

Notes on the tangent bundle with deformed complete lift metric

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Abstract: In this paper, our aim is to study some properties of the tangent bundle with a deformed complete lift metric.

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1. Introduction

In the context of Riemannian geometry, the tangent bundle TM of a Riemannian manifold (M, g) was classically equipped with the Sasaki metric g_S , which was introduced in 1958 by Sasaki [14]. The study of the relationship between the geometry of a manifold (M, g) and that of its tangent bundle TM equipped with the Sasaki metric g_S has shown some kinds of rigidity (see [7, 9]). Other (classes of) metrics defined by the various kinds of classical lifts of the metric g from M to TM were defined in [19], and then geometers obtained interesting results related to these metrics involving the different aspects and concepts of differential geometry.

If (M, J, g) is an almost Hermitian manifold, its tangent bundle TM is also an almost Hermitian manifold with almost Hermitian structure $({}^HJ, g_S)$, where HJ is the horizontal lift of J [19]. In [20] (see also [21, 22]), Zayatuev studied the almost Hermitian structure on TM given by $({}^HJ, \tilde{g})$, where the metric \tilde{g} is defined by

$$\begin{aligned}\tilde{g}({}^HX, {}^HY) &= fg(X, Y), \\ \tilde{g}({}^HX, {}^VY) &= \tilde{g}({}^VX, {}^HY) = 0, \\ \tilde{g}({}^VX, {}^VY) &= g(X, Y)\end{aligned}$$

for all vector fields X and Y on M , and $f > 0$, $f \in C^\infty(M)$. For $f = 1$, it follows that $\tilde{g} = g_S$, i.e. the metric \tilde{g} is a generalization of g_S .

For a given Riemannian metric g on a differentiable manifold M , the complete lift Cg and vertical lift Vg of g are defined respectively as follows:

$$(a) \quad \begin{aligned}{}^Cg({}^HX, {}^HY) &= 0, \\ {}^Cg({}^HX, {}^VY) &= {}^Cg({}^VX, {}^HY) = g(X, Y), \\ {}^Cg({}^VX, {}^VY) &= 0,\end{aligned}$$

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$$(b) \quad \begin{aligned} Vg(HX, HY) &= g(X, Y), \\ Vg(HX, VY) &= Vg(VX, HY) = 0, \\ Vg(VX, VY) &= 0 \end{aligned}$$

for all vector fields X and Y on M . Moreover, note that Cg is a pseudo-Riemannian metric on TM and Vg is a degenerate metric on TM . As a generalization of the complete lift metric, in the present paper, we consider a family of metrics on TM . Let (M, g) be a Riemannian manifold and f be a nonzero differentiable function on (M, g) . On TM we define a deformation of the complete lift metric by $\tilde{G}_f = Cg + V(fg)$. We call the metric \tilde{G}_f a deformed complete lift metric. This paper can be considered as a contribution to the topic, considering for study a special new family of metrics on the tangent bundle constructed from the base metric and generated by a nonzero differentiable function on M . It is worth mentioning that a metric from this new family is g -natural only if the generating function is constant. Thus, the considered family is far from being a subfamily of the class of the so-called g -natural metrics (for g -natural metrics, see [1, 4, 8, 10, 12, 13]). The synectic lift of g on M to TM is of the form $\tilde{G} = Cg + Va$ (Va -vertical lift of symmetric $(0, 2)$ -tensor field) and was first considered in [16] and then studied in [2, 5, 11]. The metric \tilde{G}_f is a particular case of the synectic lift metric \tilde{G} . However, the study of the metric \tilde{G}_f is remarkable in some sense. In fact, the metric \tilde{G}_f has permitted us to establish an almost complex Norden structure on the tangent bundle (see Theorem 5.1) and the obtained results related to the metric \tilde{G}_f by means of the conditions represented by relations involving the function f and its derivative are also quite interesting (see Theorems 3.2, 3.3, 3.5, 4.2, and 5.2). Finally, it should be noted that the properties studied in this paper have not yet been considered for the synectic lift metric.

All manifolds, tensor fields, and connections in this paper are always assumed to be differentiable of class C^∞ . Moreover, we denote by $\mathfrak{S}_q^p(M)$ the set of all tensor fields of type (p, q) on M , and by $\mathfrak{S}_q^p(TM)$ the corresponding set on TM .

2. Preliminaries

Let M be an n -dimensional Riemannian manifold with a Riemannian metric g and denote by $\pi : TM \rightarrow M$ its tangent bundle with fiber the tangent spaces to M . TM is then a $2n$ -dimensional smooth manifold and some local charts induced naturally from local charts on M may be used. Namely, a system of local coordinates (U, x^i) in M induces on TM a system of local coordinates $(\pi^{-1}(U), x^i, x^{\bar{i}} = y^{\bar{i}})$, where (x^i) , $i = 1, \dots, n$ is a local coordinate system defined in the neighborhood U and $(y^{\bar{i}})$ is the Cartesian coordinates in each tangent space $T_P M$ at an arbitrary point P in U with respect to the natural basis $\{\frac{\partial}{\partial x^i} |_P\}$. Summation over repeated indices is always implied.

