Pseudosymmetric lightlike hypersurfaces

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Abstract: We study lightlike hypersurfaces of a semi-Riemannian manifold satisfying pseudosymmetry conditions. We give sufficient conditions for a lightlike hypersurface to be pseudosymmetric and show that there is a close relationship of the pseudosymmetry condition of a lightlike hypersurface and its integrable screen distribution. We obtain that a pseudosymmetric lightlike hypersurface is a semisymmetric lightlike hypersurface or totally geodesic under certain conditions. Moreover, we give an example of pseudosymmetric lightlike hypersurfaces and investigate pseudoparallel lightlike hypersurfaces. Furthermore, we introduce Ricci-pseudosymmetric lightlike hypersurfaces, obtain characterizations, and give an example for such hypersurfaces.

Key words: Semisymmetric lightlike hypersurface, Ricci-semisymmetric lightlike hypersurface, pseudosymmetric lightlike hypersurface, pseudoparallel lightlike hypersurface

1. Introduction

Let (M, g) be a Riemannian manifold of dimension n and ∇ be the Levi-Civita connection. A Riemannian manifold is called locally symmetric if $\nabla R = 0$, where R is the Riemannian curvature tensor of M [6]. Locally symmetric Riemannian manifolds are a generalization of manifolds of constant curvature. As a generalization of locally symmetric Riemannian manifolds, semisymmetric Riemannian manifolds were defined by the condition

$$R \cdot R = 0.$$

It is known that locally symmetric manifolds are semisymmetric manifolds but the converse is not true [28]. Such manifolds were investigated by Cartan and they were locally classified by Szabo [5].

The Riemannian manifold (M, g) is called a pseudosymmetric manifold if at every point of M the following condition is satisfied: *the tensor $R \cdot R$ and $Q(g, R)$ are linearly dependent*.

The manifold (M, g) is pseudosymmetric if only if $R \cdot R = LQ(g, R)$ on the set $U = \{x \in M \mid Q(g, R) \neq 0 \ at \ x\}$, where L is some function on U.

Pseudosymmetric manifolds were discovered during the study of totally umbilical submanifolds of semisymmetric manifolds [1]. It is clear that every semisymmetric Riemannian manifold is a pseudosymmetric manifold but the converse is not true.

On the other hand, lightlike hypersurfaces of a semi-Riemannian manifold were studied by Duggal and Bejancu and they obtained a transversal bundle for such hypersurfaces to the overcome anomaly that occurred due to degenerate metric. After their book [18], many authors studied lightlike hypersurfaces by using their

approach. In [27], Şahin introduced the notion of semisymmetric lightlike hypersurfaces of a semi-Riemannian manifold and obtained many new results. After Şahin's paper, many authors have studied such surfaces in various semi-Riemannian manifolds (see [20, 21, 22, 23, 24, 29]).

In this paper, we study a more general curvature condition for lightlike hypersurfaces: pseudosymmetry conditions. We define a pseudosymmetric lightlike hypersurface, give an example, and obtain certain sufficient conditions for such hypersurfaces in Section 3, after we establish the basic information needed for the rest of the paper in Section 2. We also investigate sufficient conditions for a lightlike hypersurface (Einstein) to be pseudosymmetric. In Section 4, we study lightlike hypersurfaces by imposing a pseudoparallel condition and we observe that the situation is very different from the nondegenerate case. In Section 5, we check the Ricci-pseudosymmetry conditions for a lightlike hypersurface and provide an example of such hypersurfaces. Moreover, we show that a Ricci-pseudosymmetric lightlike hypersurface is totally geodesic under certain geometric conditions.

2. Preliminaries

In this section, we give a review on manifolds with pseudosymmetry type and lightlike hypersurfaces.

Let (M, g) be a connected n-dimensional, $n \geq 3$, semi-Riemannian manifold of class C^∞. For a $(0, k)$-tensor field T on M, $k \geq 1$, we define the $(0, k+2)$-tensors $R \cdot T$ and $Q(g, T)$ by

$$
\begin{aligned}
(R \cdot T)(X_1, ..., X_k; X, Y) &= (\tilde{R}(X, Y) \cdot T)(X_1, ..., X_k) \\
&= -T(\tilde{R}(X, Y)X_1, X_2, ..., X_k) \\
&\quad - ... - T(X_1, ..., X_{k-1}, \tilde{R}(X, Y)X_k),
\end{aligned}
\tag{2.1}
$$

and

$$
\begin{aligned}
Q(g, T)(X_1, ..., X_k; X, Y) &= ((X \wedge Y) \cdot T)(X_1, ..., X_k) \\
&= -T((X \wedge Y)X_1, X_2, ..., X_k) \\
&\quad - ... - T(X_1, ..., X_{k-1}, (X \wedge Y)X_k),
\end{aligned}
\tag{2.2}
$$

respectively, for $X_1, ..., X_k, X, Y \in \Gamma(TM)$, where $\tilde{R}$ is the curvature tensor field of M and R is the Riemannian–Christoffel tensor field given by $R(X_1, X_2, X_3, X_4) = g(\tilde{R}(X_1, X_2)X_3, X_4)$, and the endomorphisms are defined by $\tilde{R}(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$,
$(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y$. Curvature conditions, involving the form $R \cdot T = 0$, are called curvature conditions of semisymmetric type [7]. A semi-Riemannian manifold (M, g) is then said to be *semisymmetric* if it satisfies the condition $R \cdot R = 0$. It is well known that the class of semisymmetric manifolds includes the set of locally symmetric manifolds ($\nabla R = 0$) as a proper subset [2]; here, we suppose that (M, g) is a Riemmanian manifold. If M satisfies the condition

$$\nabla R = 0,$$

then M is called a *locally symmetric manifold*. A semi-Riemannian manifold (M, g) is said to be a *pseudosymmetric manifold* if at every point of M the tensor $R \cdot R$ and $Q(g, R)$ are linearly dependent. This is equivalent to the fact that the equality

$$R \cdot R = L_R Q(g, R) \tag{2.3}$$

holds on $U_R = \{x \in M : Q(g, R) \neq 0\}$ for some function L_R on U_R [10].

On the other hand, (M, g) is said to be a *Ricci-pseudosymmetric manifold* if at every point of M the tensor $R \cdot S$ and $Q(g, S)$ are linearly dependent. This is equivalent to the fact that the equality

$$R \cdot S = L_S Q(g, S), \tag{2.4}$$

holds the set $U_S = \{x \in M : Q(g, S) \neq 0\}$ for some function L_S on U_S, where S is the Ricci tensor [9]. For pseudosymmetry, Ricci-pseudosymmetry, and pseudosymmetry type curvature conditions, see also [3, 11, 10, 9, 12, 14, 16, 17, 15, 13, 26].

We now recall the main notions and formulas for lightlike hypersurfaces. For the geometry of lightlike hypersurfaces, we refer to [18] and [19]. Let (M, g) be a hypersurface of a $(m+2)$-dimensional, with $g = \bar{g}_{|M}$, $m > 0$, semi-Riemannian manifold $(\bar{M}, \bar{g})$ of index $q \geq 1$ and $T_x M$ be the tangent space of M at x. Then

$$T_x^\perp M = \{V_x \in T_x \bar{M} : \bar{g}_x(V_x, W_x) = 0, \forall W_x \in T_x M\},$$

and

$$Rad T_x M = T_x M \cap T_x^\perp M,$$

whose dimensional is the nullity degree of g. Then M is called a *lightlike hypersurface* of $\bar{M}$ if $Rad T_x M \neq \{0\}$, where $RadTM$ is called *radical distribution* [18]. We also recall that the nullity degree of g for a lightlike hypersurface of M is 1. The complementary vector bundle $S(TM)$ of $RadTM$ in TM is called *the screen bundle* of M. It is known that any screen bundle is nondegenerate. Thus, we have

$$TM = RadTM \perp S(TM), \tag{2.5}$$

where $\perp$ denotes the orthogonal direct sum. On the other hand, there exists a unique vector bundle $tr(TM)$ of rank 1 over M, such that for any nonzero section ξ of $TM^\perp$ on a coordinate neighborhood $U \subset M$, there exists a unique section N of $tr(TM)$ on U such that

$$\bar{g}(\xi, N) = 1, \bar{g}(N, N) = \bar{g}(N, X) = 0, \forall X \in \Gamma(S(TM|_U)). \tag{2.6}$$

It follows from (2.6) that $tr(TM)$ is a lightlike vector bundle such that $tr(TM)_x \cap T_x M = \{0\}$ for any $x \in M$. Thus, from (2.5) and (2.6), we have

$$
\begin{aligned}
T\bar{M}|_M &= S(TM) \oplus (TM^\perp \oplus tr(TM)) &\tag{2.7} \\
&= TM \oplus tr(TM). &\tag{2.8}
\end{aligned}
$$

Here, the complementary (nonorthogonal) vector bundle $tr(TM)$ to the tangent bundle TM in $T\bar{M}|_M$ is called *the lightlike transversal bundle* of M with respect to screen distribution $S(TM)$.

