

Hilbert series of the braid monoid MB_4 in band generators

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Abstract: L. A. Bokut gave a Gröbner–Shirshov basis of the braid group B_n in band generators. Using this presentation and solving all the ambiguities we construct a linear system for irreducible words and compute the Hilbert series of the braid monoid MB_4 .

Key words: Braid monoids, Gröbner–Shirshov basis, reducible and irreducible words, Hilbert series

1. Introduction

The braid group B_n admits the following classical presentation given by E. Artin [3]:

$$B_n = \left\langle x_1, \dots, x_{n-1} \left| \begin{array}{l} x_i x_j = x_j x_i \text{ if } |i - j| \geq 2 \\ x_{i+1} x_i x_{i+1} = x_i x_{i+1} x_i \text{ if } 1 \leq i \leq n - 2 \end{array} \right. \right\rangle.$$

Elements of B_n are *words* expressed in the generators x_1, \dots, x_{n-1} . The braid group B_n admits another presentation called the band presentation given by J. Birman, K. H. Ko, and S. J. Lee [5]. This presentation consists of the generators a_{ts} , $n \geq t > s \geq 1$, where a_{ts} represents the braid in which the t^{th} string crosses over the s^{th} string while the s^{th} and t^{th} strings cross in front of all intermediate strings. Therefore, the band presentation of the braid group B_n is given by

$$B_n = \left\langle a_{ts}, n \geq t > s \geq 1 \left| \begin{array}{l} a_{ts} a_{rq} = a_{rq} a_{ts}, (t - r)(s - r)(s - q)(t - q) > 0 \\ a_{ts} a_{sr} = a_{tr} a_{ts} = a_{sr} a_{tr}, n \geq t > s \geq 1 \end{array} \right. \right\rangle.$$

The *braid monoid* MB_n consisting of only positive crossings admits the same presentation of the braid group B_n :

$$MB_n = \left\langle a_{ts}, n \geq t > s \geq 1 \left| \begin{array}{l} a_{ts} a_{rq} = a_{rq} a_{ts}, (t - r)(s - r)(s - q)(t - q) > 0 \\ a_{ts} a_{sr} = a_{tr} a_{ts} = a_{sr} a_{tr}, n \geq t > s \geq 1 \end{array} \right. \right\rangle.$$

In [10] we constructed a linear system for the braid monoid MB_n in Artin generators and computed the Hilbert series for the braid monoids MB_3 and MB_4 . In this paper we construct a similar kind of linear system to compute the Hilbert series of MB_4 in band generators. This linear system is the key behind all the computations to compute the Hilbert series of MB_4 .

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In the Hilbert series $\frac{1}{q_n(t)}$ of MB_n (for $n = 3, 4, 5, 6$) for Artin generators, the degrees of the polynomials $q_n(t)$ are 3, 6, 10, 15 respectively (for details see [11]), whereas, in the case of band generators, the degrees of $q_n(t)$ are 2 and 3 (for $n = 3$ and 4). The advantage of the Hilbert series (for band generators) is that the growth of $q_n(t)$ for band generators is much slower than the growth for Artin generators.

2. Preliminaries

In a presentation of a monoid we fix a total order of the generators (between the generators we choose the natural order $a_{21} < a_{31} < a_{32} < a_{41} < \dots < a_{n(n-1)}$). In the monoid the relation $\alpha = \beta$ will be written as $\alpha > \beta$ in the length-lexicographic order. Let $\alpha_1 = uw$ and $\alpha_2 = vw$; then the word of the form uvw is said to be an *ambiguity* (for details see [4]). If $\alpha_1 v = u\alpha_2$ as a relation as well as in the length-lexicographic order then we say that the ambiguity uvw is solvable (or solved). Such a presentation is *complete* if and only if all the ambiguities are solvable (for details see [4], [8]). Corresponding to the relations $\alpha = \beta$, the changes $\gamma\alpha\delta \rightarrow \gamma\beta\delta$ give a rewriting system. A complete presentation is equivalent to a confluent rewriting system.

In a complete presentation (or in the general presentation) of MB_n a word containing α will be called a *reducible word* and a word that does not contain α will be called an *irreducible word* (also called normal form of the word). We will denote $B_*^{(m)}$ as the set of reducible words and $A_*^{(m)}$ as the set of irreducible words in MB_n .

