[i_G^*]{} & H^*(X, \mathbb{Q}) \\ \uparrow \cong & & \uparrow \cong \\ H_{G/N}^*(X/N, \mathbb{Q}) & \xrightarrow[i_{G/N}^*]{} & H^*(X/N, \mathbb{Q}) \end{array}$$

we see that X/N is $TNHZ$ in $(X/N)_{G/N} \to B_{G/N}$ with respect to $H^*(-, \mathbb{Q})$. □

With the assumptions of $(*)$, we have 3 corollaries.

Corollary 3.3 *If* $0 < dimH^*(X, \mathbb{Q}) < \infty$, *then* $X^G \neq \emptyset$.

Proof Suppose $X^G = \emptyset$. Then X/N is TNHZ in $(X/N)_{G/N} \to B_{G/N}$ by Theorem (3.2). Since $X^G \approx (X/N)^{G/N}$, we have

$$dimH^*(X/N, \mathbb{Q}) = dimH^*((X/N)^{G/N}, \mathbb{Q}) = 0$$

(see [2, Corollary 3.1.15]). Since $H^*(X, \mathbb{Q}) \cong H^*(X/N, \mathbb{Q})$, this contradicts the assumption. □

Next we need the notion of a rational cohomology n-manifold. A rational cohomology n-manifold is a locally compact space whose cohomological dimension over $\mathbb{Q}$ is finite, and it has locally constant cohomologies over $\mathbb{Q}$ such that it is equal to $\mathbb{Q}$ for degree n and to zero in degrees other than n. A connected rational cohomology n-manifold over X is called orientable if $H_c^*(X, \mathbb{Q}) \cong \mathbb{Q}$. Details can be found in the work of Bredon (see [6], Section V.16).

Topological n-manifolds are examples of rational cohomology n-manifolds. A nonmanifold example is the open cone over the $(n-1)$-manifold, which is not a sphere but has the rational cohomology of an $(n-1)$-sphere (for example, a real projective space of odd dimensions).

The property of being a rational cohomology manifold passes to orbit spaces under some mild conditions.

Theorem 3.4 ***(See Raymond*** [18]*) Let* N *be a second countable totally disconnected compact group acting on a connected orientable rational cohomology* n*-manifold* X. *Suppose the action of* N *on* $H_c^*(X, \mathbb{Q})$ *is trivial. Then* X/N *is a rational cohomology* n*-manifold.*

The next corollary is an explicit application to compact (c-)symplectic (cohomology) manifolds without using any geometric concepts.

Corollary 3.5 *If X is a compact connected c-symplectic orientable rational cohomology manifold, then the fixed point set is nonempty.*

Proof Let N be a compact, totally disconnected subgroup of G such that G/N is a circle. X/N is a compact connected c-symplectic rational cohomology manifold having the same dimensions as X. This follows by Raymond's theorem (see [18]) and Remark 2.7. Since the induced action of the circle, G/N, on the orbit space X/N is almost effective, this action is cohomologically effective by Remark 2.2. Suppose $X^G = \emptyset$. Then X/N is TNHZ in $(X/N)_{G/N} \to B_{G/N}$ by Theorem (3.2). Since TNHZ implies cohomologically Hamiltonian, by the result of Allday (see [1], Proposition 6.7 and Remark 6.8), the fixed point set $(X/N)^{G/N}$ (which is homeomorphic to X^G) is nonempty and it has at least 2 connected components, which contradicts the assumption. □

Next we need to recall some basic facts concerning commutative graded algebras. Let k be a field and $A = \bigoplus_{i=0}^{\infty} A_i$ be an $\mathbb{N}$-graded k-algebra. We shall assume that A is connected: i.e. $A_0 = k$. We shall also assume that A is commutative in the graded sense: i.e. for any $a \in A_i, b \in A_j, ba = (-1)^{ij}ab$. In the category of connected commutative $\mathbb{N}$-graded k-algebras the free objects are those of the form $k[x_i : i \in I] \otimes \bigwedge(y_j : j \in J)$ where $k[x_i : i \in I]$ is the polynomial ring generated by $\{x_i : i \in I\}$ where each $x_i, i \in I$, is homogeneous of positive even degree and $\bigwedge(y_j : j \in J)$ is the exterior algebra generated by $\{y_j : j \in J\}$ where each $y_j, j \in J$, is homogeneous of positive odd degree.