Suppose that ∇ and $\bar{\nabla}$ are the Levi-Civita connections of the M lightlike hypersurface and $\bar{M}$ semi-Riemannian manifold, respectively. According to (2.8), we have

$$
\begin{aligned}
\bar{\nabla}_X Y &= \nabla_X Y + h(X, Y) &\tag{2.9} \\
\bar{\nabla}_X N &= -A_N X + \nabla_X^t N, &\tag{2.10}
\end{aligned}
$$

for any $X, Y \in \Gamma(TM)$, $N \in \Gamma(tr(TM))$, where $\nabla_X Y, A_N X \in \Gamma(TM)$ and $h(X, Y), \nabla_X^t N \in \Gamma(tr(TM))$. If we set $B(X, Y) = g(h(X, Y), \xi)$ and $\tau(X) = \bar{g}(\nabla_X^t N, \xi)$, then from (2.9) and (2.10), we have

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N \tag{2.11}$$

$$\bar{\nabla}_X N = -A_N X + \tau(X)N, \tag{2.12}$$

for any $X, Y \in \Gamma(TM)$, $N \in \Gamma(tr(TM))$. A_N and B are called *the shape operator* and *the second fundamental form* of the lightlike hypersurface M, respectively.

Let P be the projection of $\Gamma(TM)$ on $\Gamma(S(TM))$. Then we have

$$\nabla_X PY = \nabla_X^* PY + C(X, PY)\xi \tag{2.13}$$

$$\nabla_X \xi = -A_\xi^* X + \tau(X)\xi, \tag{2.14}$$

for any $X, Y \in \Gamma(TM)$, where $\nabla_X^* PY, A_\xi^* X \in \Gamma(S(TM))$ and C is a 1-form on U defined by

$$C(X, PY) = \bar{g}(\nabla_X PY, N). \tag{2.15}$$

$C, A_\xi^* X$, and ∇^* are called *the local second fundamental form, the local shape operator, and the induced connection* on $S(TM)$, respectively. Then we have the following assertions:

$$g(A_N Y, PW) = C(Y, PW), g(A_N Y, N) = 0, B(X, \xi) = 0, \tag{2.16}$$

$$g(A_\xi^* X, PY) = B(X, PY), g(A_\xi^* X, N) = 0, \tag{2.17}$$

for $X, Y, W \in \Gamma(TM), \xi \in \Gamma(TM^\perp)$, and $N \in \Gamma(tr(TM))$.

Now let M be a lightlike hypersurface of a semi-Euclidean space $R_q^{(n+2)}$. The Gauss equation of M is then given by

$$R(X, Y)Z = B(Y, Z)A_N X - B(X, Z)A_N Y, \tag{2.18}$$

for any $X, Y, Z \in \Gamma(TM)$ and $N \in \Gamma(tr(TM))$, where R is curvature tensor field of M.

Let M be a lightlike hypersurface of semi-Euclidean $(m+2)$-space. Then the Ricci tensor Ric of M is given by

$$Ric(X, Y) = -\Sigma_{i=1}^m \varepsilon_i \{B(X, Y)C(W_i, W_i) - g(A_\xi^* Y, A_N X)\}, \varepsilon_i = \pm 1 \tag{2.19}$$

for any $X, Y \in \Gamma(TM)$, $N \in \Gamma(tr(TM))$ and $\{W_{i=1}^m\}$ is an orthonormal basis of $S(TM)$.

Let M be a lightlike hypersurface of a semi-Euclidean space. We say that M is a semisymmetric if the following condition is satisfied:

$$(R(X, Y) \cdot R)(X_1, X_2, X_3, X_4) = 0 \tag{2.20}$$

for any $X, Y, X_1, X_2, X_3, X_4 \in \Gamma(TM)$ [27]. Additionally, a lightlike hypersurface M is called a *Ricci semisymmetric* lightlike hypersurface if the following condition is satisfied:

$$(R(X, Y) \cdot Ric)(X_1, X_2) = 0 \tag{2.21}$$

for any $X, Y, X_1, X_2 \in \Gamma(TM)$ [27].

3. Pseudosymmetric lightlike hypersurfaces in semi-Euclidean spaces

In this section, we consider pseudosymmetric lightlike hypersurfaces in a semi-Euclidean space. We give a nontrivial example, obtain certain sufficient conditions for lightlike hypersurfaces to be pseudosymmetric, and show that under certain conditions a pseudosymmetric lightlike hypersurface is totally geodesic. We also relate the pseudosymmetry condition of the leaves of integrable screen distribution with the pseudosymmetry condition of lightlike hypersurfaces.

Definition 3.1 *Let M be a lightlike hypersurface of a semi-Euclidean space. We say that M is a pseudosymmetric lightlike hypersurface if the tensors of $R \cdot R$ and $Q(g, R)$ are linearly dependent at $\forall p \in M$. This is equivalent to $R \cdot R = L_R Q(g, R)$ on $U_R = \{p \in M | Q(g, R) \neq 0\}$, where L_R is some function on U_R.*

First of all, we give a nontrivial example of pseudosymmetric lightlike hypersurface in $\mathbf{R}_1^4$.

Example 3.2 *Let M be a hypersurface in $\mathbf{R}_1^4$ given by*

$$x_1 = u_1 \sec u_3 \ , \ x_2 = u_1 \cos(u_2 + u_3) \ , \ x_3 = u_1 \sin(u_2 + u_3) \ , \ x_4 = u_1 \tan u_3,$$

where $\mathbf{R}_1^4$ is semi-Euclidean space of signature $(-, +, +, +)$ with respect to canonical basis

$$\{\partial x_1, \partial x_2, \partial x_3, \partial x_4\}$$

and $u_1 \neq 0$; $u_3, u_2 + u_3 \in (0, \frac{\pi}{2})$. Then TM is spanned by

$$
\begin{aligned}
Z_1 &= \sec u_3 \partial x_1 + \cos(u_2 + u_3)\partial x_2 + \sin(u_2 + u_3)\partial x_3 + \tan u_3 \partial x_4 \\
Z_2 &= -u_1 \sin(u_2 + u_3)\partial x_2 + u_1 \cos(u_2 + u_3)\partial x_3 \\
Z_3 &= u_1 \sec u_3 \tan u_3 \partial x_1 - u_1 \sin(u_2 + u_3)\partial x_2 + u_1 \cos(u_2 + u_3)\partial x_3 + u_1 \sec^2 u_3 \partial x_4.
\end{aligned}
$$

Thus, the induced metric tensor of M is given by

$$
\begin{aligned}
\partial s^2 &= 0\partial u_1^2 + u_1^2(\partial u_2^2 + \partial u_2 \partial u_3 + (1 + \sec^2 u_3)\partial u_3^2) \\
&= u_1^2(\partial u_2^2 + \partial u_2 \partial u_3 + (1 + \sec^2 u_3)\partial u_3^2).
\end{aligned}
$$

Hence, M is a warped product lightlike hypersurface with $RadTM = Span\{Z_1\}$ and $S(TM) = Span\{Z_2, Z_3\}$. Then the lightlike transversal vector bundle of M is spanned by

$$N = -\frac{1}{2}(\sec u_3 \partial x_1 - \cos(u_2 + u_3)\partial x_2 - \sin(u_2 + u_3)\partial x_3 + \tan u_3 \partial x_4).$$

By direct computations, we then get $\eta(Z_2) = 0$, $\eta(Z_3) = 0$ and $\eta([Z_2, Z_3]) = 0$. Thus, $S(TM)$ is integrable. Now, by using the Gauss formula, we obtain

$$B(Z_2, Z_2) = -u_1 \ , \ B(Z_2, Z_3) = -u_1 \ , \ B(Z_3, Z_3) = -u_1 - u_1 \sec^2 u_3.$$

On the other hand, from the Weingarten formula (2.12), we obtain

$$A_N Z_2 = -\frac{1}{2u_1} Z_2 \ , \ A_N Z_3 = -\frac{1}{u_1} Z_2 + \frac{1}{2u_1} Z_3.$$

Then, from the above equations, we get

$$(R \cdot R)(Z_2, Z_3, Z_2, Z_3; Z_2, Z_3) = -\frac{u_1^2 \sec^2 u_3}{2},$$

and

$$Q(g, R)(Z_2, Z_3, Z_2, Z_3; Z_2, Z_3) = u_1^4 \sec^2 u_3.$$

Thus, we have

$$(R \cdot R)(Z_2, Z_3, Z_2, Z_3; Z_2, Z_3) = -\frac{1}{2u_1^2} Q(g, R)(Z_2, Z_3, Z_2, Z_3; Z_2, Z_3).$$