Let U and V be nonempty words; then the word $Ua_{ij}V$ will be denoted as $Ua_{ij} \times_{ij} a_{ij}V$.

Definition 2.1 [9] *Let G be a finitely generated group and S be a finite set of generators of G . The word length $l_S(g)$ of an element $g \in G$ is the smallest integer n for which there exists $s_1, \dots, s_n \in S \cup S^{-1}$ such that $g = s_1 \cdots s_n$.*

Definition 2.2 [9] *Let G be a finitely generated group and S be a finite set of generators of G . The growth function of the pair (G, S) associates to an integer $k \geq 0$ the number $a(k)$ of elements $g \in G$ such that $l_S(g) = k$ and the corresponding spherical growth series or the Hilbert series is given by $P_G(t) = \sum_{k=0}^{\infty} a(k)t^k$.*

For a sequence $\{s_k\}_{k \geq 1}$ of positive numbers, we define the growth rate:

Definition 2.3 *Let r be a positive real number; then the growth rate r of the sequence $\{s_k\}_{k \geq 1}$ of positive numbers is defined as*

$$\overline{\lim}_k \exp\left(\frac{\log s_k}{k}\right).$$

In 2008, L. A. Bokut [6] gave the Gröbner–Shirshov basis (GSB) of B_n in band generators. The notion of this basis is in [2, 4, 7, 8, 12] under different names: complete presentation, presentation with solvable ambiguities, Gröbner–Shirshov basis, rewriting system, and so on. In [1] we proved that a subset of the GSB of B_n given by Bokut [6] is a GSB of MB_n . Using the notations (used in [1]) (t, s) for generator a_{ts} and $V_{[t,s]}$ or $W_{[t,s]}$ for words in (k, l) such that $t \geq k > l \geq s$, we proved in [1] that:

Theorem 2.4 [1] *A GSB of braid monoid MB_n consists of the following relations:*

$$(k, l)(i, j) = (i, j)(k, l), \quad k > l > i > j,$$

$$(k, l)V_{[j-1,1]}(i, j) = (i, j)(k, l)V_{[j-1,1]}, \quad k > i > j > l,$$

$$(t_3, t_2)(t_2, t_1) = (t_2, t_1)(t_3, t_1),$$

$$(t_3, t_1)V_{[t_2-1,1]}(t_3, t_2) = (t_2, t_1)(t_3, t_1)V_{[t_2-1,1]},$$

$$(t, s)V_{[t_2-1,1]}(t_2, t_1)W_{[t_3-1,t_1]}(t_3, t_1) = (t_3, t_2)(t, s)V_{[t_2-1,1]}(t_2, t_1)W'_{[t_3-1,t_1]},$$

$$(t_3, s)V_{[t_2-1,1]}(t_2, t_1)W_{[t_3-1,t_1]}(t_3, t_1) = (t_2, s)(t_3, s)V_{[t_2-1,1]}(t_2, t_1)W'_{[t_3-1,t_1]},$$

for $t_3 > t_2 > t_1$, $t > t_3, t_2 > s$, and $W_{[t_3-1,t_1]}(t_3, t_1) = (t_3, t_1)W'_{[t_3-1,t_1]}$ where

$$W'_{[t_3-1,t_1]} = W_{[t_3-1,t_1]}|_{(p,q) \mapsto (p,q)} \text{ if } q \neq t_1; (p, t_1) \mapsto (t_3, p).$$

3. Hilbert series of MB_4 in Birman–Ko–Lee generators

For the band presentation of the braid monoid MB_3 we gave its Hilbert series in [1] as

$$P_M^{(3)}(t) = \frac{1}{(1-t)(1-2t)}.$$

In this paper we compute the Hilbert series of MB_4 (in band presentation). From 1 we have the following band-presentation of MB_4 :

$$\langle a_{43}, a_{42}, a_{41}, a_{32}, a_{31}, a_{21} | R_1^{(3)}, R_2^{(3)}, R_i^{(4)}, i = 3, \dots, 10 \rangle,$$

where $R_1^{(3)} : a_{31}a_{32} = a_{21}a_{31}$, $R_2^{(3)} : a_{32}a_{21} = a_{21}a_{31}$, $R_3^{(4)} : a_{41}a_{32} = a_{32}a_{41}$,

