Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2014) 38: 985 – 993

doi:10.3906/mat-1310-24

Gradient estimates for the porous medium type equation on smooth metric measure space

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Received: 13.10.2013 • **Accepted:** 19.03.2014 • **Published Online:** 24.10.2014 • **Printed:** 21.11.2014

Abstract: The porous medium equation arises in different applications to model diffusive phenomena. In this paper, we obtain several gradient estimates for some porous medium type equations on smooth metric measure space with N-Bakry-Emery Ricci tensor bounded from below. In particular, we improve and generalize some current gradient estimates for the porous medium equations.

Key words: Gradient estimates, porous medium type equation, smooth metric measure space, positive solution

1. Introduction

A smooth metric measure space is a triple, $(M^n, g, e^{-f}dv)$, where M^n is a complete n-dimensional Riemannian manifold with metric g, f is a smooth real-valued function on M^n, and dv is the Riemannian volume density. Smooth metric measure spaces carry a natural analog of the Laplace-Beltrami operator $\triangle$, the f-Laplacian, which is also called drifting Laplacian or Witten-Laplacian, defined for a function u by $\triangle_f u = \triangle u - g(\nabla f, \nabla u) = \triangle u - \langle \nabla f, \nabla u\rangle$. The N-Bakry-Emery Ricci tensor is defined by $Ric_f^N = Ric + \mathrm{Hess} f - \frac{1}{N} df \otimes df$. A natural question about smooth metric measure space is which of the results about the Ricci tensor and the Laplace-Beltrami operator can be extended to the N-Bakry-Emery Ricci tensor and the f-Laplacian. For example, in [15], Yang discussed the gradient estimates for the following parabolic equation,

$$\frac{\partial}{\partial t} u = \triangle u + au \log u + bu,$$

on Riemannian manifolds. In [8], Huang and Ma considered the gradient estimates for the following parabolic equation,

$$\frac{\partial}{\partial t} u = \triangle_f u + au \log u + bu,$$

on smooth metric measure spaces. Inspired by the discussions of the gradient estimate of the harmonic function and positive solutions to linear heat flow on Riemannian manifolds, the authors in [1–3] discussed the gradient estimates of the f-harmonic function and positive solutions to linear heat flow on smooth metric measure spaces.

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This research was supported by the Natural Science Foundation of Hunan province (No.14JJ2120), Scientific Research Fund of Hunan Provincial Education Department (No.14A020), and was partly supported by the Key Constructional Discipline of Hunan Province (No.[2011]76).
2010 *AMS Mathematics Subject Classification:* Primary: 35B45; Secondary: 58J05.

Motivated by the study of eigenvalue estimates for the Laplace-Beltrami operator on Riemannian manifolds, the authors in [4,5,11,12,14] studied the eigenvalue estimates for drifting Laplacians on smooth metric measure spaces.

In [10], Lu et al. obtained some gradient estimates for the following porous medium equation on Riemannian manifolds with Ricci curvature bounded from below:

$$u_t = \triangle u^m, \tag{1.1}$$

where $m > 1$. In [6], Huang and Li improved the results in [10]. In [7], Huang and Li studied the following porous medium type equation,

$$u_t = \triangle_f u^m, \tag{1.2}$$

on smooth metric measure space. Under the assumption that the N-dimensional Bakry-Emery Ricci curvature is bounded from below, Huang and Li obtained some gradient estimates that generalized the results in [6] and [10].