Using $A_N Z_1 = 0$, we derive

$$(R \cdot R)(Z_1, Z_3, Z_2, Z_3; Z_2, Z_3) = 0 \ , \ (R \cdot R)(Z_2, Z_1, Z_2, Z_3; Z_2, Z_3) = 0,$$
$$(R \cdot R)(Z_2, Z_3, Z_1, Z_3; Z_2, Z_3) = 0 \ , \ (R \cdot R)(Z_2, Z_3, Z_2, Z_1; Z_2, Z_3) = 0,$$

where $Z_1 \in \Gamma(RadTM)$. Similarly, we obtain

$$Q(g, R)(Z_1, Z_3, Z_2, Z_3; Z_2, Z_3) = 0 \ , \ Q(g, R)(Z_2, Z_1, Z_2, Z_3; Z_2, Z_3) = 0,$$
$$Q(g, R)(Z_2, Z_3, Z_1, Z_3; Z_2, Z_3) = 0 \ , \ Q(g, R)(Z_2, Z_3, Z_2, Z_1; Z_2, Z_3) = 0,$$

where $Z_1 \in \Gamma(RadTM)$. Thus, M is a totally umbilical pseudosymmetric lightlike hypersurface.

In the sequel, we give 2 sufficient conditions for a lightlike hypersurface to be pseudosymmetric.

Theorem 3.3 *Let M be a nontotally geodesic lightlike hypersurface of a semi-Euclidean space with integrable screen distribution such that $A_N \neq 0, A_\xi^* \neq 0$. If $B(X,Y)A_N^2 Z = g(X,Y)A_N Z$ and $B(X,Y)A_\xi^* A_N Z = g(X,Y)A_\xi^* Z$, then M is a pseudosymmetric lightlike hypersurface such that $L_R = 1$, where $X, Y, Z \in \Gamma(TM)$.*

Proof From the hypothesis, for $X, Y, Z, W, U \in \Gamma(TM)$, we get

$$B(X,Y)A_N^2 Z = g(X,Y)A_N Z \quad \Rightarrow \quad g(B(X,Y)A_N^2 Z, W) = g(g(X,Y)A_N Z, W)$$
$$\Rightarrow \quad B(X,Y)g(A_N^2 Z, W) = g(X,Y)g(A_N Z, W) \tag{3.1}$$

and

$$B(X,Y)A_\xi^* A_N Z = g(X,Y)A_\xi^* Z \quad \Rightarrow \quad g(B(X,Y)A_\xi^* A_N Z, U) = g(g(X,Y)A_\xi^* Z, U)$$
$$\Rightarrow \quad B(X,Y)B(A_N Z, U) = g(X,Y)B(Z, U). \tag{3.2}$$

Taking into account (2.18), (3.1), and (3.2) in (2.1), we have

$$(R \cdot R)(X_1, X_2, X_3, X_4; X, Y)$$

$$= -R(R(X,Y)X_1, X_2, X_3, X_4) - R(X_1, R(X,Y)X_2, X_3, X_4)$$

$$- R(X_1, X_2, R(X,Y)X_3, X_4) - R(X_1, X_2, X_3, R(X,Y)X_4)$$

$$= -B(Y, X_1)[g(A_N^2 X, X_4)B(X_2, X_3) - B(A_N X, X_3)g(A_N X_2, X_4)]$$

$$+ B(X, X_1)[g(A_N^2 Y, X_4)B(X_2, X_3) - B(A_N Y, X_3)g(A_N X_2, X_4)]$$

$$- B(Y, X_2)[B(A_N X, X_3)g(A_N X_1, X_4) - g(A_N^2 X, X_4)B(X_1, X_3)]$$

$$+ B(X, X_2)[B(A_N Y, X_3)g(A_N X_1, X_4) - g(A_N^2 Y, X_4)B(X_1, X_3)]$$

$$- B(Y, X_3)[B(X_2, A_N X)g(A_N X_1, X_4) - B(X_1, A_N X)g(A_N X_2, X_4)]$$

$$+ B(X, X_3)[B(X_2, A_N Y)g(A_N X_1, X_4) - B(X_1, A_N Y)g(A_N X_2, X_4)]$$

$$- B(Y, X_4)[g(A_N X_1, A_N X)B(X_2, X_3) - g(A_N X_2, A_N X)B(X_1, X_3)]$$

$$+ B(X, X_4)[g(A_N X_1, A_N Y)B(X_2, X_3) - g(A_N X_2, A_N Y)B(X_1, X_3)]$$

$$= -g(Y, X_1)[g(A_N X, X_4)B(X_2, X_3) - B(X, X_3)g(A_N X_2, X_4)]$$

$$+ g(X, X_1)[g(A_N Y, X_4)B(X_2, X_3) - B(Y, X_3)g(A_N X_2, X_4)]$$

$$- g(Y, X_2)[B(X, X_3)g(A_N X_1, X_4) - g(A_N X, X_4)B(X_1, X_3)]$$

$$+ g(X, X_2)[B(Y, X_3)g(A_N X_1, X_4) - g(A_N Y, X_4)B(X_1, X_3)]$$

$$- g(Y, X_3)[B(X_2, X)g(A_N X_1, X_4) - B(X_1, X)g(A_N X_2, X_4)]$$

$$+ g(X, X_3)[B(X_2, Y)g(A_N X_1, X_4) - B(X_1, Y)g(A_N X_2, X_4)]$$

$$- g(Y, X_4)[g(A_N X_1, X)B(X_2, X_3) - g(A_N X_2, X)B(X_1, X_3)]$$

$$+ g(X, X_4)[g(A_N X_1, Y)B(X_2, X_3) - g(A_N X_2, Y)B(X_1, X_3)]$$

$$= Q(g, R)(X_1, X_2, X_3, X_4; X, Y),$$

for $X_1, X_2, X_3, X_4, X, Y \in \Gamma(TM)$. This completes the proof. $\quad\square$

As a result of the above theorem, we have the following corollary.

Corollary 3.4 *Let M be a lightlike hypersurface of a semi-Euclidean space such that $A_N \neq 0, A_\xi^* \neq 0$. If $B(X,Y)A_N^2 Z = g(X,Y)A_N Z$ and $B(X,Y)A_\xi^* A_N Z = g(X,Y)A_\xi^* Z$ on M, then $B(A_N Z, U) = g(Z, U)$ on M, where $X, Y, Z, U \in \Gamma(TM)$.*

Proof From (3.1), for $X, Y, Z, W, U \in \Gamma(TM)$, we have

$$B(X, Y) = \frac{g(X, Y)}{g(A_N^2 Z, W)} g(A_N Z, W). \tag{3.3}$$

On the other hand, from (3.2), we write

$$B(X,Y) = \frac{g(X,Y)}{B(A_N Z, U)} B(Z,U). \tag{3.4}$$

Thus, using (3.3) and (3.4), we have

$$g(X,Y)g(A_N Z, W)B(A_N Z, U) = g(X,Y)g(A_N^2 Z, W)B(Z,U),$$

which is equivalent to

$$g(X,Y)[g(A_N Z, W)B(A_N Z, U) - g(A_N^2 Z, W)B(Z,U)] = 0.$$

This implies that

$$g(A_N Z, W)B(A_N Z, U) - g(B(Z,U)A_N^2 Z, W) = 0. \tag{3.5}$$

However, from assumption, we have

$$B(X,Y)A_N^2 Z = g(X,Y)A_N Z.$$

Thus, (3.5) can be written as

$$g(A_N Z, W)[B(A_N Z, U) - g(Z,U)] \quad = \quad 0,$$

which completes the proof. □

The next theorem shows that the pseudosymmetry condition of a lightlike hypersurface is related to the pseudosymmetry of its integrable screen distribution.