$R_4^{(4)} : a_{41}a_{42} = a_{21}a_{41}$, $R_5^{(4)} : a_{41}a_{43} = a_{31}a_{41}$, $R_6^{(4)} : a_{42}a_{21} = a_{21}a_{41}$, $R_7^{(4)} : a_{42}a_{43} = a_{32}a_{42}$, $R_8^{(4)} : a_{43}a_{21} = a_{21}a_{43}$, $R_9^{(4)} : a_{43}a_{31} = a_{31}a_{41}$ and $R_{10}^{(4)} : a_{43}a_{32} = a_{32}a_{42}$ are the given basic relations.

For the braid monoid MB_4 we give another form of Theorem 2.4 that is directly used to compute the Hilbert series of MB_4 . This form is obtained by solving all the ambiguities in the presentation of MB_4 .

Proposition 3.1 *A complete presentation of MB_4 for band generators is given by*

$$\langle a_{43}, a_{42}, a_{41}, a_{32}, a_{31}, a_{21} | R_1^{(3)}, R_2^{(3)}, R_{19}^{(3)}, R_i^{(4)}, i = 3, \dots, 18 \rangle,$$

where $R_1^{(3)}, R_2^{(3)}, R_i^{(4)}, i = 3, \dots, 10$ are the basic relations and the new relations $R_{11}^{(4)}, \dots, R_{18}^{(4)}$ are given as follows:

$$R_{11}^{(4)} : a_{41}a_{21}^{s+1}a_{31} = a_{32}a_{41}a_{21}a_{32}^s,$$

$$R_{12}^{(4)} : a_{41}a_{21}^{s+1}a_{41} = a_{21}a_{41}a_{21}a_{42}^s,$$

$$R_{13}^{(4)} : a_{41}a_{21}^r a_{32}^{s+1}a_{42} = a_{31}a_{41}a_{21}^r a_{32}^s a_{43}^s,$$

$$R_{14}^{(4)} : a_{41}a_{21}^r a_{43} = a_{31}a_{41}a_{21}^r,$$

$$R_{15}^{(4)} : a_{42}a_{32}^{s+1}a_{42} = a_{32}a_{42}a_{32}a_{43}^s,$$

$$R_{16}^{(4)} : a_{41}a_{21}^{s+1}W(32)a_{41} = a_{21}a_{41}a_{21}W'(32),$$

$$R_{17}^{(4)} : a_{41}a_{31}W(31)a_{41} = a_{31}a_{41}a_{31}W'(31),$$

$$R_{18}^{(4)} : a_{42}a_{31}W(31)a_{41} = a_{32}a_{42}a_{31}W'(31),$$

$$R_{19}^{(4)} : a_{31}a_{21}^{s+1}a_{31} = a_{21}a_{31}a_{21}a_{32}^s,$$

where r is a positive and s is a nonnegative integer, $W(3k)$ an irreducible word in MB_3 starting with a_{3k} , ($k = 1, 2$) and $W'(3k) = W(3k) : a_{32} \rightarrow a_{32}$, $a_{21} \rightarrow a_{41}$, $a_{31} \rightarrow a_{43}$ (as mentioned in Theorem 2.4).

Proof It is obvious from solving all the ambiguities and from the presentation given in [1]. Hence the proof is omitted. □

As defined above, the set $A_*^{(m)}$ denotes the set of irreducible words and $B_*^{(m)}$ the set of reducible words in MB_m . In particular, $B_{ij \cdot kl; uv}^{(m)}$ denotes the set of reducible words starting with $a_{ij}a_{kl}$ and ending with a_{uv} and $B_{ij \cdot kl; pq; uv}^{(m)}$ denotes the set of reducible words starting with $a_{ij}a_{kl}^{s+1}a_{pq}$ and ending with a_{uv} . Hence we have the following sets of reducible words in MB_4 :

$$B_{31 \cdot 21; 31}^{(3)} = \{a_{31}a_{21}^{s+1}a_{31}\}, \quad B_{42 \cdot 32; 42}^{(4)} = \{a_{42}a_{32}^{s+1}a_{42}\}, \quad B_{41 \cdot 21; 41}^{(4)} = \{a_{41}a_{21}^{s+1}a_{41}\},$$