Let $X = X^i \frac{\partial}{\partial x^i}$ be the local expression in U of a vector field X on M . The vertical lift VX , the horizontal lift HX , and the complete lift CX of X are then given respectively by

$$VX = X^i \partial_{\bar{i}}, \tag{2.1}$$

$$HX = X^i \partial_i - y^j \Gamma_{jk}^i X^k \partial_{\bar{i}} \tag{2.2}$$

and

$$CX = X^i \partial_i + y^j \partial_j X^i \partial_{\bar{i}} \tag{2.3}$$

with respect to the induced coordinates, where $\partial_i = \frac{\partial}{\partial x^i}$, $\partial_{\bar{i}} = \frac{\partial}{\partial y^i}$ and Γ_{jk}^i are the coefficients of the Levi-Civita connection ∇ of g .

The bracket operation of vertical and horizontal vector fields is given by the following formulas:

$$\begin{cases} [{}^H X, {}^H Y] = {}^H[X, Y] - {}^V(R(X, Y)u) \\ [{}^H X, {}^V Y] = {}^V(\nabla_X Y) \\ [{}^V X, {}^V Y] = 0 \end{cases}$$

for all $X, Y \in \mathfrak{X}_0^1(M)$ [3], where R is the Riemannian curvature of g defined by

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}.$$

Given a (p, q) -tensor field S on M , $q > 1$, we then consider a tensor field $\gamma S \in \mathfrak{S}_{q-1}^p(TM)$ on $\pi^{-1}(U)$ by

$$\gamma S = (y^s S_{s_{i_2} \dots i_q}^{j_1 \dots j_p}) \partial_{\bar{j}_1} \otimes \dots \otimes \partial_{\bar{j}_p} \otimes dx^{i_2} \otimes \dots \otimes dx^{i_q}$$

with respect to the induced coordinates (x^i, y^i) ([19], p. 12). The tensor field γS defined on each $\pi^{-1}(U)$ determines global tensor field on TM . For any $A \in \mathfrak{S}_1^1(M)$, we easily see that γA has components, with respect to the induced coordinates (x^i, y^i) ,

$$(\gamma A) = y^s A_s^i \partial_{\bar{i}}. \tag{2.4}$$

With the connection ∇ of g on M , we can introduce on each induced coordinate neighborhood $\pi^{-1}(U)$ of TM a frame field that is very useful in our computation. The adapted frame on $\pi^{-1}(U)$ consists of the following $2n$ linearly independent vector fields:

$$\begin{aligned} E_j &= \partial_j - y^s \Gamma_{sj}^h \partial_{\bar{h}}, \\ E_{\bar{j}} &= \partial_{\bar{j}}. \end{aligned}$$

We write the adapted frame as $\{E_\beta\} = \{E_j, E_{\bar{j}}\}$. A direct calculation proves the following lemma.

Lemma 2.1 ([19], p. 101) *The Lie brackets of the adapted frame of TM satisfy the following identities:*

$$\begin{cases} [E_j, E_i] = y^s R_{ijs}^m E_{\bar{h}} \\ [E_j, E_{\bar{i}}] = \Gamma_{ji}^h E_{\bar{h}} \\ [E_{\bar{j}}, E_{\bar{i}}] = 0 \end{cases} \tag{2.5}$$

where R_{ijb}^a denotes the components of the curvature tensor of M .

Using Eqs. (2.1), (2.2), (2.3), and (2.4), we have

$$\begin{aligned} {}^H X &= X^j E_j, \\ {}^V X &= X^j E_{\bar{j}}, \\ {}^C X &= X^j E_j + y^s \nabla_s X^j E_{\bar{j}} \end{aligned}$$

and

$$\gamma A = y^s A_s^j E_{\bar{j}}$$

with respect to the adapted frame $\{E_\beta\}$.

3. The Levi-Civita connection of the deformed complete lift metric \tilde{G}_f

In this section, we give the Levi-Civita connection $\tilde{\nabla}$ of the tangent bundle TM with the deformed complete lift metric \tilde{G}_f and study fiber-preserving Killing vector fields on TM . The deformed complete lift metric \tilde{G}_f is defined by

$$\begin{aligned} \tilde{G}({}^H X, {}^H Y) &= fg(X, Y) \\ \tilde{G}({}^H X, {}^V Y) &= \tilde{G}({}^V X, {}^H Y) = g(X, Y) \\ \tilde{G}({}^V X, {}^V Y) &= 0 \end{aligned}$$

for all $X, Y \in \mathfrak{S}_0^1(M)$. We now give expressions of the deformed complete lift metric \tilde{G}_f and its inverse \tilde{G}_f^{-1} with respect to the adapted frame $\{E_\beta\}$:

$$\tilde{G}_f = \begin{pmatrix} fg_{ij} & g_{ij} \\ g_{ij} & 0 \end{pmatrix} \text{ and } \tilde{G}_f^{-1} = \begin{pmatrix} 0 & g^{jh} \\ g^{jh} & -fg^{jh} \end{pmatrix}. \tag{3.1}$$