Theorem 3.5 *Let M be a lightlike hypersurfaces of semi-Euclidean space such that $B(X,Y)A_\xi^* A_N Z = g(X,Y)A_\xi^* Z$ and $S(TM)$ is integrable. M is then pseudosymmetric if and only if the integral manifold of screen distribution is pseudosymmetric.*

Proof Using (2.18) and (3.2), we obtain

$$g(R(X,Y)PZ, PW) = B(A_N Y, PZ)B(A_N X, PW) - B(A_N X, PZ)B(A_N Y, PW), \tag{3.6}$$

for any $X,Y,Z,W \in \Gamma(TM)$. Then, by straightforward computations, we have

$$\begin{aligned}
g(R(X,Y)PZ, PW) &= g(R^*(X,Y)PZ, PW) - B(A_N Y, PZ)B(A_N X, PW) \\
&+ B(A_N X, PZ)B(A_N Y, PW),
\end{aligned} \tag{3.7}$$

for any $X,Y,Z,W \in \Gamma(TM)$. From (3.6) and (3.7), we get

$$g(R(X,Y)PZ, PW) = \frac{1}{2}g(R^*(X,Y)PZ, PW). \tag{3.8}$$

On the other hand, from (2.16) and (2.18), we have

$$g(R(X,Y)PZ, N) = 0, \tag{3.9}$$

$\forall X, Y, Z \in \Gamma(TM), N \in \Gamma(tr(TM))$. Hence, (3.8) and (3.9) imply that

$$R(X,Y)PZ = \frac{1}{2} R^*(X,Y)PZ. \tag{3.10}$$

Thus, using algebraic properties of the curvature tensor field, we get

$$(R(X,Y) \cdot R)(U,V,W,Z) = \frac{1}{4}(R^*(X,Y) \cdot R^*)(U,V,W,Z), \tag{3.11}$$

for any $X, Y, U, V, W \in \Gamma(S(TM))$. On the other hand, from (3.10), we have

$$Q(g,R)(U,V,W,Z;X,Y) = \frac{1}{2}Q(g,R^*)(U,V,W,Z;X,Y), \tag{3.12}$$

for any $X, Y, U, V, W \in \Gamma(S(TM))$. Thus, if M is pseudosymmetric, from (3.11) and (3.12), $S(TM)$ is pseudosymmetric. The converse is clear from (3.11) and (3.12). $\qquad\square$

When M and $S(TM)$ are totally umbilical, we obtain the following theorem.

Theorem 3.6 *Let M be a pseudosymmetric lightlike hypersurface of a semi-Euclidean space. If M and $S(TM)$ are totally umbilical, then M is a semisymmetric lightlike hypersurface.*

Proof Let M be a pseudosymmetric lightlike hypersurface of a semi-Euclidean space, i.e. $R \cdot R = L_R Q(g, R)$. Using (2.18) in (2.1), for $X_2, X_3, X_4, X, Y \in \Gamma(TM)$ and $X_1 = \xi \in \Gamma(RadTM)$, we have

$$
\begin{aligned}
&(R \cdot R)(\xi, X_2, X_3, X_4; X, Y) \\
&= -B(Y,\xi)[g(A_N^2 X, X_4)B(X_2, X_3) - B(A_N X, X_3)g(A_N X_2, X_4)] \\
&\quad + B(X,\xi)[g(A_N^2 Y, X_4)B(X_2, X_3) - B(A_N Y, X_3)g(A_N X_2, X_4)] \\
&\quad - B(Y, X_2)[B(A_N X, X_3)g(A_N \xi, X_4) - g(A_N^2 X, X_4)B(\xi, X_3)] \\
&\quad + B(X, X_2)[B(A_N Y, X_3)g(A_N \xi, X_4) - g(A_N^2 Y, X_4)B(\xi, X_3)] \\
&\quad - B(Y, X_3)[B(X_2, A_N X)g(A_N \xi, X_4) - B(\xi, A_N X)g(A_N X_2, X_4)] \\
&\quad + B(X, X_3)[B(X_2, A_N Y)g(A_N \xi, X_4) - B(\xi, A_N Y)g(A_N X_2, X_4)] \\
&\quad - B(Y, X_4)[g(A_N \xi, A_N X)B(X_2, X_3) - g(A_N X_2, A_N X)B(\xi, X_3)] \\
&\quad + B(X, X_4)[g(A_N \xi, A_N Y)B(X_2, X_3) - g(A_N X_2, A_N Y)B(\xi, X_3)] \\
&= -B(Y, X_2)B(A_N X, X_3)g(A_N \xi, X_4) + B(X, X_2)B(A_N Y, X_3)g(A_N \xi, X_4) \\
&\quad - B(Y, X_3)B(X_2, A_N X)g(A_N \xi, X_4) + B(X, X_3)B(X_2, A_N Y)g(A_N \xi, X_4) \\
&\quad - B(Y, X_4)g(A_N \xi, A_N X)B(X_2, X_3) + B(X, X_4)g(A_N \xi, A_N Y)B(X_2, X_3).
\end{aligned}
$$

Since M and $S(TM)$ are totally umbilical, $B(X,Y) = \rho g(X,Y)$ and $C(X,Y) = \lambda g(X,Y)$, for $X,Y \in \Gamma(TM)$, where ρ and λ are smooth functions. The above equation then becomes

$$(R \cdot R)(\xi, X_2, X_3, X_4; X, Y)$$

$$= -\rho g(Y, X_2)\rho g(A_N X, X_3)C(\xi, X_4) + \rho g(X, X_2)\rho g(A_N Y, X_3)C(\xi, X_4)$$

$$- \rho g(Y, X_3)\rho g(X_2, A_N X)C(\xi, X_4) + \rho g(X, X_3)\rho g(X_2, A_N Y)C(\xi, X_4)$$

$$- \rho g(Y, X_4)C(\xi, A_N X)\rho g(X_2, X_3) + \rho g(X, X_4)C(\xi, A_N Y)\rho g(X_2, X_3)$$

$$= -\rho g(Y, X_2)\rho g(A_N X, X_3)\lambda g(\xi, X_4) + \rho g(X, X_2)\rho g(A_N Y, X_3)\lambda g(\xi, X_4)$$

$$- \rho g(Y, X_3)\rho g(X_2, A_N X)\lambda g(\xi, X_4) + \rho g(X, X_3)\rho g(X_2, A_N Y)\lambda g(\xi, X_4)$$

$$- \rho g(Y, X_4)\lambda g(\xi, A_N X)\rho g(X_2, X_3) + \rho g(X, X_4)\lambda g(\xi, A_N Y)\rho g(X_2, X_3)$$

$$= \rho^2 \lambda Q(g, R)(\xi, X_2, X_3, X_4; X, Y)$$

$$= 0.$$

In a similar way, we obtain

$$
\begin{aligned}
(R \cdot R)(X_1, \xi, X_3, X_4; X, Y) &= 0, (R \cdot R)(X_1, X_2, \xi, X_4; X, Y) = 0, \\
(R \cdot R)(X_1, X_2, X_3, \xi; X, Y) &= 0, (R \cdot R)(X_1, X_2, X_3, X_4; \xi, Y) = 0, \\
(R \cdot R)(X_1, X_2, X_3, X_4; X, \xi) &= 0,
\end{aligned}
$$

and

$$
\begin{aligned}
Q(g, R)(X_1, \xi, X_3, X_4; X, Y) &= 0, Q(g, R)(X_1, X_2, \xi, X_4; X, Y) = 0, \\
Q(g, R)(X_1, X_2, X_3, \xi; X, Y) &= 0, Q(g, R)(X_1, X_2, X_3, X_4; \xi, Y) = 0, \\
Q(g, R)(X_1, X_2, X_3, X_4; X, \xi) &= 0.
\end{aligned}
$$

On the other hand, for $X_1 = PX_1$, we have

$$(R \cdot R)(PX_1, X_2, X_3, X_4; X, Y)$$

$$= -B(Y, PX_1)[g(A_N^2 X, X_4)B(X_2, X_3) - B(A_N X, X_3)g(A_N X_2, X_4)]$$

$$+ B(X, PX_1)[g(A_N^2 Y, X_4)B(X_2, X_3) - B(A_N Y, X_3)g(A_N X_2, X_4)]$$

$$- B(Y, X_2)[B(A_N X, X_3)g(A_N PX_1, X_4) - g(A_N^2 X, X_4)B(PX_1, X_3)]$$

$$+ B(X, X_2)[B(A_N Y, X_3)g(A_N PX_1, X_4) - g(A_N^2 Y, X_4)B(PX_1, X_3)]$$

$$- B(Y, X_3)[B(X_2, A_N X)g(A_N PX_1, X_4) - B(PX_1, A_N X)g(A_N X_2, X_4)]$$

$$+ B(X, X_3)[B(X_2, A_N Y)g(A_N PX_1, X_4) - B(PX_1, A_N Y)g(A_N X_2, X_4)]$$

$$- B(Y, X_4)[g(A_N PX_1, A_N X)B(X_2, X_3) - g(A_N X_2, A_N X)B(PX_1, X_3)]$$

$$+ B(X, X_4)[g(A_N PX_1, A_N Y)B(X_2, X_3) - g(A_N X_2, A_N Y)B(PX_1, X_3)],$$

where P is the projection morphism of $\Gamma(TM)$ on $\Gamma S(TM)$. Totally umbilical M and $S(TM)$ imply that

$$(R \cdot R)(PX_1, X_2, X_3, X_4; X, Y)$$

$$= -\rho g(Y, PX_1)[C(A_N X, X_4)\rho g(X_2, X_3) - \rho C(X, X_3)g(A_N X_2, X_4)]$$