Inspired by [6] and [7], we make further discussions of gradient estimates for positive solutions of equation (1.2). For this purpose, we let

$$v = \frac{m}{m-1}u^{m-1}, \quad a = \frac{(n+N)(m-1)}{(n+N)(m-1)+2}, \quad b = \frac{n+N}{(n+N)(m-1)+2}. \tag{1.3}$$

Theorem 1.1 *Let $(M^n, g, e^{-f}dv)$ be a smooth metric measure space. Suppose that u is a positive solution to (1.2). If $Ric_f^N(B_p(2R)) \geq -K$ and $K \geq 0$, then on the ball $B_p(R)$ we have*

$$\frac{|\nabla v|^2}{v} - \alpha(t)\frac{v_t}{v} \leq \frac{1}{1+a(\alpha-1)}\{\frac{CHa\alpha^2}{R^2}(\frac{bm^2\alpha^2}{2(\alpha-1)(1-a)} + 3 + \sqrt{K}R) + \frac{a\alpha^2}{t}\}, \tag{1.4}$$

where C is a constant depending only on n and

$$H = \sup_{B_p(2R)\times[0,T]}(m-1)v, \quad \alpha(t) = e^{2HKt}.$$

Corollary 1.2 *Let $(M^n, g, e^{-f}dv)$ be a smooth metric measure space with $Ric_f^N \geq -K$ and $K \geq 0$. Suppose that (M^n, g) is a complete noncompact Riemannian manifold and u is a positive solution to (1.2), then*

$$\frac{|\nabla v|^2}{v} - \alpha(t)\frac{v_t}{v} \leq \frac{1}{1+a(\alpha-1)}\frac{a\alpha^2(t)}{t}, \tag{1.5}$$

where $\alpha(t) = e^{2SKt}$, $S = \sup_{M^n\times[0,T]}(m-1)v$.

Theorem 1.3 *Let $(M^n, g, e^{-f}dv)$ be a smooth metric measure space. Suppose that u is a positive solution to (1.2). If $Ric_f^N(B_p(2R)) \geq -K$ and $K \geq 0$, then on the ball $B_p(R)$ we have*

$$\frac{|\nabla v|^2}{v} - \alpha(t)\frac{v_t}{v} - \varphi(t) \leq \frac{CHa\alpha^2(t)}{R^2}[(3+\sqrt{K}R) + \frac{bm^2\alpha^2(t)}{\tanh(HKt)}], \tag{1.6}$$

where C is a constant depending only on n, $\alpha(t)=1+\frac{2}{3}SKt$, $S=\sup_{M^n\times[0,T]}(m-1)v$, and

$$H=\sup_{B_p(2R)\times[0,T]}(m-1)v, \quad \varphi(t)=\frac{a}{t}+aSK+\frac{a}{3}(SK)^2t+\frac{\lambda}{t^2}, \quad \lambda\geq 0.$$

Corollary 1.4 *Let $(M^n,g,e^{-f}dv)$ be a smooth metric measure space with $Ric_f^N\geq -K$ and $K\geq 0$. Suppose that (M^n,g) is a complete noncompact Riemannian manifold and u is a positive solution to (1.2), then*

$$\frac{|\nabla v|^2}{v}-\alpha(t)\frac{v_t}{v}-\varphi(t)\leq 0, \tag{1.7}$$

where $\varphi(t)$ and $\alpha(t)$ are defined in Theorem 1.3.

Remark 1 *Obviously, Theorem 1.1 and Corollary 1.2 in this paper are better than Theorem 1.3 and Corollary 1.4 in [6], respectively.*

Remark 2 *Theorem 1.3 and Corollary 1.4 in this paper are more general than Theorem 1.4 and Corollary 1.7 in [6], respectively.*

2. Some lemmas

To prove Theorem 1.1 and Theorem 1.3, we need some lemmas. Suppose that u is a positive solution to (1.2). Let $v=\frac{m}{m-1}u^{m-1}$. Direct calculation shows that

$$v_t=mu^{m-2}\triangle_f u^m=(m-1)v\triangle_f v+|\nabla v|^2. \tag{2.1}$$

Since $v\neq 0$, then (2.1) is equivalent to

$$\frac{v_t}{v}=(m-1)\triangle_f v+\frac{|\nabla v|^2}{v}. \tag{2.2}$$

Let $L=\partial_t-(m-1)v\triangle_f$ and $F=\frac{|\nabla v|^2}{v}-\alpha\frac{v_t}{v}-\varphi$. We have the following.