Theorem 3.1 *Let (M, g) be a Riemannian manifold and TM be its tangent bundle equipped with the deformed complete lift metric \tilde{G}_f . The corresponding Levi-Civita connection $\tilde{\nabla}$ then satisfies the following equations:*

$$\begin{cases} \tilde{\nabla}_{E_i} E_j = \Gamma_{ij}^h E_h + \{y^s R_{sij}{}^h + \frac{1}{2}(f_i \delta_j^h + f_j \delta_i^h - g_{ij} f^h)\} E_{\bar{h}}, \\ \tilde{\nabla}_{E_i} E_{\bar{j}} = \Gamma_{ij}^h E_{\bar{h}}, \\ \tilde{\nabla}_{E_{\bar{i}}} E_j = 0, \\ \tilde{\nabla}_{E_{\bar{i}}} E_{\bar{j}} = 0 \end{cases}$$

with respect to the adapted frame $\{E_\beta\}$. Here, Γ_{ij}^h and $R_{sij}{}^h$ are respectively the components of the Levi-Civita connection and Riemannian curvature of g , and $f_i = \partial_i f$, $f^h = g^{mh} f_m$.

Proof Using $\tilde{\nabla}_{E_\alpha} E_\beta = \tilde{\Gamma}_{\alpha\beta}^\gamma E_\gamma$ and the Koszul formula

$$\begin{aligned} 2\tilde{G}_f(\tilde{\nabla}_{\tilde{X}} \tilde{Y}, \tilde{Z}) &= \tilde{X}(\tilde{G}_f(\tilde{Y}, \tilde{Z})) + \tilde{Y}(\tilde{G}_f(\tilde{Z}, \tilde{X})) - \tilde{Z}(\tilde{G}_f(\tilde{X}, \tilde{Y})) - \tilde{G}_f(\tilde{X}, [\tilde{Y}, \tilde{Z}]) \\ &\quad + \tilde{G}_f(\tilde{Y}, [\tilde{Z}, \tilde{X}]) + \tilde{G}_f(\tilde{Z}, [\tilde{X}, \tilde{Y}]) \end{aligned}$$

for all $\tilde{X}, \tilde{Y}, \tilde{Z} \in \mathfrak{S}_0^1(TM)$, one can verify the Koszul formula for pairs $\tilde{X} = E_i, E_{\bar{i}}$ and $\tilde{Y} = E_j, E_{\bar{j}}$ and $\tilde{Z} = E_k, E_{\bar{k}}$. In calculations, the formulas in Eqs. (2.5) and (3.1) and the first Bianchi identity for R should be applied. We omit standard calculations. \square

If we denote the horizontal and vertical projections by \mathcal{H} and \mathcal{V} , respectively, we can then say the following:

- i) The vertical distribution VTM is totally geodesic in TTM if $\mathcal{H}\tilde{\nabla}_{E_{\bar{i}}} E_{\bar{j}} = 0$,
- ii) The horizontal distribution HTM is totally geodesic in TTM if $\mathcal{V}\tilde{\nabla}_{E_i} E_j = 0$.

Hence, in view of Theorem 3.1, we can say the following result.

Theorem 3.2 *Let (M, g) be a Riemannian manifold and TM be its tangent bundle equipped with the deformed complete lift metric \tilde{G}_f . Then:*

- i) *The vertical distribution VTM is totally geodesic in TTM .*
- ii) *The horizontal distribution HTM is totally geodesic in TTM if and only if (M, g) is locally flat and $f = C$ (const.).*

Proof *i)* From the last relation in Theorem 3.1, we get $\mathcal{H}\tilde{\nabla}_{E_{\bar{i}}}E_{\bar{j}} = 0$. Thus, VTM is totally geodesic.

ii) From the first relation in Theorem 3.1, we have

$$\mathcal{V}\tilde{\nabla}_{E_i}E_j = \{y^s R_{sij}{}^h + \frac{1}{2}(f_i\delta_j^h + f_j\delta_i^h - g_{ij}f^h)\}E_{\bar{h}}. \tag{3.2}$$

Let $\mathcal{V}\tilde{\nabla}_{E_i}E_j = 0$, i.e. the horizontal distribution HTM be totally geodesic, and then by Eq. (3.2) we get

$$y^s R_{sij}{}^h + \frac{1}{2}(f_i\delta_j^h + f_j\delta_i^h - g_{ij}f^h) = 0. \tag{3.3}$$

Operating $\partial_{\bar{k}}$ to Eq. (3.3), we obtain $R_{kij}{}^h = 0$, i.e. (M, g) is locally flat. In this case, Eq. (3.3) reduces to

$$f_i\delta_j^h + f_j\delta_i^h - g_{ij}f^h = 0,$$

from which, contracting i and h , it follows that $\partial_j f = f_j = 0$. Since this is true for any j , we can say $f = C$ (*const.*).