$$+ \rho g(X, PX_1)[C(A_N Y, X_4)\rho g(X_2, X_3) - \rho C(Y, X_3)g(A_N X_2, X_4)]$$

$$- \rho g(Y, X_2)[\rho C(X, X_3)g(A_N PX_1, X_4) - C(A_N X, X_4)\rho g(PX_1, X_3)]$$

$$+ \rho g(X, X_2)[\rho C(Y, X_3)g(A_N PX_1, X_4) - C(A_N Y, X_4)\rho g(PX_1, X_3)]$$

$$- \rho g(Y, X_3)[\rho C(X_2, X)g(A_N PX_1, X_4) - \rho C(PX_1, X)g(A_N X_2, X_4)]$$

$$+ \rho g(X, X_3)[\rho C(X_2, Y)g(A_N PX_1, X_4) - \rho C(PX_1, Y)g(A_N X_2, X_4)]$$

$$- \rho g(Y, X_4)[C(A_N PX_1, X)\rho g(X_2, X_3) - C(A_N X_2, X)\rho g(PX_1, X_3)]$$

$$+ \rho g(X, X_4)[C(A_N PX_1, Y)\rho g(X_2, X_3) - C(A_N X_2, Y)\rho g(PX_1, X_3)]$$

$$= -\rho g(Y, PX_1)[\lambda^2 g(X, X_4)\rho g(X_2, X_3) - \rho \lambda g(X, X_3)\lambda g(X_2, X_4)]$$

$$+ \rho g(X, PX_1)[\lambda^2 g(Y, X_4)\rho g(X_2, X_3) - \rho \lambda g(Y, X_3)\lambda g(X_2, X_4)]$$

$$- \rho g(Y, X_2)[\rho \lambda g(X, X_3)\lambda g(PX_1, X_4) - \lambda^2 g(X, X_4)\rho g(PX_1, X_3)]$$

$$+ \rho g(X, X_2)[\rho \lambda g(Y, X_3)\lambda g(PX_1, X_4) - \rho \lambda^2 g(Y, X_4)\rho g(PX_1, X_3)]$$

$$- \rho g(Y, X_3)[\rho \lambda g(X_2, X)\lambda g(PX_1, X_4) - \rho \lambda g(PX_1, X)\lambda g(X_2, X_4)]$$

$$+ \rho g(X, X_3)[\rho \lambda g(X_2, Y)\lambda g(PX_1, X_4) - \rho \lambda g(PX_1, Y)\lambda g(X_2, X_4)]$$

$$- \rho g(Y, X_4)[\lambda^2 g(PX_1, X)\rho g(X_2, X_3) - \lambda^2 g(X_2, X)\rho g(PX_1, X_3)]$$

$$+ \rho g(X, X_4)[\lambda^2 g(PX_1, Y)\rho g(X_2, X_3) - \lambda^2 g(X_2, Y)\rho g(PX_1, X_3)]$$

$$= \rho^2 \lambda^2 \{-g(Y, PX_1)[g(X, X_4)g(X_2, X_3) - g(X, X_3)g(X_2, X_4)]$$

$$+ g(X, PX_1)[g(Y, X_4)g(X_2, X_3) - g(Y, X_3)g(X_2, X_4)]$$

$$- g(Y, X_2)[g(X, X_3)g(PX_1, X_4) - g(X, X_4)g(PX_1, X_3)]$$

$$+ g(X, X_2)[g(Y, X_3)g(PX_1, X_4) - g(Y, X_4)g(PX_1, X_3)]$$

$$- g(Y, X_3)[g(X_2, X)g(PX_1, X_4) - g(PX_1, X)g(X_2, X_4)]$$

$$+ g(X, X_3)[g(X_2, Y)g(PX_1, X_4) - g(PX_1, Y)g(X_2, X_4)]$$

$$- g(Y, X_4)[g(PX_1, X)g(X_2, X_3) - g(X_2, X)g(PX_1, X_3)]$$

$$+ g(X, X_4)[g(PX_1, Y)g(X_2, X_3) - g(X_2, Y)g(PX_1, X_3)]\}$$

$$= \rho^2 \lambda^2 Q(g, R)(PX_1, X_2, X_3, X_4; X, Y)$$

$$= 0.$$

In a similar way, we obtain

$$(R \cdot R)(X_1, PX_2, X_3, X_4; X, Y) = 0, (R \cdot R)(X_1, X_2, PX_3, X_4; X, Y) = 0,$$
$$(R \cdot R)(X_1, X_2, X_3, PX_4; X, Y) = 0, (R \cdot R)(X_1, X_2, X_3, X_4; PX, Y) = 0,$$
$$(R \cdot R)(X_1, X_2, X_3, X_4; X, PY) = 0,$$

and

$$Q(g, R)(X_1, PX_2, X_3, X_4; X, Y) = 0, Q(g, R)(X_1, X_2, PX_3, X_4; X, Y) = 0,$$
$$Q(g, R)(X_1, X_2, X_3, PX_4; X, Y) = 0, Q(g, R)(X_1, X_2, X_3, X_4; PX, Y) = 0,$$
$$Q(g, R)(X_1, X_2, X_3, X_4; X, PY) = 0,$$

which completes the proof. □

Theorem 3.7 *Let M be a pseudosymmetric lightlike hypersurface of a semi-Euclidean space. Then, either M is totally geodesic or $A_\xi^* X_2$ and $A_N \xi$ are linearly dependent such that $A_N \xi$ is a nonnull vector field.*

Proof Suppose that M is a pseudosymmetric lightlike hypersurface of a semi-Euclidean space. Taking $X_1 = \xi$ and using (2.18) in (2.1) and (2.2), we have

$$(R \cdot R)(\xi, X_2, X_3, X_4; X, Y)$$
$$= -B(Y, X_2)B(A_N X, X_3)g(A_N \xi, X_4) + B(X, X_2)B(A_N Y, X_3)g(A_N \xi, X_4)$$
$$- B(Y, X_3)B(X_2, A_N X)g(A_N \xi, X_4) + B(X, X_3)B(X_2, A_N Y)g(A_N \xi, X_4)$$
$$- B(Y, X_4)g(A_N \xi, A_N X)B(X_2, X_3) + B(X, X_4)g(A_N \xi, A_N Y)B(X_2, X_3)$$

and

$$Q(g, R)(\xi, X_2, X_3, X_4; X, Y)$$
$$= -g(Y, X_2)B(X, X_3)g(A_N \xi, X_4) + g(X, X_2)B(Y, X_3)g(A_N \xi, X_4)$$
$$- g(Y, X_3)B(X_2, X)g(A_N \xi, X_4) + g(X, X_3)B(X_2, Y)g(A_N \xi, X_4)$$
$$- g(Y, X_4)g(A_N \xi, X)B(X_2, X_3) + g(X, X_4)g(A_N \xi, Y)B(X_2, X_3).$$

Then, for $Y = \xi$, we have $Q(g, R)(\xi, X_2, X_3, X_4; X, \xi) = 0$ and

$$(R \cdot R)(\xi, X_2, X_3, X_4; X, \xi)$$
$$= B(X, X_2)B(A_N \xi, X_3)g(A_N \xi, X_4) + B(X, X_3)B(X_2, A_N \xi)g(A_N \xi, X_4)$$
$$+ B(X, X_4)g(A_N \xi, A_N \xi)B(X_2, X_3)$$
$$= 0 \tag{3.13}$$

due to M being a pseudosymmetric lightlike hypersurface. If we get $X_2 = X_3 = X_4$, (3.13) is equivalent to

$$(R \cdot R)(\xi, X_2, X_3, X_4; X, \xi)$$
$$= B(X, X_2)B(A_N \xi, X_2)g(A_N \xi, X_2) + B(X, X_2)B(X_2, A_N \xi)g(A_N \xi, X_2)$$
$$+ B(X, X_2)g(A_N \xi, A_N \xi)B(X_2, X_2)$$
$$= 0.$$

Hence, we obtain

$$B(X, X_2)[2B(A_N\xi, X_2)g(A_N\xi, X_2) + g(A_N\xi, A_N\xi)B(X_2, X_2)] = 0,$$

and using (2.17), we have

$$B(X, X_2)[2g(A_N\xi, A_\xi^* X_2)g(A_N\xi, X_2) + g(A_N\xi, A_N\xi)g(A_\xi^* X_2, X_2)] = 0.$$

Thus, the proof is complete. □

Let $(\bar{M}, \bar{g})$ be an $(m+2)$-dimensional semi-Riemannian manifold. $\bar{M}$ is Einstein if $\bar{S} = \bar{k}\bar{g}$, and $\bar{k}$ is a constant, where S is a Ricci tensor. Moreover, $\bar{M}$ is Einstein if and only if $\bar{k} = \bar{r}/(m+2)$, where $\bar{r}$ is the scalar curvature of $\bar{M}$. Obviously, a geometric concept of a lightlike Einstein hypersurface $(M, g, S(TM))$ must involve its scalar curvature. Therefore, for a well-defined concept of a lightlike Einstein hypersurface M one should assure that M admits a symmetric Ricci tensor from which an induced scalar curvature can be calculated [19].