Lemma 2.1 *Suppose that u is a positive solution to (1.2). Then*

$$L(\frac{v_t}{v})=(m-1)\frac{v_t}{v}\triangle_f v+\frac{v_t}{v}\frac{|\nabla v|^2}{v}+2m\nabla v\nabla(\frac{v_t}{v}), \tag{2.3}$$

$$\begin{aligned} L(\frac{|\nabla v|^2}{v})\leq{}& 2(m-1)\frac{|\nabla v|^2}{v}\triangle_f v-\frac{2(m-1)}{N+n}|\triangle_f v|^2\\ &-2(m-1)Ric_f^N(\nabla v,\nabla v)+2m\nabla v\nabla(\frac{|\nabla v|^2}{v})+\frac{|\nabla v|^4}{v^2}. \end{aligned} \tag{2.4}$$

Proof Direct calculation shows that

$$\nabla(\frac{v_t}{v})=\frac{\nabla v_t}{v}-\frac{v_t\nabla v}{v^2}. \tag{2.5}$$

Therefore, we get

$$\triangle(\frac{v_t}{v})=\frac{1}{v}\triangle v_t-\frac{v_t}{v^2}\triangle v-\frac{2}{v^2}\nabla v\nabla v_t+\frac{2v_t}{v^3}|\nabla v|^2, \tag{2.6}$$

and

$$(m-1)v\langle\nabla f,\nabla(\frac{v_t}{v})\rangle=(m-1)\langle\nabla f,\nabla v_t\rangle-(m-1)\frac{v_t}{v}\langle\nabla f,\nabla v\rangle. \tag{2.7}$$

By (2.2), we have

$$\partial_t(\frac{v_t}{v})=(m-1)\partial_t(\triangle_f v)+\partial_t(\frac{|\nabla v|^2}{v})=(m-1)\triangle_f v_t+\frac{2}{v}\nabla v\nabla v_t-\frac{v_t}{v}\frac{|\nabla v|^2}{v}. \tag{2.8}$$

According to (2.4), (2.5), (2.6), (2.7), and (2.8), we conclude that

$$L(\frac{v_t}{v})=\partial_t(\frac{v_t}{v})-(m-1)v\triangle_f(\frac{v_t}{v})=(m-1)\frac{v_t}{v}\triangle_f v+\frac{v_t}{v}\frac{|\nabla v|^2}{v}+2m\nabla v\nabla(\frac{v_t}{v}).$$

On the other hand, by (2.1) we get

$$\begin{aligned}\partial_t(\frac{|\nabla v|^2}{v})&=\frac{2v\nabla v\nabla v_t-|\nabla v|^2v_t}{v^2}\\&=\frac{2\nabla v}{v}\nabla((m-1)v\triangle_f v+|\nabla v|^2)-\frac{|\nabla v|^2}{v^2}((m-1)v\triangle_f v+|\nabla v|^2)\\&=(m-1)\frac{|\nabla v|^2}{v}\triangle_f v+2(m-1)\nabla v\nabla\triangle_f v+\frac{2\nabla v}{v}\nabla|\nabla v|^2-\frac{|\nabla v|^4}{v^2}.\end{aligned} \tag{2.9}$$

Direct calculation shows that

$$\begin{aligned}\triangle_f\frac{|\nabla v|^2}{v}&=\triangle\frac{|\nabla v|^2}{v}-\langle\nabla f,\nabla\frac{|\nabla v|^2}{v}\rangle\\&=\frac{1}{v}\triangle|\nabla v|^2-\frac{|\nabla v|^2}{v^2}\triangle v-\frac{2}{v^2}\nabla v\nabla|\nabla v|^2+\frac{2}{v^3}|\nabla v|^4-\frac{1}{v}\langle\nabla f,\nabla|\nabla v|^2\rangle+\frac{|\nabla v|^2}{v^2}\langle\nabla f,\nabla v\rangle\\&=\frac{1}{v}\triangle_f|\nabla v|^2-\frac{|\nabla v|^2}{v^2}\triangle_f v-\frac{2}{v^2}\nabla v\nabla|\nabla v|^2+\frac{2}{v^3}|\nabla v|^4.\\&=\frac{1}{v}\triangle_f|\nabla v|^2-\frac{|\nabla v|^2}{v^2}\triangle_f v-\frac{2\nabla v}{v}\nabla(\frac{|\nabla v|^2}{v}).\end{aligned} \tag{2.10}$$