Conversely, if (M, g) is locally flat and $f = C$ (*const.*), then by Eq. (3.2), $\mathcal{V}\tilde{\nabla}_{E_i}E_j = 0$, i.e. HTM is totally geodesic. □

The Levi-Civita connection ${}^C\nabla$ of the complete lift metric Cg is given by

$$\begin{cases} {}^C\nabla_{E_i} E_j = \Gamma_{ij}^h E_h + y^s R_{sij}{}^h E_{\bar{h}}, \\ {}^C\nabla_{E_i} E_{\bar{j}} = \Gamma_{ij}^h E_{\bar{h}}, \\ {}^C\nabla_{E_{\bar{i}}} E_j = 0, \\ {}^C\nabla_{E_{\bar{i}}} E_{\bar{j}} = 0 \end{cases}$$

with respect to the adapted frame $\{E_{\beta}\}$. On comparing the Levi-Civita connections of the complete lift metric Cg and the deformed complete lift metric \tilde{G}_f , we have the result below.

Theorem 3.3 *Let (M, g) be a Riemannian manifold and TM be its tangent bundle. The Levi-Civita connections of the complete lift metric Cg and the deformed complete lift metric \tilde{G}_f coincide if and only if $f = C$ (*const.*).*

Proof The Levi-Civita connections of the complete lift metric Cg and the deformed complete lift metric \tilde{G}_f coincide if and only if $f_i\delta_j^h + f_j\delta_i^h - g_{ij}f^h = 0$, from which $f = C$ (*const.*) (see *(ii)* in proof of Theorem 3.1). □

Next, we study fiber-preserving Killing vector fields on TM with respect to the deformed complete lift metric \tilde{G}_f . Let \tilde{X} have the components $(V^h, V^{\bar{h}})$ with respect to the adapted frame $\{E_{\beta}\}$; then \tilde{X} is fiber-preserving if and only if V^h depends only on the variables (x^h) . First, we state following lemma.

Lemma 3.4 *The Lie derivative of the deformed complete lift metric \tilde{G}_f with respect to the fiber-preserving vector field $\tilde{X} = V^h E_h + V^{\bar{h}} E_{\bar{h}}$ is in the following form:*

$$\begin{aligned} L_{\tilde{X}}\tilde{G}_f &= \{L_V(fg_{ij}) - g_{im}(y^b V^c R_{jcb}{}^m - V^{\bar{b}}\Gamma_{bj}^m - (E_j V^{\bar{m}})) \\ &\quad - g_{mj}(y^b V^c R_{icb}{}^m - V^{\bar{b}}\Gamma_{bi}^m - (E_i V^{\bar{m}}))\}dx^i dx^j \\ &\quad + 2\{L_V g_{ij} - g_{im}\nabla_j V^m - g_{im}(E_{\bar{j}} V^{\bar{m}})\}dx^i \delta y^j \end{aligned}$$

where $L_V g_{ij}$ and $\nabla_i V^m$ denote the components of the Lie derivative $L_V g$ and the covariant derivative of V , respectively.

Proof The proof is similar to that in [17], so we omit it. □

Theorem 3.5 *Let (M, g) be a Riemannian manifold and TM be its tangent bundle equipped with the deformed complete lift metric \tilde{G}_f . A vector field \tilde{X} on TM is a fiber-preserving Killing vector field with respect to \tilde{G}_f if and only if*

$$\tilde{X} = {}^C V + {}^V B + \gamma A,$$

where $B = (B^h)$, $V = (V^h) \in \mathfrak{S}_0^1(M)$ and $A = (A_i^h) \in \mathfrak{S}_1^1(M)$ such that

- (i) $A = (A_s^m) = -g^{im}(L_V g_{is})$,
- (ii) $L_V \Gamma_{ij}^m = 0$, i.e. V is an infinitesimal affine transformation on M ,
- (iii) $(V^m \partial_m f)g_{ij} + f L_V g_{ij} + L_B g_{ij} = 0$.