For a lightlike Einstein hypersurface, we give the following corollary.

Corollary 3.8 *Let M be a lightlike Einstein hypersurface of a semi-Euclidean space. If $R \cdot R = Q(S, R)$, then M is a pseudosymmetric lightlike hypersurface, where S is the Ricci tensor of M.*

Proof Suppose that M is a lightlike Einstein hypersurface of a semi-Euclidean space. We then obtain

$$\begin{aligned}
&Q(S, R)(X_1, X_2, X_3, X_4; X, Y) \\
&= -R((X \wedge_S Y)X_1, X_2, X_3, X_4) - R(X_1, (X \wedge_S Y)X_2, X_3, X_4) \\
&\quad - R(X_1, X_2, (X \wedge_S Y)X_3, X_4) - R(X_1, X_2, X_3, (X \wedge_S Y)X_4) \\
&= -S(Y, X_1)R(X, X_2, X_3, X_4) + S(X, X_1)R(Y, X_2, X_3, X_4) \\
&\quad - S(Y, X_2)R(X_1, X, X_3, X_4) + S(X, X_2)R(X_1, Y, X_3, X_4) \\
&\quad - S(Y, X_3)R(X_1, X_2, X, X_4) + S(X, X_3)R(X_1, X_2, Y, X_4) \\
&\quad - S(Y, X_4)R(X_1, X_2, X_3, X) + S(X, X_4)R(X_1, X_2, X_3, Y),
\end{aligned} \tag{3.14}$$

for $X_1, X_2, X_3, X_4, X, Y \in \Gamma(TM)$.

From the hypothesis and (3.14), we have

$$\begin{aligned}
&(R \cdot R)(X_1, X_2, X_3, X_4; X, Y) \\
&= Q(S, R)(X_1, X_2, X_3, X_4; X, Y) \\
&= -\lambda g(Y, X_1)R(X, X_2, X_3, X_4) + \lambda g(X, X_1)R(Y, X_2, X_3, X_4) \\
&\quad - \lambda g(Y, X_2)R(X_1, X, X_3, X_4) + \lambda g(X, X_2)R(X_1, Y, X_3, X_4) \\
\\
&\quad - \lambda g(Y, X_3)R(X_1, X_2, X, X_4) + \lambda g(X, X_3)R(X_1, X_2, Y, X_4) \\
&\quad - \lambda g(Y, X_4)R(X_1, X_2, X_3, X) + \lambda g(X, X_4)R(X_1, X_2, X_3, Y) \\
&= \lambda Q(g, R)(X_1, X_2, X_3, X_4; X, Y).
\end{aligned}$$

Thus, the proof is complete. □

4. Pseudoparallel lightlike hypersurfaces in semi-Euclidean spaces

In this section, we consider pseudoparallel lightlike hypersurfaces in a semi-Euclidean space by adapting the well-known pseudoparallel notion; see [4].

Definition 4.1 *Let M be a lightlike hypersurface of a semi-Euclidean space. We say that M is a pseudoparallel lightlike hypersurface if the tensors of $R \cdot h$ and $Q(g, h)$ are linearly dependent at $\forall p \in M$. This is equivalent to $R \cdot h = L_h Q(g, h)$ on $U_h = \{p \in M | Q(g, h) \neq 0\}$, where L_h is some function on U_h and h is the second fundamental form of M.*

Theorem 4.2 *Let M be a lightlike hypersurface of a semi-Euclidean space and A_N be symmetric with respect to B. If τ-parallel and $B(X,Y)A_\xi^* A_N Z = g(X,Y)A_\xi^* Z$, then M is a pseudoparallel lightlike hypersurface such that $L_h = 1$, where $X, Y, Z \in \Gamma(TM)$ and τ is 1-form on M.*

Proof For $X_1, X_2, X, Y \in \Gamma(TM)$, we have

$$
\begin{aligned}
&(R(X,Y) \cdot h)(X_1, X_2) \\
&= R^\perp(X,Y)h(X_1, X_2) - h(R(X,Y)X_1, X_2) - h(X_1, R(X,Y)X_2)
\end{aligned}
$$

(4.1)

$$
\begin{aligned}
&= [(\nabla_X \tau)Y - (\nabla_Y \tau)X]h(X_1, X_2) - h(X, A_{h(X_1, X_2)}Y) \\
&+ h(Y, A_{h(X_1, X_2)}X) - B(Y, X_1)B(A_N X, X_2)N \\
&+ B(X, X_1)B(A_N Y, X_2)N - B(Y, X_2)B(X_1, A_N X)N \\
&+ B(X, X_2)B(X_1, A_N Y)N \\
&= [(\nabla_X \tau)Y - (\nabla_Y \tau)X - B(X, A_N Y) + B(Y, A_N X)]B(X_1, X_2)N \\
&- B(Y, X_1)B(A_N X, X_2)N + B(X, X_1)B(A_N Y, X_2)N \\
&- B(Y, X_2)B(X_1, A_N X)N + B(X, X_2)B(X_1, A_N Y)N.
\end{aligned}
$$

(4.2)

From the hypothesis and (3.2), we obtain

$$
\begin{aligned}
(R(X,Y) \cdot h)(X_1, X_2) &= -B(Y, X_1)B(A_N X, X_2)N + B(X, X_1)B(A_N Y, X_2)N \\
&- B(Y, X_2)B(X_1, A_N X)N + B(X, X_2)B(X_1, A_N Y)N \\
&= -g(Y, X_1)B(X, X_2)N + g(X, X_1)B(Y, X_2)N \\
&- g(Y, X_2)B(X_1, X)N + g(X, X_2)B(X_1, Y)N \\
&= Q(g, h)(X_1, X_2; X, Y),
\end{aligned}
$$

which shows the assertion. $\square$

Corollary 4.3 *Let M be a lightlike hypersurface of a semi-Euclidean space such that A_N is symmetric with respect to B. If τ-parallel and $B(X,Y)A_\xi^* A_N Z = g(X,Y)A_\xi^* Z$, then M is a Ricci symmetric pseudoparallel lightlike hypersurface.*

Proof Since A_N is symmetric with respect to B, we have $B(A_N X, Y) = B(X, A_N Y)$, for any $X, Y \in \Gamma(TM)$. Thus, we get $Ric(X, Y) = Ric(Y, X)$. Hence, the proof is obvious from Theorem 4.2. □

Definition 4.4 *Let $M \subset E_s^{n+1}$ be a hypersurface such that $g(A_\xi X, Y) = h(X, Y)$ for the second fundamental form h of M. If the following condition is satisfied:*

$$(R(X, Y) \cdot h)(U, V) = 0, \quad \forall X, Y, U, V \in \Gamma(TM)$$

then M is called a semiparallel hypersurface [8].

Theorem 4.5 *Let M be a pseudoparallel lightlike hypersurface of a semi-Euclidean space and A_N be symmetric with respect to B. If M and $S(TM)$ are totally umbilical and τ-parallel, then M is a semiparallel lightlike hypersurface.*

Proof Suppose that M and $S(TM)$ are totally umbilical. Then, from (4.2) and the hypothesis, we obtain

$$
\begin{aligned}
(R(X, Y) \cdot h)(X_1, X_2) &= -B(Y, X_1)B(A_N X, X_2)N + B(X, X_1)B(A_N Y, X_2)N \\
&\quad - B(Y, X_2)B(X_1, A_N X)N + B(X, X_2)B(X_1, A_N Y)N \\
&= -\rho g(Y, X_1)\rho \lambda g(X, X_2)N + \rho g(X, X_1)\rho \lambda g(Y, X_2)N \\
&\quad - \rho g(Y, X_2)\rho \lambda g(X_1, X)N + \rho g(X, X_2)\rho \lambda(X_1, Y)N \\
&= \rho^2 \lambda Q(g, h)(X_1, X_2; X, Y) \\
&= 0,
\end{aligned}
$$

for $X_1, X_2, X, Y \in \Gamma(TM)$. This completes the proof. □

Corollary 4.6 *Let M be a pseudoparallel lightlike hypersurface of a semi-Euclidean space. If M and $S(TM)$ are totally umbilical and $R^\perp$ vanishes, then M is a semiparallel lightlike hypersurface.*

Proof The proof follows from (4.2). □

Theorem 4.7 *Let M be a pseudoparallel lightlike hypersurface of a semi-Euclidean space such that A_N is symmetric with respect to B. If τ-parallel, then either M is totally geodesic or $B(A_N \xi, X_1) = 0$, $X_1 \in \Gamma(TM)$, $\xi \in \Gamma(RadTM)$.*