According to (2.9) and (2.10), we obtain

$$\begin{aligned}L(\frac{|\nabla v|^2}{v})&=\partial_t(\frac{|\nabla v|^2}{v})-(m-1)v\triangle_f\frac{|\nabla v|^2}{v}\\&=2(m-1)(\frac{|\nabla v|^2}{v}\triangle_f v+\nabla v\nabla\triangle_f v-\frac{1}{2}\triangle_f|\nabla v|^2)+2m\nabla v\nabla(\frac{|\nabla v|^2}{v})+\frac{|\nabla v|^4}{v^2}.\end{aligned} \tag{2.11}$$

According to [9,13], we have

$$\frac{1}{2}\triangle_f|\nabla v|^2\geq\frac{1}{N+n}|\triangle_f v|^2+\nabla v\nabla\triangle_f v+Ric_f^N(\nabla v,\nabla v). \tag{2.12}$$

By (2.11) and (2.12), we conclude that (2.4) is true. □

Lemma 2.2 *([7]) The function F satisfies the following inequality:*

$$L(F) \leq -\frac{2(m-1)}{N+n}|\triangle_f v|^2 - 2(m-1)Ric_f^N(\nabla v, \nabla v) + 2m\nabla v \nabla F$$

$$-((m-1)\triangle_f v)^2 + (1-\alpha)(\frac{v_t}{v})^2 - \alpha' \frac{v_t}{v} - \varphi'. \tag{2.13}$$

Proof For the reader's convenience, we give the details of the proof of Lemma 2.2. By (2.3) and (2.4), we have

$$L(F) = L(\frac{|\nabla v|^2}{v}) - \alpha L(\frac{v_t}{v}) - \alpha' \frac{v_t}{v} - \varphi'$$

$$\leq 2(m-1)\frac{|\nabla v|^2}{v}\triangle_f v - \frac{2(m-1)}{N+n}|\triangle_f v|^2 - 2(m-1)Ric_f^N(\nabla v, \nabla v)$$

$$+2m\nabla v \nabla(\frac{|\nabla v|^2}{v}) + \frac{|\nabla v|^4}{v^2} - \alpha(m-1)\frac{v_t}{v}\triangle_f v - \alpha\frac{v_t}{v}\frac{|\nabla v|^2}{v}$$

$$-2\alpha m \nabla v \nabla(\frac{v_t}{v}) - \alpha' \frac{v_t}{v} - \varphi'. \tag{2.14}$$

By the definition of F, we have

$$2m\nabla v\nabla(\frac{|\nabla v|^2}{v}) - 2\alpha m \nabla v \nabla(\frac{v_t}{v}) = 2m\nabla v \nabla(\frac{|\nabla v|^2}{v} - \alpha\frac{v_t}{v}) = 2m\nabla v\nabla F. \tag{2.15}$$

According to (2.2), we get

$$\frac{|\nabla v|^4}{v^2} - \alpha\frac{v_t}{v}\frac{|\nabla v|^2}{v} = (1-\alpha)\frac{|\nabla v|^2}{v}\frac{v_t}{v} - (m-1)\frac{|\nabla v|^2}{v}\triangle_f v. \tag{2.16}$$