Proof A vector field \tilde{X} is a Killing vector field on TM with respect to \tilde{G}_f if and only if $L_{\tilde{X}} \tilde{G}_f = 0$. By virtue of Lemma 3.4, we say that $\tilde{X} = V^h E_h + V^{\bar{h}} E_{\bar{h}}$ is a fiber-preserving Killing vector field on TM with respect to \tilde{G}_f if and only if the following relations hold:

$$\begin{aligned} L_V(fg_{ij}) - g_{im}(y^b V^c R_{jcb}{}^m - V^{\bar{b}} \Gamma_{bj}^m - (E_j V^{\bar{m}})) \\ - g_{mj}(y^b V^c R_{icb}{}^m - V^{\bar{b}} \Gamma_{bi}^m - (E_i V^{\bar{m}})) = 0 \end{aligned} \tag{3.4}$$

and

$$L_V g_{ij} - g_{im} \nabla_j V^m - g_{im} (E_{\bar{j}} V^{\bar{m}}) = 0. \tag{3.5}$$

From Eq. (3.5), since $(E_{\bar{j}} V^{\bar{m}})$ depends only on the variables (x^h) , we can put

$$V^{\bar{h}} = y^s A_s^h + B^h, \tag{3.6}$$

where A_s^h and B^h are certain functions that depend only on the variables (x^h) and also respectively are the components of a (1,1)-tensor field A and a vector field B on M . Hence, the fiber-preserving Killing vector field \tilde{X} on TM can be expressed in the following form:

$$\tilde{X} = V^h E_h + V^{\bar{h}} E_{\bar{h}} = V^h E_h + \{y^s A_s^h + B^h\} E_{\bar{h}}.$$

Substituting Eq. (3.6) into Eqs. (3.4) and (3.5), we obtain

$$L_V(fg_{ij}) + g_{im} \nabla_j B^m + g_{jm} \nabla_i B^m = 0, \tag{3.7}$$

$$V^c (R_{cjsi} + R_{cisi}) + g_{im} \nabla_j A_s^m + g_{jm} \nabla_i A_s^m = 0, \tag{3.8}$$

$$L_V g_{ij} - g_{im} \nabla_j V^m - g_{im} A_j^m = 0. \tag{3.9}$$

From Eq. (3.9), we have

$$A_s^m = \nabla_s V^m - g^{im}(L_V g_{is}).$$

Substituting the above equation into Eq. (3.8), we get

$$\begin{aligned} 0 &= V^c(R_{cjsi} + R_{cisi}) + \nabla_j(g_{im}\nabla_s V^m - L_V g_{is}) + \nabla_i(g_{jm}\nabla_s V^m - L_V g_{sj}) \\ &= V^c(R_{cjsi} + R_{cisi}) + g_{im}\nabla_j\nabla_s V^m - \nabla_j L_V g_{is} + g_{jm}\nabla_i\nabla_s V^m - \nabla_i L_V g_{sj} \\ &= V^c(R_{cjsi} + R_{cisi}) + g_{im}(L_V \Gamma_{js}^m - V^c R_{cjs}^m) - (L_V \Gamma_{ij}^m g_{ms} + L_V \Gamma_{js}^m g_{im}) \\ &\quad + g_{jm}(L_V \Gamma_{is}^m - V^c R_{cis}^m) - (L_V \Gamma_{ij}^m g_{ms} + L_V \Gamma_{is}^m g_{jm}) \\ &= -2(L_V \Gamma_{ij}^m)g_{ms}, \end{aligned}$$

from which it follows that $L_V \Gamma_{ij}^m = 0$, i.e. V is an infinitesimal affine transformation on M .

From Eq. (3.7), we have

$$\begin{aligned} 0 &= L_V(fg_{ij}) + g_{im}\nabla_j B^m + g_{jm}\nabla_i B^m \\ &= (V^m \partial_m f)g_{ij} + fL_V g_{ij} + L_B g_{ij}. \end{aligned}$$

Conversely, if B^h , V^h , and A_i^h are given so that they satisfy (i)-(iii), we easily see that

$$\begin{aligned} \tilde{X} &= V^h E_h + V^{\bar{h}} E_{\bar{h}} = V^h E_h + \{y^s(\nabla_s V^h - g^{ih}(L_V g_{is})) + B^h\} E_{\bar{h}} \\ &= {}^C V + {}^V B + \gamma A \end{aligned}$$

is a fiber-preserving Killing vector field on (TM, \tilde{G}_f) . □

4. Some curvature properties of the deformed complete lift metric \tilde{G}_f

We now turn our attention to the Riemannian curvature tensor \tilde{R} of the tangent bundle TM equipped with the deformed complete lift metric \tilde{G}_f . The Riemannian curvature tensor of the tangent bundle with the deformed complete lift metric \tilde{G}_f is defined by

$$\tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z} = \tilde{\nabla}_{\tilde{X}}\tilde{\nabla}_{\tilde{Y}}\tilde{Z} - \tilde{\nabla}_{\tilde{Y}}\tilde{\nabla}_{\tilde{X}}\tilde{Z} - \tilde{\nabla}_{[\tilde{X}, \tilde{Y}]}\tilde{Z}$$

for all $\tilde{X}, \tilde{Y}, \tilde{Z} \in \mathfrak{S}_0^1(TM)$. Using Eq. (2.5) and Theorem 3.1, standard calculations give the following proposition.