$$B_{41 \cdot 21; 43}^{(4)} = \{a_{41}a_{21}^r a_{43}\}, \quad B_{41 \cdot 21; 31}^{(4)} = \{a_{41}a_{21}^{s+1}a_{31}\},$$

$$B_{41 \cdot 21; 42}^{(4)} = \{a_{41}a_{21}^r a_{32}^{s+1}a_{42}\}, \quad B_{41 \cdot 21 \cdot 32; 41}^{(4)} = \{a_{41}a_{21}^{s+1}W(32)a_{41}\},$$

$$B_{41 \cdot 31; 41}^{(4)} = \{a_{41}a_{31}W(31)a_{41}\}, \quad B_{42 \cdot 31; 41}^{(4)} = \{a_{42}a_{31}W(31)a_{41}\}.$$

We are using the other notions as follows:

- We denote the set $\{a_{21}, a_{21}^2, a_{21}^3, \dots\}$ by $A_{21}^{(2)}$.
- $A_{ij}^{(n)}$ denotes the set of irreducible words starting with a_{ij} and $A_{ij}^{(n)}$ denotes $\{a_{ij}, a_{ij}^2, a_{ij}^3, \dots\}$.
- $A_{nj \cdot kl}^{(n)}$ denotes the set of irreducible words starting with $a_{nj}a_{kl}$, where $j = 2, 3$, $k = 2, 3$ and $l = 1, 2$.
- The Hilbert series of $B_*^{(m)}$, $A_*^{(m)}$, and MB_4 are denoted by $Q_*^{(m)}$, $P_*^{(m)}$, and $P_M^{(4)}(t)$ respectively. It is obvious that $\frac{P_*^{(k)}}{k(k-1)} = P_{21}^{(2)}$, where $P_{21}^{(2)} = \frac{t}{1-t}$.

Note that as $B_{31 \cdot 21; 31}^{(3)} = \{a_{31}\} \times A_{21}^{(2)} \times \{a_{31}\}$ and $P_{21}^{(2)} = \frac{t}{1-t}$, hence $Q_{31 \cdot 21; 31}^{(3)} = \frac{t^3}{1-t}$.

Now we construct a linear system for reducible words in MB_4 .

Proposition 3.2 *The following equalities hold for reducible words in MB_4 .*

- 1) $Q_{41 \cdot 21; 31}^{(4)} = \frac{t^3}{1-t}$,
- 2) $Q_{41 \cdot 21; 41}^{(4)} = \frac{t^3}{1-t}$,
- 3) $Q_{41 \cdot 21 \cdot 32; 41}^{(4)} = \frac{t^4}{(1-t)(1-2t)}$,
- 4) $Q_{41 \cdot 21; 43}^{(4)} = \frac{t^3}{1-t}$,
- 5) $Q_{41 \cdot 21; 42}^{(4)} = \frac{t^4}{(1-t)^2}$,
- 6) $Q_{41 \cdot 31; 41}^{(4)} = \frac{t^3}{1-2t}$,
- 7) $Q_{42 \cdot 32; 42}^{(4)} = \frac{t^3}{1-t}$,
- 8) $Q_{42 \cdot 31; 41}^{(4)} = \frac{t^3}{1-2t}$,

Proof Using simply the decomposition of words we have:

- 1) $B_{41 \cdot 21; 31}^{(4)} = \{a_{41}a_{21}^{r+1}a_{31}\} = \{a_{41}\} \times A_{21}^{(2)} \times \{a_{31}\}$ implies $Q_{41 \cdot 21; 31}^{(4)} = \frac{t^3}{1-t}$.
- 2) $B_{41 \cdot 21; 41}^{(4)} = \{a_{41}a_{21}^{r+1}a_{41}\} = \{a_{41}\} \times A_{21}^{(2)} \times \{a_{41}\}$ gives us $Q_{41 \cdot 21; 41}^{(4)} = \frac{t^3}{1-t}$.
- 3) The decomposition $B_{41 \cdot 21 \cdot 32; 41}^{(4)} = \{a_{41}a_{21}^{r+1}W(32)a_{41}\} = \{a_{41}\} \times A_{21}^{(2)} \times A_{32}^{(3)} \times \{a_{41}\}$ gives the Hilbert series $Q_{41 \cdot 21 \cdot 31; 41}^{(4)} = \frac{t^4}{(1-t)(1-2t)}$.
- 4) $B_{41 \cdot 21; 43}^{(4)} = \{a_{41}a_{21}^r a_{43}\} = \{a_{41}\} \times A_{21}^{(2)} \times \{a_{43}\}$ implies $Q_{41 \cdot 21; 43}^{(4)} = \frac{t^3}{1-t}$.
- 5) The decomposition $B_{41 \cdot 21; 42}^{(4)} = \{a_{41}a_{21}^r a_{32}^{s+1}a_{42}\} = \{a_{41}\} \times A_{21}^{(2)} \times A_{32}^{(3)} \times \{a_{41}\}$ gives the Hilbert series $Q_{41 \cdot 21; 42}^{(4)} = \frac{t^4}{(1-t)^2}$.
- 6) $B_{41 \cdot 31; 41}^{(4)} = \{a_{41}a_{31}W(31)a_{41}\} = \{a_{41}\} \times A_{31}^{(3)} \times \{a_{41}\}$ implies $Q_{41 \cdot 31; 41}^{(4)} = \frac{t^3}{1-2t}$.
- 7) $B_{42 \cdot 32; 42}^{(4)} = \{a_{42}a_{32}^{r+1}a_{42}\} = \{a_{42}\} \times A_{32}^{(3)} \times \{a_{42}\}$ implies $Q_{42 \cdot 32; 42}^{(4)} = \frac{t^3}{1-t}$.
- 8) $B_{42 \cdot 31; 41}^{(4)} = \{a_{42}a_{31}W(31)a_{41}\} = \{a_{42}\} \times A_{31}^{(3)} \times \{a_{41}\}$ implies $Q_{42 \cdot 31; 41}^{(4)} = \frac{t^3}{1-2t}$. □

Next we construct a linear system for canonical forms in MB_4 .

Proposition 3.3 *The following equalities hold for irreducible words in MB_4 .*

- 1) $P_{31}^{(4)} = \frac{t}{1-2t} \left(1 + \sum_{i=1}^3 P_{4i}^{(4)}\right),$
- 2) $P_{32}^{(4)} = \frac{t}{1-2t} \left(1 + \sum_{i=1}^3 P_{4i}^{(4)}\right),$
- 3) $P_{21}^{(4)} = \frac{t(1-2t)}{1-t} \left(1 + \sum_{i=1}^3 P_{4i}^{(4)}\right),$
- 4) $P_{41}^{(4)} = t + tP_{41}^{(4)} + \sum_{i=2}^3 P_{41 \cdot i1}^{(4)},$
- 5) $P_{42}^{(4)} = t + t \sum_{i=1}^2 P_{4i}^{(4)} + \sum_{i=1}^2 P_{42 \cdot 3i}^{(4)},$
- 6) $P_{43}^{(4)} = t + t \sum_{i=1}^3 P_{4i}^{(4)},$
- 7) $P_{41 \cdot 31}^{(4)} = tP_{31}^{(4)} - \frac{t^2}{1-2t}P_{41}^{(4)},$
- 8) $P_{42 \cdot 31}^{(4)} = tP_{31}^{(4)} - \frac{t^2}{1-2t}P_{41}^{(4)},$
- 9) $P_{42 \cdot 32}^{(4)} = tP_{32}^{(4)} - \frac{t^2}{1-t}P_{42}^{(4)},$
- 10) $P_{41 \cdot 21}^{(4)} = tP_{21}^{(4)} - \frac{t^2}{1-t}P_{31}^{(4)} - \frac{t^2}{1-2t}P_{41}^{(4)} - \frac{t^3}{(1-t)^2}P_{42}^{(4)} - \frac{t^2}{1-t}P_{43}^{(4)}.$