Using (2.2) again, we arrive at

$$(m-1)\frac{|\nabla v|^2}{v}\triangle_f v - \alpha(m-1)\frac{v_t}{v}\triangle_f v + (1-\alpha)\frac{|\nabla v|^2}{v}\frac{v_t}{v}$$

$$= \frac{|\nabla v|^2}{v}(\frac{v_t}{v} - \frac{|\nabla v|^2}{v}) - \alpha\frac{v_t}{v}(\frac{v_t}{v} - \frac{|\nabla v|^2}{v}) + (1-\alpha)\frac{|\nabla v|^2}{v}\frac{v_t}{v}$$

$$= 2\frac{|\nabla v|^2}{v}\frac{v_t}{v} - \frac{|\nabla v|^4}{v^2} - \alpha(\frac{v_t}{v})^2$$

$$= -(\alpha\frac{v_t}{v} - \frac{|\nabla v|^2}{v})^2 + (1-\alpha)(\frac{v_t}{v})^2$$

$$= -((m-1)\triangle_f v)^2 + (1-\alpha)(\frac{v_t}{v})^2. \tag{2.17}$$

Putting (2.15), (2.16), and (2.17) into (2.14), we conclude that (2.13) is true.

3. The proof of Theorem 1.1

Let $\overline{F}=\frac{|\nabla v|^2}{v}-\alpha\frac{v_t}{v}$, where $\alpha=e^{2HKt}$. We consider $\overline{F}$ in the geodesic ball $B_p(2R)$, which is centered at p with radius $2R$. Since $Ric_f^N(B_p(2R))\geq -K$, by (2.13) and the definition of H and a, we have

$$L(\overline{F})\leq \frac{-2}{(m-1)(N+n)}((m-1)\triangle_f v)^2+2vK\frac{|\nabla v|^2}{v}+2m\nabla v\nabla\overline{F}$$

$$-((m-1)\triangle_f v)^2+(1-\alpha)(\frac{v_t}{v})^2-\alpha'\frac{v_t}{v}$$

$$\leq -\frac{1}{a}((m-1)\triangle_f v)^2+2HK\frac{|\nabla v|^2}{v}+2m\nabla v\nabla\overline{F}+(1-\alpha)(\frac{v_t}{v})^2-\alpha'\frac{v_t}{v}.$$

Since $L(\alpha^{-1}\overline{F})=(\alpha^{-1})'\overline{F}+\alpha^{-1}L(\overline{F})$ and $\alpha'=2HK\alpha$, then

$$L(\alpha^{-1}\overline{F})\leq -\frac{1}{a}\alpha^{-1}((m-1)\triangle_f v)^2+2m\alpha^{-1}\nabla v\nabla\overline{F}+(1-\alpha)\alpha^{-1}(\frac{v_t}{v})^2. \tag{3.1}$$

By (2.2) and the definition of $\overline{F}$, we get

$$(m-1)\triangle_f v=(\alpha^{-1}-1)\frac{|\nabla v|^2}{v}-\alpha^{-1}\overline{F},\quad \alpha\frac{v_t}{v}=\frac{|\nabla v|^2}{v}-\overline{F}. \tag{3.2}$$

Putting (3.2) into (3.1), we obtain

$$L(\alpha^{-1}\overline{F})\leq -\frac{(1-\alpha)^2}{a\alpha^3}\frac{|\nabla v|^4}{v^2}-\frac{1}{a\alpha^3}\overline{F}^2+\frac{2(1-\alpha)}{a\alpha^3}\frac{|\nabla v|^2}{v}\overline{F}+\frac{2m}{\alpha}\nabla v\nabla\overline{F}$$

$$+\frac{1-\alpha}{\alpha^3}\frac{|\nabla v|^4}{v^2}+\frac{1-\alpha}{\alpha^3}\overline{F}^2-\frac{2(1-\alpha)}{\alpha^3}\frac{|\nabla v|^2}{v}\overline{F}. \tag{3.3}$$