Proposition 4.1 *Let (M, g) be a Riemannian manifold and \tilde{R} be the Riemannian curvature tensor of the tangent bundle TM equipped with the deformed complete lift metric \tilde{G}_f . Then the following formulas hold:*

$$\begin{aligned}
 \tilde{R}(E_{\bar{m}}, E_{\bar{i}})E_{\bar{j}} &= 0, \\
 \tilde{R}(E_{\bar{m}}, E_{\bar{i}})E_j &= 0, \\
 \tilde{R}(E_m, E_{\bar{i}})E_{\bar{j}} &= 0, \\
 \tilde{R}(E_m, E_{\bar{i}})E_j &= R_{mij}{}^h E_{\bar{h}}, \\
 \tilde{R}(E_m, E_i)E_{\bar{j}} &= R_{mij}{}^h E_{\bar{h}}, \\
 \tilde{R}(E_m, E_i)E_j &= R_{mij}{}^h E_h + [y^s(\nabla_m R_{sij}{}^h - \nabla_i R_{smj}{}^h) \\
 &\quad + \frac{1}{2}((\nabla_m f_i - \nabla_i f_m)\delta_j^h + (\nabla_m f_j)\delta_i^h - (\nabla_i f_j)\delta_m^h \\
 &\quad + g_{mj}(\nabla_i f^h) - g_{ij}(\nabla_m f^h))]E_{\bar{h}}
 \end{aligned}
 \tag{4.1}$$

with respect to the adapted frame $\{E_\beta\}$.

Next we compare the geometries of the manifold (M, g) and its tangent bundle (TM, \tilde{G}_f) .

Theorem 4.2 *Let (M, g) be a Riemannian manifold and TM be its tangent bundle equipped with the deformed complete lift metric \tilde{G}_f . Then TM is locally flat if and only if M is locally flat and the function f satisfies the condition*

$$(\nabla_m f_i - \nabla_i f_m)\delta_j^h + (\nabla_m f_j)\delta_i^h - (\nabla_i f_j)\delta_m^h + g_{mj}(\nabla_i f^h) - g_{ij}(\nabla_m f^h) = 0,
 \tag{4.2}$$

where ∇ is the Levi-Civita connection of g .

Proof It follows from Proposition 4.1 that if (M, g) is locally flat and the condition in Eq. (4.2) holds, then (TM, \tilde{G}_f) is locally flat. Conversely, if we assume $\tilde{R} = 0$, by means of the last equation in Eq. (4.1) in the point $(x, 0)$ we get

$$\begin{aligned}
 R_{mij}{}^h E_h + \frac{1}{2}((\nabla_m f_i - \nabla_i f_m)\delta_j^h + (\nabla_m f_j)\delta_i^h - (\nabla_i f_j)\delta_m^h \\
 + g_{mj}(\nabla_i f^h) - g_{ij}(\nabla_m f^h))E_{\bar{h}} = \tilde{R}_{(x,0)}(E_m, E_i)E_j = 0,
 \end{aligned}$$

from which $R_{mij}{}^h = 0$ and $(\nabla_m f_i - \nabla_i f_m)\delta_j^h + (\nabla_m f_j)\delta_i^h - (\nabla_i f_j)\delta_m^h + g_{mj}(\nabla_i f^h) - g_{ij}(\nabla_m f^h) = 0$. This completes the proof. \square

Next, we calculate the Ricci tensor and the scalar curvature of (TM, \tilde{G}_f) . Let $\tilde{R}_{IJ} = \tilde{R}_{MIJ}{}^M$ denote the Ricci tensor of the deformed complete lift metric \tilde{G}_f . It follows that, from the equations in (4.1), the components of the Ricci tensor \tilde{R}_{IJ} are characterized by

$$\begin{aligned}
 \tilde{R}_{ij} &= 2R_{ij}, \quad \tilde{R}_{\bar{i}j} = 0, \\
 \tilde{R}_{i\bar{j}} &= 0, \quad \tilde{R}_{\bar{i}\bar{j}} = 0.
 \end{aligned}
 \tag{4.3}$$

Let \tilde{X} and \tilde{Y} be vector fields of TM . The curvature operator $\tilde{R}(\tilde{X}, \tilde{Y})$ is a differential operator on TM . Similarly, for vector fields X and Y of M , $R(X, Y)$ is a differential operator on M . Now we operate

the curvature operator to the Ricci tensor. The tensors $(\widetilde{R}(\widetilde{X}, \widetilde{Y})\widetilde{Ric})(\widetilde{Z}, \widetilde{W})$ and $(R(X, Y)Ric)(Z, W)$ have coefficients