Proof Suppose that M is a pseudoparallel lightlike hypersurface, i.e.

$$(R(X, Y) \cdot h)(X_1, X_2) = \lambda Q(g, h)(X_1, X_2; X, Y).$$

Then, from the hypothesis, we have

$$
\begin{aligned}
&- B(Y, X_1)B(A_N X, X_2)N + B(X, X_1)B(A_N Y, X_2)N \\
&- B(Y, X_2)B(X_1, A_N X)N + B(X, X_2)B(X_1, A_N Y)N \\
&+ \lambda g(Y, X_1)B(X, X_2)N - \lambda g(X, X_1)B(Y, X_2)N \\
&+ \lambda g(Y, X_2)B(X_1, X)N - \lambda g(X, X_2)B(X_1, Y)N \\
&= 0, \quad (4.3)
\end{aligned}
$$

for $X_1, X_2, X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(RadTM)$. Thus, for $Y = \xi$, we have

$$B(X, X_1)B(A_N\xi, X_2) + B(X, X_2)B(X_1, A_N\xi) = 0.$$

Here, for $X_1 = X_2$, we arrive at $B(X, X_1)B(A_N\xi, X_1) = 0$. Thus, the proof is complete. □

Corollary 4.8 *Let M be a pseudoparallel lightlike hypersurface of a semi-Euclidean space. If $R^\perp$ vanishes, then either M is totally geodesic or $B(A_N\xi, X_1) = 0$, $X_1 \in \Gamma(TM)$, $\xi \in \Gamma(RadTM)$.*

Proof The proof follows from Theorem 4.7. □

5. Ricci-pseudosymmetric lightlike hypersurfaces in semi-Euclidean spaces

In this section, we introduce Ricci-pseudosymmetric lightlike hypersurfaces, obtain sufficient conditions for a lightlike hypersurface to be Ricci-semisymmetric, and investigate relations between such hypersurfaces and Ricci-semisymmetric and totally geodesic hypersurfaces. We also obtain Ricci-generalized pseudoparallel lightlike hypersurfaces by imposing certain conditions on 1-form τ.

Definition 5.1 *Let M be a lightlike hypersurface of a semi-Euclidean space. We say that M is a Ricci-pseudosymmetric lightlike hypersurface if the tensors of $R \cdot S$ and $Q(g, S)$ are linearly dependent at $\forall p \in M$. This is equivalent to $R \cdot S = L_S Q(g, S)$ on $U_S = \{p \in M | Q(g, S) \neq 0\}$, where L_S is some function on U_S and S is a Ricci tensor.*

Example 5.2 *Let M be a hypersurface in $\mathbf{R}_2^4$ given by*

$$x_1 = u_1 \sec u_3 \ , \ x_2 = u_1 \tan u_2 \ , \ x_3 = u_1 \sec u_2 \ , \ x_4 = u_1 \tan u_3,$$

where $\mathbf{R}_2^4$ is semi-Euclidean space of signature $(-, -, +, +)$ with respect to canonical basis

$$\{\partial x_1, \partial x_2, \partial x_3, \partial x_4\}$$

and $u_1 \neq 0$; $u_2, u_3 \in (0, \frac{\pi}{2})$. TM is then spanned by

$$
\begin{aligned}
Z_1 &= \sec u_3 \partial x_1 + \tan u_2 \partial x_2 + \sec u_2 \partial x_3 + \tan u_3 \partial x_4 \\
Z_2 &= u_1 \sec^2 u_2 \partial x_2 + u_1 \sec u_2 \tan u_2 \partial x_3 \\
Z_3 &= u_1 \sec u_3 \tan u_3 \partial x_1 + u_1 \sec^2 u_3 \partial x_4.
\end{aligned}
$$

Hence, the induced metric tensor of M is given by

$$
\begin{aligned}
\partial s^2 &= 0\partial u_1^2 + u_1^2(-\sec^2 u_2 \partial u_2^2 + \sec^2 u_3 \partial u_3^2) \\
&= u_1^2(-\sec^2 u_2 \partial u_2^2 + \sec^2 u_3 \partial u_3^2).
\end{aligned}
$$

Thus, M is a warped product lightlike hypersurface with $RadTM = Span\{Z_1\}$ and $S(TM) = Span\{Z_2, Z_3\}$. The lightlike transversal vector bundle of M is then spanned by

$$N = \frac{1}{2}(-\sec u_3 \partial x_1 + \tan u_2 \partial x_2 + \sec u_2 \partial x_3 - \tan u_3 \partial x_4).$$

Here, for $\bar{\nabla}_{Z_2} Z_3 = 0$ *and similarly* $\bar{\nabla}_{Z_3} Z_2 = 0$, *we have* $[Z_2, Z_3] = 0$. *Then, by direct computations, we get* $\eta(Z_2) = 0$, $\eta(Z_3) = 0$ *and* $\eta([Z_2, Z_3]) = 0$. *Thus,* $S(TM)$ *is integrable. Now, by using the Gauss formula, we obtain*

$$B(Z_2, Z_2) = u_1 \sec^2 u_2 \ , \ B(Z_2, Z_3) = 0 \ , \ B(Z_3, Z_3) = -u_1 \sec^2 u_3.$$

On the other hand, from the Weingarten formula (2.12), we obtain

$$A_N Z_2 = -\frac{1}{2u_1} Z_2 \ , \ A_N Z_3 = \frac{1}{2u_1} Z_3.$$

Then, from the above equations, we show that

$$(R \cdot S)(Z_2, Z_3; Z_2, Z_3) = \alpha u_1 \sec^2 u_2 \sec^2 u_3,$$

where $\alpha = \Sigma_{i=1}^n \varepsilon_i C(W_i, W_i)$, $\{W_i\}_{i=1}^n$ *is a basis of* $S(TM)$ *and*

$$Q(g, S)(Z_2, Z_3; Z_2, Z_3) = u_1^2 \sec^2 u_2 \sec^2 u_3.$$

Thus, we have

$$(R \cdot S)(Z_2, Z_3; Z_2, Z_3) = \frac{\alpha}{u_1} Q(g, S)(Z_2, Z_3; Z_2, Z_3).$$

Using $A_N Z_1 = 0$, *we obtain*

$$\begin{aligned}
(R \cdot S)(Z_1, Z_3; Z_2, Z_3) &= 0 \ , \ (R \cdot S)(Z_2, Z_1; Z_2, Z_3) = 0, \\
(R \cdot S)(Z_2, Z_3; Z_1, Z_3) &= 0 \ , \ (R \cdot S)(Z_2, Z_3; Z_2, Z_1) = 0,
\end{aligned}$$

where $Z_1 \in \Gamma(RadTM)$. *Similarly, we obtain*

$$\begin{aligned}
Q(g, S)(Z_1, Z_3; Z_2, Z_3) &= 0 \ , \ Q(g, S)(Z_2, Z_1; Z_2, Z_3) = 0, \\
Q(g, S)(Z_2, Z_3; Z_1, Z_3) &= 0 \ , \ Q(g, S)(Z_2, Z_3; Z_2, Z_1) = 0,
\end{aligned}$$

where $Z_1 \in \Gamma(RadTM)$. *Then we can say that* M *is a totally umbilical Ricci-pseudosymmetric lightlike hypersurface.*

Theorem 5.3 *Let* M *be a lightlike hypersurface of a semi-Euclidean space such that* A_N *is symmetric with respect to* B. *If* $B(X, Y) A_\xi^* A_N Z = g(X, Y) A_\xi^* Z$, *then* M *is a Ricci-pseudosymmetric lightlike hypersurface such that* $L_S = 1$, *where* $X, Y, Z \in \Gamma(TM)$.