Definition 3.6 *A connected commutative graded algebra A is said to be finitely generated if there is a homogeneous epimorphism of k-algebras of degree zero*

$$\pi : B = k[x_1, ..., x_r] \otimes \bigwedge(y_1, ..., y_s) \to A$$

where each x_i (resp y_j) is homogeneous of positive even (resp. odd) degree. Then $J = Ker\pi$ is called the ideal of relations. We shall refer to the exact sequence

$$0 \to J \to B \to A \to 0$$

as a presentation of A.

Let G be a circle group and $R_G = H^*(B_G, \mathbb{Q})$. It is well known that $R_G = \mathbb{Q}[w]$, $degw = 2$. Furthermore, $\mathbb{Q}$ is an R_G-module via the standard augmentation homomorphism $R_G \to \mathbb{Q}$ defined by $w \to 1$. The next theorem, proven by Chang (see [8, pp. 245246]), is one of the steps of the corollary of Theorem 2 in [8].

Theorem 3.7 *(**See Chang** [8]) Let X be a space, and suppose that $H^*(X, \mathbb{Q})$ has a $\mathbb{Q}$-algebra presentation*

$$0 \to J \to \mathbb{Q}[x_1, ..., x_g] \otimes \bigwedge(y_1, ..., y_h) \to H^*(X, \mathbb{Q}) \to 0$$

where $x_1, ..., x_g$ are generators of positive even degree, $y_1, ..., y_h$ are generators of odd degree, and the ideal of relations $J = (f_1, ..., f_m, e_1, ..., e_n)$, where $f_1, ..., f_m$ are relations of even degree and $e_1, ..., e_n$ are relations of odd degree.

Let G be a circle group, and suppose that G is acting on X so that X is $TNHZ$ in $X_G \to B_G$ with respect to $H^*(-,\mathbb{Q})$.

Then $H^*_G(X,\mathbb{Q})$ has an R_G-algebra presentation as follows, and there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J_G & \longrightarrow & R_G[X_1,...,X_g]\otimes\bigwedge(Y_1,...,Y_h) & \longrightarrow & H^*_G(X,\mathbb{Q}) & \longrightarrow & 0\\ && \big\downarrow && \big\downarrow \varphi && \big\downarrow i^*_G && \\ 0 & \longrightarrow & J & \longrightarrow & \mathbb{Q}[x_1,...,x_g]\otimes\bigwedge(y_1,...,y_h) & \longrightarrow & H^*(X,\mathbb{Q}) & \longrightarrow & 0 \end{array}$$

where $J_G = (F_1,...,F_m,E_1,...,E_n), \varphi(X_i) = x_i$ for $1 \le i \le g, \varphi(Y_i) = y_i$ for $1 \le i \le h, \varphi(F_i) = f_i$ for $1 \le i \le m$, and $\varphi(E_i) = e_i$ for $1 \le i \le n$.

We will prove that a similar result holds for solenoid actions on a compact space without a fixed point.

Recall that for a solenoid G, there is a totally disconnected closed subgroup N such that G/N is a circle group. It is well known that $H^*(B_{G/N},\mathbb{Q}) = \mathbb{Q}[w]$, $degw = 2$.

Let $B_\pi : B_G \to B_{G/N}$ be the mapping induced by canonical epimorphism $\pi : G \to G/N$. In the following, let $R_{G/N} = \mathbb{Q}[w]$ and $R_G = \mathbb{Q}[v]$, $v = B^*_\pi(w)$ where $B^*_\pi : H^*(B_{G/N},\mathbb{Q}) \to H^*(B_G,\mathbb{Q})$.