Proof We compute the Hilbert series inductively. Here we use the series of the irreducible words of MB_3 , which we have computed in [1]. The series are: $P_{31}^{(3)} = \frac{t}{1-2t}$, $P_{32}^{(3)} = \frac{t}{1-2t}$, $P_{21}^{(3)} = \frac{t(1-2t)}{1-t}$. If \sqcup denotes the disjoint union of sets, then using the GSB of MB_4 and the decomposition of words we have:

1) $A_{31}^{(4)} = A_{31}^{(3)} \sqcup (A_{31}^{(3)} \times A_{41}^{(4)}) \sqcup (A_{31}^{(3)} \times A_{42}^{(4)}) \sqcup (A_{31}^{(3)} \times A_{43}^{(4)})$. This gives us

$$\begin{aligned} P_{31}^{(4)} &= \left(1 + \sum_{i=1}^3 P_{4i}^{(4)}\right) P_{31}^{(3)} \\ &= \frac{t}{1-2t} \left(1 + \sum_{i=1}^3 P_{4i}^{(4)}\right). \end{aligned}$$

2) $A_{32}^{(4)} = A_{32}^{(3)} \sqcup (A_{32}^{(3)} \times A_{41}^{(4)}) \sqcup (A_{32}^{(3)} \times A_{42}^{(4)}) \sqcup (A_{32}^{(3)} \times A_{43}^{(4)})$ implies

$$\begin{aligned} P_{32}^{(4)} &= \left(1 + \sum_{i=1}^3 P_{4i}^{(4)}\right) P_{32}^{(3)} \\ &= \frac{t}{1-2t} \left(1 + \sum_{i=1}^3 P_{4i}^{(4)}\right). \end{aligned}$$

3) $A_{21}^{(4)} = A_{21}^{(3)} \sqcup (A_{21}^{(3)} \times A_{41}^{(4)}) \sqcup (A_{21}^{(3)} \times A_{42}^{(4)}) \sqcup (A_{21}^{(3)} \times A_{43}^{(4)})$ implies

$$\begin{aligned} P_{21}^{(4)} &= \left(1 + \sum_{i=1}^3 P_{4i}^{(4)}\right) P_{21}^{(3)} \\ &= \frac{t(1-2t)}{1-t} \left(1 + \sum_{i=1}^3 P_{4i}^{(4)}\right). \end{aligned}$$

The set $A_{4i}^{(4)}$ consists of all the words starting with the generator a_{4i} . Therefore, the set $\{a_{4i}\} \times A_{4i}^{(4)}$ is a subset of $A_{4i}^{(4)}$ consisting of all the words starting with a_{4i}^2 . We apply this concept in the proofs of (4), (5), and (6).

4) The set $A_{41}^{(4)}$ is a disjoint union of the sets $\{a_{41}\}$, $\{a_{41}\} \times A_{41}^{(4)}$, $A_{41 \cdot 21}^{(4)}$, and $A_{41 \cdot 31}^{(4)}$, i.e. $A_{41}^{(4)} = \{a_{41}\} \sqcup (\{a_{41}\} \times A_{41}^{(4)}) \sqcup A_{41 \cdot 21}^{(4)} \sqcup A_{41 \cdot 31}^{(4)}$. Therefore, we have

$$P_{41}^{(4)} = t + tP_{41}^{(4)} + \sum_{i=2}^3 P_{41 \cdot i1}^{(4)}.$$

Similarly, we have

5) $A_{42}^{(4)} = \{a_{42}\} \sqcup (\{a_{42}\} \times A_{42}^{(4)}) \sqcup (\{a_{42}\} \times A_{41}^{(4)}) \sqcup A_{42 \cdot 31}^{(4)} \sqcup A_{42 \cdot 32}^{(4)}$ implies

$$P_{42}^{(4)} = t + t \sum_{i=1}^2 P_{4i}^{(4)} + \sum_{i=1}^2 P_{42 \cdot 3i}^{(4)}.$$