According to (2.4) and (2.5) in [8], we can construct a cut-off function ϕ such that $0\leq\phi\leq 1, \sup(\phi)\subset B_p(2R), \phi|_{B_p(R)}=1$ and

$$\frac{|\nabla\phi|^2}{\phi}\leq\frac{C}{R^2},\quad -\triangle_f\phi\leq\frac{C}{R^2}(1+R\sqrt{K}), \tag{3.4}$$

where C is a constant depending only on n. Set $G=t\phi\alpha^{-1}\overline{F}$. Assume that G achieves its maximum at the point $(x_0,s)\in B_p(2R)\times[0,T]$ and assume $G(x_0,s)>0$. By the maximum principle, we have

$$\nabla G=0,\quad L(G)\geq 0,\quad \nabla(\alpha^{-1}\widetilde{F})=-\frac{\alpha^{-1}\widetilde{F}}{\phi}\nabla\phi$$

at the point (x_0,s), and

$$0\leq L(G)=s\phi L(\alpha^{-1}\overline{F})-(m-1)v\frac{\triangle_f\phi}{\phi}G+2(m-1)v\frac{|\nabla\phi|^2}{\phi^2}G+\frac{G}{s}$$

$$\leq s\phi\{-\frac{1}{a\alpha^3}\overline{F}^2+\frac{2(1-\alpha)}{a\alpha^3}\frac{|\nabla v|^2}{v}\overline{F}+\frac{2m}{\alpha}\nabla v\nabla\overline{F}+\frac{1-\alpha}{\alpha^3}\overline{F}^2-\frac{2(1-\alpha)}{\alpha^3}\frac{|\nabla v|^2}{v}\overline{F}\}$$

$$
\begin{aligned}
&-(m-1)v\frac{\triangle_f\phi}{\phi}G+2(m-1)v\frac{|\nabla\phi|^2}{\phi^2}G+\frac{G}{s}\\
=&-\frac{1}{a\alpha s\phi}G^2+\frac{2(1-\alpha)}{a\alpha^2}\frac{|\nabla v|^2}{v}G-2m\nabla v\frac{\nabla\phi}{\phi}G+\frac{1-\alpha}{\alpha s\phi}G^2-\frac{2(1-\alpha)}{\alpha^2}\frac{|\nabla v|^2}{v}G\\
&-(m-1)v\frac{\triangle_f\phi}{\phi}G+2(m-1)v\frac{|\nabla\phi|^2}{\phi^2}G+\frac{G}{s}\\
\leq&-\frac{1+a(\alpha-1)}{a\alpha s\phi}G^2+\frac{2(1-\alpha)}{a\alpha^2}\frac{|\nabla v|^2}{v}G-\frac{2(1-\alpha)}{\alpha^2}\frac{|\nabla v|^2}{v}G\\
&+2\frac{mH^{\frac{1}{2}}}{(m-1)^{\frac{1}{2}}}\frac{|\nabla\phi|}{\phi}\frac{|\nabla v|}{v^{\frac{1}{2}}}G-H\frac{\triangle_f\phi}{\phi}G+2H\frac{|\nabla\phi|^2}{\phi^2}G+\frac{G}{s}. \qquad (3.5)
\end{aligned}
$$

Multiplying both sides of (3.5) by $\frac{a\alpha s\phi}{(1+a(\alpha-1))G}$, we get

$$
\begin{aligned}
G(x,T)\leq G(x_0,s)\leq&\frac{1}{1+a(\alpha-1)}\{\frac{2(1-\alpha)(1-a)}{a\alpha}s\phi\frac{|\nabla v|^2}{v}\\
&+2\frac{mH^{\frac{1}{2}}}{(m-1)^{\frac{1}{2}}}as\phi|\nabla\phi|\frac{|\nabla v|}{v^{\frac{1}{2}}}-Has\alpha\triangle_f\phi+2Has\alpha\frac{|\nabla\phi|^2}{\phi}+a\alpha\phi\}. \qquad (3.6)
\end{aligned}
$$

By (3.4), (3.5), and the inequality $2xy\leq x^2+y^2$, similar to the proof of Theorem 1.2 in [6] we have

$$
\overline{F}(x,T)\leq\frac{1}{1+a(\alpha-1)}\{(\frac{bm^2Ha\alpha^4}{2(\alpha-1)(1-a)}+3Ha\alpha^2)\frac{C}{R^2}+Ha\alpha^2\sqrt{K}\frac{C}{R}+\frac{a\alpha^2}{T}\}.
$$