$$((\widetilde{R}(\widetilde{X}, \widetilde{Y})\widetilde{Ric})(\widetilde{Z}, \widetilde{W}))_{\alpha\beta\gamma\theta} = \widetilde{R}_{\alpha\beta\gamma}{}^\varepsilon \widetilde{R}_{\varepsilon\theta} + \widetilde{R}_{\alpha\beta\theta}{}^\varepsilon \widetilde{R}_{\gamma\varepsilon}$$

and

$$((R(X, Y)Ric)(Z, W))_{ijkl} = R_{ijk}{}^p R_{pl} + R_{ijl}{}^p R_{kp},$$

respectively. By putting $\alpha = i, \beta = j, \gamma = k, \theta = l$, it follows that

$$\begin{aligned} ((\widetilde{R}(\widetilde{X}, \widetilde{Y})\widetilde{Ric})(\widetilde{Z}, \widetilde{W}))_{ijkl} &= 2R_{ijk}{}^p R_{pl} + 2R_{ijl}{}^p R_{kp} \\ &= 2((R(X, Y)Ric)(Z, W))_{ijkl}, \end{aligned}$$

with all of the others being 0. Therefore, we get the following.

Theorem 4.3 *Let (M, g) be a Riemannian manifold and TM be its tangent bundle equipped with the deformed complete lift metric \widetilde{G}_f . Then $(\widetilde{R}(\widetilde{X}, \widetilde{Y})\widetilde{Ric})(\widetilde{Z}, \widetilde{W}) = 0$ if and only if $(R(X, Y)Ric)(Z, W) = 0$.*

Let $\widetilde{S}c$ denote the scalar curvature of the deformed complete lift metric \widetilde{G}_f . It also follows from Eqs. (3.1) and (4.3) that the scalar curvature of the deformed complete lift metric \widetilde{G}_f is given by

$$\widetilde{S}c = \widetilde{G}_f^{IJ} \widetilde{R}_{IJ} = \widetilde{G}_f^{ij} \widetilde{R}_{ij} + \widetilde{G}_f^{i\bar{j}} \widetilde{R}_{i\bar{j}} + \widetilde{G}_f^{\bar{i}j} \widetilde{R}_{\bar{i}j} + \widetilde{G}_f^{\bar{i}\bar{j}} \widetilde{R}_{\bar{i}\bar{j}} = 0.$$

Hence, we have the following.

Theorem 4.4 *Let (M, g) be a Riemannian manifold and TM be its tangent bundle equipped with the deformed complete lift metric \widetilde{G}_f . Then (TM, \widetilde{G}_f) is space of constant scalar curvature 0.*

5. Kähler-Norden structures on the tangent bundle

Let (M_{2n}, J) be an almost complex manifold with an almost complex structure J . A pseudo-Riemannian metric g of signature (n, n) on M_{2n} is called a Norden metric if

$$g(JX, JY) = -g(X, Y)$$

or equivalently

$$g(JX, Y) = g(X, JY)$$

for any $X, Y \in \mathfrak{S}_0^1(M_{2n})$. Next, the triple (M_{2n}, J, g) is called an almost complex Norden manifold. A Kähler-Norden (anti-Kähler) manifold can be defined as a triple (M_{2n}, J, g) that consists of a smooth manifold M_{2n} endowed with an almost complex structure J and a Norden metric g such that $\nabla J = 0$, where ∇ is the Levi-Civita connection of g . It is well known that the condition $\nabla J = 0$ is equivalent to C-holomorphicity (analyticity) of the Norden metric g [6], i.e. $\Phi_J g = 0$, where Φ_J is the Tachibana operator [15, 18]: $(\Phi_J g)(X, Y, Z) = (JX)(g(Y, Z)) - X(g(JY, Z)) + g((L_Y J)X, Z) + g(Y, (L_Z - J)X)$. Moreover, note that $G(Y, Z) = g(JY, Z)$ is the twin Norden metric. Since in dimension 2 a Kähler-Norden manifold is flat, we assume in the sequel that $\dim M \geq 4$.

Let M be a manifold with an almost complex structure J and a pseudo-Riemannian metric g . The horizontal lift ${}^HJ \in \mathfrak{S}_1^1(TM)$ is defined by the formulas

$$\begin{cases} {}^HJ({}^HX) = {}^H(JX), \\ {}^HJ({}^VX) = {}^V(JX) \end{cases}$$

for any $X \in \mathfrak{S}_0^1(M)$ and $J \in \mathfrak{S}_1^1(M)$. Moreover, it is well known that if J is an almost complex structure on M , then its horizontal lift HJ is an almost complex structure on TM [19]. Also note that the signature of the deformed complete lift metric \tilde{G}_f is (n, n) . We calculate

$$A(\tilde{X}, \tilde{Y}) = \tilde{G}_f({}^HJ\tilde{X}, \tilde{Y}) - \tilde{G}_f(\tilde{X}, {}^HJ\tilde{Y})$$