Corollary 3.8 *Suppose we have assumptions of $(*)$ and suppose that $H^*(X,\mathbb{Q})$ has a $\mathbb{Q}$-algebra presentation*

$$0 \to J \to \mathbb{Q}[x_1,...,x_g] \otimes \bigwedge(y_1,...,y_h) \to H^*(X,\mathbb{Q}) \to 0$$

where $x_1,...,x_g$ are generators of positive even degree, $y_1,...,y_h$ are generators of odd degree, and the ideal of relations $J = (f_1,...,f_m,e_1,...,e_n)$, where $f_1,...,f_m$ are relations of even degree and $e_1,...,e_n$ are relations of odd degree.

If $X^G = \emptyset$, then $H^*_G(X,\mathbb{Q})$ has an R_G-algebra presentation as follows and there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J_G & \longrightarrow & R_G[\bar{X}_1,...,\bar{X}_g]\otimes\bigwedge(\bar{Y}_1,...,\bar{Y}_h) & \longrightarrow & H^*_G(X,\mathbb{Q}) & \longrightarrow & 0\\ && \big\downarrow && \big\downarrow \varphi && \big\downarrow i^*_G && \\ 0 & \longrightarrow & J & \longrightarrow & \mathbb{Q}[x_1,...,\mathrm{x}_g]\otimes\bigwedge(y_1,...,\mathrm{y}_h) & \longrightarrow & H^*(X,\mathbb{Q}) & \longrightarrow & 0 \end{array}$$

where $J_G = (F_1,...,F_m,E_1,...,E_n), \varphi(\bar{X}_i) = x_i$ for $1 \le i \le g, \varphi(\bar{Y}_i) = y_i$ for $1 \le i \le h, \varphi(F_i) = f_i$ for $1 \le i \le m$, and $\varphi(E_i) = e_i$ for $1 \le i \le n$.

Proof Since the orbit projection $\pi : X \to X/N$ induces an isomorphism $\pi^* : H^*(X/N,\mathbb{Q}) \simeq H^*(X,\mathbb{Q})$, we consider

$$0 \to J \to \mathbb{Q}[x_1,...,x_g] \otimes \bigwedge(y_1,...,y_h) \to H^*(X/N,\mathbb{Q}) \to 0$$

as a $\mathbb{Q}$-algebra presentation of $H^*(X/N,\mathbb{Q})$. On the other hand, X/N is TNHZ in $(X/N)_{G/N} \to B_{G/N}$ with respect to $H^*(-,\mathbb{Q})$ by Theorem 3.2. It follows by Chang's result that $H^*_{G/N}(X/N,\mathbb{Q})$ has an $R_{G/N}$-presentation

$$f : R_{G/N}[X_1,...,X_g] \otimes \bigwedge(Y_1,...,Y_h) \to H^*_{G/N}(X/N,\mathbb{Q})$$

as in Theorem 3.7. Since X/N is TNHZ, it follows that

$$(\alpha,\pi)^*: H^*_{G/N}(X/N,\mathbb{Q}) \to H^*_G(X,\mathbb{Q})$$

is surjective where $\alpha: G \to G/N, \pi: X \to X/N$.

Let $(\alpha,\pi)^*(X_i) = \bar{X}_i$, $i = 1,...,g$ and $(\alpha,\pi)^*(Y_j) = \bar{Y}_j$, $j = 1,...,h$. We define a homogeneous epimorphism of $\mathbb{Q}$-algebras of degree zero:

$$R_G[\bar{X}_1,...,\bar{X}_g] \otimes \bigwedge(\bar{Y}_1,...,\bar{Y}_h) \to H^*_G(X,\mathbb{Q}),$$

$$\bar{X}_i \mapsto (\alpha,\pi)^*(f(X_i)),$$

$$\bar{Y}_j \mapsto (\alpha,\pi)^*(f(Y_j)).$$

It is easy to check that this epimorphism satisfies all the conditions we need. □

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