Since T is arbitrary, we obtain

$$
\overline{F}(x,T)\leq\frac{1}{1+a(\alpha-1)}\{\frac{CHa\alpha^2}{R^2}(\frac{bm^2\alpha^2}{2(\alpha-1)(1-a)}+3+\sqrt{K}R)+\frac{a\alpha^2}{T}\}.
$$

Thus, the proof of Theorem 1.1 is complete. Letting $R\to\infty$ in (1.4), we get (1.5). Therefore, we conclude that Corollary 1.2 is true. □

4. The proof of Theorem 1.3

We find that $\varphi(t)=\frac{a}{t}+aSK+\frac{a}{3}(SK)^2t+\frac{\lambda}{t^2}$ and $\alpha(t)=1+\frac{2}{3}SKt$ satisfy the following equations:

$$
\begin{cases}
-\frac{2}{t}\varphi+a(\frac{1}{t}+HK)^2-\varphi'=0,\\
2(\frac{1}{t}+HK)-\alpha'=\frac{2\alpha}{t}.
\end{cases} \qquad (4.1)
$$

On the other hand, by (2.2) and the definition of F, $\varphi(t)$, and $\alpha(t)$ we get

$$
\begin{aligned}
(m-1)\triangle_f v+a(\frac{1}{t}+HK)=&-\frac{1}{\alpha}\{F+\varphi+(\alpha-1)\frac{|\nabla v|^2}{v}-a\alpha(\frac{1}{t}+HK)\}\\
=&-\frac{1}{\alpha}\{F+(\alpha-1)\frac{|\nabla v|^2}{v}-(\frac{2a}{3}HK+\frac{a}{3}(HK)^2t-\frac{\lambda}{t^2})\}.
\end{aligned}
$$

Therefore, similar to (5.4) in [6] and some discussions in Section 2 and Section 3, we have

$$L(F) \leq -\frac{1}{a\alpha^2}\{F + (\alpha-1)\frac{|\nabla v|^2}{v} - (\frac{2a}{3}HK + \frac{a}{3}(HK)^2 t - \frac{\lambda}{t^2})\}^2 - \frac{2}{t}F + 2m\nabla v \nabla F.$$

Construct a cut-off function ϕ as that in Section 3. Define $G = \beta(t)\phi F$. Assume that G achieves its maximum at the point $(x_0, s) \in B_p(2R) \times [0, T]$ and assume $G(x_0, s) > 0$. Similar to (5.6) in [6], we conclude that

$$G(x_0, s) \leq \frac{\alpha\beta}{s}\phi\{\frac{2}{3}(2HKs + (HK)^2 s^2 - \frac{\lambda}{s}) + s\alpha^2(\frac{\beta'}{\beta} - \frac{2}{s})\}$$

$$+a\alpha^2\beta H\{-\triangle_f \phi + 2\frac{|\nabla\phi|^2}{\phi} + \frac{bm^2\alpha^2}{2(\alpha-1)}\frac{|\nabla\phi|^2}{\phi}\}. \tag{4.2}$$

Let $\beta(t) = \tanh(HKt)$, similar to [6] we have

$$\frac{2}{3}(2HKs + (HK)^2 s^2 - \frac{\lambda}{s}) + s\alpha^2(\frac{\beta'}{\beta} - \frac{2}{s}) \leq 0. \tag{4.3}$$

By (3.4), (4.2), and (4.3), we get

$$G(x_0, s) \leq \{Ma\alpha^2(T)(3\frac{C}{R^2} + \frac{C\sqrt{K}}{R}) + \frac{CMabm^2\alpha^4(T)}{R^2}\}.$$

Thus, similar to the discussions of (5.7) in [6], we affirm that Theorem 1.3 holds. Letting $R \to \infty$ in (1.6), we get (1.7). Therefore, we conclude that Corollary 1.4 is true.

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