Proof For $X_1, X_2, X, Y \in \Gamma(TM)$, we have

$$(R \cdot S)(X_1, X_2; X, Y)$$

$$= -S(R(X,Y)X_1, X_2) - S(X_1, R(X,Y)X_2)$$

$$= -B(Y, X_1)S(A_N X, X_2) + B(X, X_1)S(A_N Y, X_2)$$

$$- B(Y, X_2)S(X_1, A_N X) + B(X, X_2)S(X_1, A_N Y)$$

$$= -B(Y, X_1)\{-\Sigma_{i=1}^{n}\varepsilon_i[B(A_N X, X_2)C(W_i, W_i) - B(X_2, A_N^2 X)]\}$$

$$+ B(X, X_1)\{-\Sigma_{i=1}^{n}\varepsilon_i[B(A_N Y, X_2)C(W_i, W_i) - B(X_2, A_N^2 Y)]\}$$

$$- B(Y, X_2)\{-\Sigma_{i=1}^{n}\varepsilon_i[B(X_1, A_N X)C(W_i, W_i) - B(A_N X, A_N X_1)]\}$$

$$+ B(X, X_2)\{-\Sigma_{i=1}^{n}\varepsilon_i[B(X_1, A_N Y)C(W_i, W_i) - B(A_N Y, A_N X_1)]\}, \qquad (5.1)$$

where $\{W_i\}_{i=1}^{n}$ is a basis of $S(TM)$. Thus, using (3.2) and the hypothesis in (5.1), we obtain

$$
\begin{aligned}
(R \cdot S)(X_1, X_2; X, Y) &= \alpha g(Y, X_1)B(X, X_2) - g(Y, X_1)B(X_2, A_N X) \\
&- \alpha g(X, X_1)B(Y, X_2) + g(X, X_1)B(X_2, A_N Y) \\
&+ \alpha g(Y, X_2)B(X_1, X) - g(Y, X_2)B(X, A_N X_1) \\
&- \alpha g(X, X_2)B(X_1, Y) + g(X, X_2)B(Y, A_N X_1) \\
&= Q(g, S)(X_1, X_2; X, Y), \qquad (5.2)
\end{aligned}
$$

where $\alpha = \Sigma_{i=1}^{n}\varepsilon_i C(W_i, W_i)$. Thus, the proof is complete. □

Theorem 5.4 *Let M be a Ricci-pseudosymmetric lightlike hypersurface of a semi-Euclidean space. If M and $S(TM)$ are totally umbilical, then M is a Ricci-semisymmetric lightlike hypersurface.*

Proof Suppose that M and $S(TM)$ are totally umbilical. Then we have

$$
\begin{aligned}
(R \cdot S)(X_1, X_2; X, Y) &= \rho^2 \lambda[\alpha g(Y, X_1)g(X, X_2) - \lambda g(Y, X_1)g(X_2, X) \\
&- \alpha g(X, X_1)g(Y, X_2) + \lambda g(X, X_1)g(X_2, Y) \\
&+ \alpha g(Y, X_2)g(X_1, X) - \lambda g(Y, X_2)g(X, X_1) \\
&- \alpha g(X, X_2)g(X_1, Y) + \lambda g(X, X_2)g(Y, X_1)] \\
&= \rho \lambda Q(g, S)(X_1, X_2; X, Y) \\
&= 0,
\end{aligned}
$$

for $X_1, X_2, X, Y \in \Gamma(TM)$, where λ and ρ are smooth functions. This completes the proof. □

Theorem 5.5 *Let M be a Ricci-pseudosymmetric lightlike hypersurface of a semi-Euclidean space. Then either M is totally geodesic or $B(A_N \xi, A_N \xi) = 0$.*

Proof From the hypothesis, $(R \cdot S)(X_1, X_2; X, Y) = \lambda Q(g, S)(X_1, X_2; X, Y)$, for $X_1, X_2, X, Y \in \Gamma(TM)$. Taking $X = \xi \in \Gamma(RadTM)$, we obtain

$$\alpha B(Y, X_1)B(A_N\xi, X_2) - B(Y, X_1)B(X_2, A_N^2\xi) + \alpha B(Y, X_2)B(X_1, A_N\xi)$$
$$-B(Y, X_2)B(A_N\xi, A_N X_1) + \lambda g(Y, X_1)B(X_2, A_N\xi) = 0.$$

Then, for $X_1 = \xi$, we get $B(Y, X_2)B(A_N\xi, A_N\xi) = 0$. Thus, the proof is complete. □

Theorem 5.6 *Let M be a pseudosymmetric lightlike hypersurface of a semi-Euclidean space such that $S(\xi, X) = 0$, $\forall X \in \Gamma(TM), \xi \in \Gamma(RadTM)$ and $A_N\xi$ is a nonnull vector field, where S is a Ricci tensor. M is then totally geodesic.*

Proof Suppose that M is a pseudosymmetric lightlike hypersurface, i.e.

$$(R \cdot R)(X_1, X_2, X_3, X_4; X, Y) = \lambda Q(g, R)(X_1, X_2, X_3, X_4; X, Y),$$

for $X_1, X_2, X_3, X_4, X, Y \in \Gamma(TM)$. Then, using (2.18) in (2.1) and (2.2), for $X_2 = \xi$, we have

$$B(Y, X_1)B(A_N X, X_3)g(A_N\xi, X_4) - B(X, X_1)B(A_N Y, X_3)g(A_N\xi, X_4)$$
$$+B(Y, X_3)B(X_1, A_N X)g(A_N\xi, X_4) - B(X, X_3)B(X_1, A_N Y)g(A_N\xi, X_4)$$
$$+B(Y, X_4)B(X_1, X_3)g(A_N\xi, A_N X) - B(X, X_4)B(X_1, X_3)g(A_N\xi, A_N Y)$$
$$-\lambda[g(Y, X_1)B(X, X_3)g(A_N\xi, X_4) - g(X, X_1)B(Y, X_3)g(A_N\xi, X_4)$$
$$+g(Y, X_3)B(X_1, X)g(A_N\xi, X_4) - g(X, X_3)B(X_1, Y)g(A_N\xi, X_4)$$
$$+g(Y, X_4)B(X_1, X_3)g(A_N\xi, X) - g(X, X_4)B(X_1, X_3)g(A_N\xi, Y)] = 0.$$

For $X = \xi$, we obtain

$$B(Y, X_1)B(A_N\xi, X_3)g(A_N\xi, X_4) + B(Y, X_3)B(X_1, A_N\xi)g(A_N\xi, X_4)$$
$$+B(Y, X_4)B(X_1, X_3)g(A_N\xi, A_N\xi) = 0. \tag{5.3}$$

For $S(\xi, X) = B(X, A_N\xi) = 0$ from the hypothesis, (5.3) is equivalent to

$$B(Y, X_4)B(X_1, X_3)g(A_N\xi, A_N\xi) = 0.$$

Since $A_N\xi$ is nonnull, taking $Y = X_1$ and $X_4 = X_3$, we have $B(X_1, X_3) = 0$. Thus, M is totally geodesic. □

Finally, we introduce Ricci-generalized pseudoparallel lightlike hypersurfaces and obtain a sufficient condition for such hypersurfaces. For the notion of Ricci-generalized pseudoparallel hypersurfaces, see [25].

Definition 5.7 *Let M be a lightlike hypersurface of a semi-Euclidean space. We say that M is a Ricci-generalized pseudoparallel lightlike hypersurface if the tensors of $R \cdot h$ and $Q(S, h)$ are linearly dependent at $\forall p \in M$. This is equivalent to $R \cdot h = LQ(S, h)$ on $U = \{p \in M | Q(S, h) \neq 0\}$, where L is some function on U.*

Theorem 5.8 *Let M be a lightlike hypersurface of a semi-Euclidean space such that A_N is symmetric with respect to B. If τ-parallel, then M is a Ricci-generalized pseudoparallel lightlike hypersurface such that $\lambda = -1$.*

Proof For $X_1, X_2, X, Y \in \Gamma(TM)$, we get

$$
\begin{aligned}
Q(S,h)(X_1,X_2;X,Y) &= \Sigma_{i=1}^n \varepsilon_i[B(Y,X_1)C(W_i,W_i) - B(X_1,A_NY)]h(X,X_2) \\
&- \Sigma_{i=1}^n \varepsilon_i[B(X,X_1)C(W_i,W_i) - B(X_1,A_NX)]h(Y,X_2) \\
&+ \Sigma_{i=1}^n \varepsilon_i[B(Y,X_2)C(W_i,W_i) - B(X_2,A_NY)]h(X_1,X) \\
&- \Sigma_{i=1}^n \varepsilon_i[B(X,X_2)C(W_i,W_i) - B(X_2,A_NX)]h(X_1,Y) \\
&= \alpha B(Y,X_1)B(X,X_2)N - B(X_1,A_NY)B(X,X_2)N \\
&- \alpha B(X,X_1)B(Y,X_2)N + B(X_1,A_NX)B(Y,X_2)N
\end{aligned}
$$

$$
\begin{aligned}
&+ \alpha B(Y,X_2)B(X_1,X)N - B(X_2,A_NY)B(X_1,X)N \\
&- \alpha B(X,X_2)B(X_1,Y)N + B(X_2,A_NX)B(X_1,Y)N \\
&= -[-B(X_2,A_NX)B(X_1,Y)N + B(X_2,A_NY)B(X_1,X)N \\
&- B(Y,X_2)B(X_1,A_NX)N + B(X,X_2)B(X_1,A_NY)N] \\
&= -(R \cdot h)(X_1,X_2;X,Y),
\end{aligned}
$$

where $\alpha = \Sigma_{i=1}^n \varepsilon_i C(W_i,W_i)$. This completes the proof. $\square$

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