for any $\tilde{X}, \tilde{Y} \in \mathfrak{S}_0^1(TM)$, and we then get

$$\begin{aligned} A({}^HX, {}^HY) &= \tilde{G}_f({}^HJ{}^HX, {}^HY) - \tilde{G}_f({}^HX, {}^HJ{}^HY) \\ &= \tilde{G}_f({}^H(JX), {}^HY) - \tilde{G}_f({}^HX, {}^H(JY)) \\ &= fg(JX, Y) - fg(X, JY), \\ A({}^VX, {}^HY) &= \tilde{G}_f({}^HJ{}^VX, {}^HY) - \tilde{G}_f({}^VX, {}^HJ{}^HY) \\ &= \tilde{G}_f({}^V(JX), {}^HY) - \tilde{G}_f({}^VX, {}^H(JY)) \\ &= g(JX, Y) - g(X, JY), \\ A({}^VX, {}^VY) &= \tilde{G}_f({}^HJ{}^VX, {}^VY) - \tilde{G}_f({}^VX, {}^HJ{}^VY) \\ &= \tilde{G}_f({}^V(JX), {}^VY) - \tilde{G}_f({}^VX, {}^V(JY)) \\ &= 0. \end{aligned}$$

The last equations show that \tilde{G}_f is pure with respect to HJ if and only if g is pure with respect to J . Hence, we have the following theorem.

Theorem 5.1 *Let (M, g) be a pseudo-Riemannian manifold and TM be its tangent bundle equipped with the deformed complete lift metric \tilde{G}_f and the almost complex structure HJ . The triple $(TM, {}^HJ, \tilde{G}_f)$ is an almost complex Norden manifold if and only if the triple (M, J, g) is an almost complex Norden manifold.*

Determining both the deformed complete lift metric \tilde{G}_f and the almost complex structure HJ , and using the facts ${}^VX(fg(Y, Z)) = 0$ and ${}^HX(fg(Y, Z)) = X(f)g(Y, Z) + fX(g(Y, Z))$, we calculate

$$\begin{aligned} (\Phi_{HJ}\tilde{G}_f)(\tilde{X}, \tilde{Y}, \tilde{Z}) &= ({}^HJ\tilde{X})(\tilde{G}_f(\tilde{Y}, \tilde{Z})) - \tilde{X}(\tilde{G}_f({}^HJ\tilde{Y}, \tilde{Z})) \\ &\quad + \tilde{G}_f((L_{\tilde{Y}}{}^HJ)\tilde{X}, \tilde{Z}) + \tilde{G}_f(\tilde{Y}, (L_{\tilde{Z}}{}^HJ)\tilde{X}) \end{aligned}$$

for all $\tilde{X}, \tilde{Y}, \tilde{Z} \in \mathfrak{S}_0^1(TM)$. We then obtain the following equations:

$$\begin{aligned} (\Phi_{HJ}\tilde{G}_f)(^V X, ^V Y, ^H Z) &= 0, \\ (\Phi_{HJ}\tilde{G}_f)(^V X, ^V Y, ^V Z) &= 0, \\ (\Phi_{HJ}\tilde{G}_f)(^V X, ^H Y, ^V Z) &= 0, \\ (\Phi_{HJ}\tilde{G}_f)(^V X, ^H Y, ^H Z) &= g((\nabla_Y J)X, Z) + g(Y, (\nabla_Z J)X), \\ (\Phi_{HJ}\tilde{G}_f)(^H X, ^V Y, ^H Z) &= (\Phi_{Jg})(X, Y, Z) - g((\nabla_Y J)X, Z), \\ (\Phi_{HJ}\tilde{G}_f)(^H X, ^V Y, ^V Z) &= 0, \\ (\Phi_{HJ}\tilde{G}_f)(^H X, ^H Y, ^H Z) &= (JX)(f)g(Y, Z) - X(f)g(JY, Z) + f((\Phi_{Jg})(X, Y, Z)) \\ &\quad + g(JR(Y, X)u - R(Y, JX)u, Z) \\ &\quad + g(Y, JR(Z, X)u - R(Z, JX)u), \\ (\Phi_{HJ}\tilde{G}_f)(^H X, ^H Y, ^V Z) &= (\Phi_{Jg})(X, Y, Z) - g(Y, (\nabla_Z J)X). \end{aligned}$$

It is well known that the equation $\Phi_{Jg} = 0$ is equivalent to $\nabla J = 0$, and the Riemann curvature R of a Kähler-Norden manifold is totally pure. Therefore, from the equations above, we have the following result.

Theorem 5.2 *Let (M, g) be a pseudo-Riemannian manifold and TM be its tangent bundle equipped with the deformed complete lift metric \tilde{G}_f and the almost complex structure $^H J$. The triple $(TM, ^H J, \tilde{G}_f)$ is a Kähler-Norden manifold if and only if the triple (M, J, g) is a Kähler-Norden manifold and the function f satisfies the condition*

$$(JX)(f)g(Y, Z) - X(f)g(JY, Z) = 0.$$

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