6) $A_{43}^{(4)} = \{a_{43}\} \sqcup (\{a_{43}\} \times A_{41}^{(4)}) \sqcup (\{a_{43}\} \times A_{42}^{(4)}) \sqcup (\{a_{43}\} \times A_{43}^{(4)})$ implies

$$P_{43}^{(4)} = t + t \sum_{i=1}^3 P_{4i}^{(4)}.$$

7) $A_{41\cdot 31}^{(4)} = \{a_{41}\} \times A_{31}^{(4)} \setminus (B_{41\cdot 31;41} \times_{41} A_{41}^{(4)})$ implies

$$P_{41\cdot 31}^{(4)} = tP_{31}^{(4)} - \frac{t^2}{1-2t}P_{41}^{(4)}.$$

8) $A_{42\cdot 31}^{(4)} = \{a_{42}\} \times A_{31}^{(4)} \setminus (B_{42\cdot 31;41} \times_{41} A_{41}^{(4)})$ implies

$$P_{42\cdot 31}^{(4)} = tP_{31}^{(4)} - \frac{t^2}{1-2t}P_{41}^{(4)}.$$

9) $A_{42\cdot 32}^{(4)} = \{a_{42}\} \times A_{32}^{(4)} \setminus (B_{42\cdot 32;42} \times_{42} A_{42}^{(4)})$ implies

$$P_{42\cdot 32}^{(4)} = tP_{32}^{(4)} - \frac{t^2}{1-2t}P_{42}^{(4)}.$$

10) $A_{41\cdot 21}^{(4)} = \{a_{41}\} \times A_{21}^{(4)} \setminus [(B_{41\cdot 21;41} \times_{41} A_{41}^{(4)}) \sqcup (B_{41\cdot 21;31} \times_{31} A_{31}^{(4)}) \sqcup (B_{41\cdot 21;43} \times_{43} A_{43}^{(4)}) \sqcup (B_{41\cdot 21;42} \times_{42} A_{42}^{(4)})]$
 implies

$$P_{41\cdot 21}^{(4)} = tP_{21}^{(4)} - \frac{t^2}{1-t}P_{31}^{(4)} - \frac{t^2}{1-2t}P_{41}^{(4)} - \frac{t^3}{(1-t)^2}P_{42}^{(4)} - \frac{t^2}{1-t}P_{43}^{(4)}.$$

□

Theorem 3.4 *The Hilbert series of the braid monoid MB_4 in band generators is given by*

$$P_M^{(4)}(t) = \frac{1}{(1-t)(1-5t+5t^2)}.$$

Proof Solving the system of linear equations constructed in Proposition 3.3 we get $P_{21}^{(4)} = \frac{t}{(1-t)(1-5t+5t^2)}$, $P_{31}^{(4)} = \frac{t}{1-5t+5t^2}$, $P_{32}^{(4)} = \frac{t}{1-5t+5t^2}$, $P_{41}^{(4)} = \frac{t-2t^2}{1-5t+5t^2}$, $P_{42}^{(4)} = \frac{t-t^2}{1-5t+5t^2}$, $P_{43}^{(4)} = \frac{t-2t^2}{1-5t+5t^2}$. Therefore, we have the Hilbert series of the braid monoid MB_4 as

$$\begin{aligned} P_M^{(4)}(t) &= 1 + P_{21}^{(4)} + \sum_{i=1}^2 P_{3i}^{(4)} + \sum_{j=1}^3 P_{4j}^{(4)}; \\ &= \frac{1}{(1-t)(1-5t+5t^2)}. \end{aligned}$$

□

Corollary 3.5 *The growth rate of braid monoid MB_4 (in Birman-Ko-Lee generators) is 3.618.*

Proof By partial fractions we have $\frac{1}{(1-t)(1-5t+5t^2)} = \frac{1}{1-t} - \frac{\sqrt{5}}{1-\frac{5-\sqrt{5}}{2}t} + \frac{\sqrt{5}}{1-\frac{5+\sqrt{5}}{2}t}$. The only term that contributes in approximation of the series is $\frac{\sqrt{5}}{1-\frac{5+\sqrt{5}}{2}t}$ and $\frac{\sqrt{5}}{1-\frac{5+\sqrt{5}}{2}t} = \sqrt{5}(1 + \frac{5+\sqrt{5}}{2}t + (\frac{5+\sqrt{5}}{2})^2t^2 + \dots)$. Therefore, the growth function is $a_k^{(4)} = \sqrt{5}(\frac{5+\sqrt{5}}{2})^k$ and hence the growth rate of MB_4 is $\frac{5+\sqrt{5}}{2}$ (approximately equal to 3.618). The growth rate of MB_3 is 2. □

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