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Stability of perturbed dynamic system on time scales with initial time difference

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Abstract: The behavior of solutions of a perturbed dynamic system with respect to an original unperturbed dynamic system, which have initial time difference, are investigated on arbitrary time scales. Notions of stability, asymptotic stability, and instability with initial time difference are introduced. Sufficient conditions of stability properties are given with the help of Lyapunov-like functions.

Key words: Time scales, stability, Lyapunov-like functions, comparison results, initial time difference

1. Introduction

In [1, 7], Hilger introduced the theory of time scales, closed subsets of $\mathbb{R}$, to unify the theory of differential and difference equations into a single set-up and to extend these theories to other kinds of so-called dynamic equations. This extension gives us a chance to consider the continuous and discrete cases simultaneously.

Stability theory is one of the important branches of the theory of differential equations. Numerous studies have been done about this theory [4, 8]. Some of these results were extended to dynamic equations on time scales [9]. An important problem in stability theory is to determine which stability properties of a particular differential system are preserved under sufficiently small perturbations. This problem was investigated in several ways in [4, 5, 6, 8, 9]. However, the possibility of making errors in initial time as well as in initial position needs to be considered. When such a change of initial time for each solution is considered, then the problem of measuring the difference between any 2 solutions starting at different times arises. The solution of this interesting problem was investigated in [10, 11, 12] in different ways.

In the present paper, the problem of determining the behavior of solutions of a perturbed dynamic equation with respect to those of an original unperturbed dynamic system that have initial time difference (ITD) is studied on arbitrary time scales. A more general result is obtained such that it can be applied in discrete and continuous cases simultaneously.

The paper is organized as follows: in Section 2, basic concepts and definitions are given. In Section 3, comparison results and stability properties of the perturbed dynamic equation with respect to the original unperturbed dynamic system that have ITD are proven. To illustrate the main results, an example is given in Section 4. Finally, concluding remarks are given in Section 5.

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2. Preliminaries

In this section, we give a brief introduction of time scales. More details can be found in [3, 2, 9].

A time scale is an arbitrary nonvoid closed subset of real numbers and is denoted by the symbol $\mathbb{T}$. We assume throughout that a time scale $\mathbb{T}$ has the topology that it inherits from the real numbers with the standard topology and, for our future purposes, unbounded from above with $t_0 \geq 0$ as a minimal element.

Since a time scale is not necessarily connected, the forward and backward jump operators are defined on $\mathbb{T}$ as follows.

Definition 2.1 *The mappings* $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ *defined by*

$$\sigma(t) = \inf \{s \in \mathbb{T}, \, s > t\} \quad \textit{and} \quad \rho(t) = \sup \{s \in \mathbb{T}, \, s < t\}$$

are called the forward jump operator and backward jump operator, respectively.

A point $t \in \mathbb{T}$ is called right-dense if $\sigma(t) = t$, right-scattered if $\sigma(t) > t$, left-dense if $\rho(t) = t$, left-scattered if $\rho(t) < t$, dense if $t = \sigma(t) = \rho(t)$, and isolated if $\sigma(t) > t > \rho(t)$.

The following function measures the gap between a point t and its right neighbor.

Definition 2.2 *The mapping* $\mu : \mathbb{T} \to \mathbb{R}_+$ *defined by*

$$\mu(t) = \sigma(t) - t$$

is called the graininess *function.*

If a time scale $\mathbb{T}$ has a maximal element that is also left-scattered, then it is called a degenerate point. $\mathbb{T}^\kappa$ represents the set of all nondegenerate points of $\mathbb{T}$. This set cuts off an eventually existing isolated maximum of $\mathbb{T}$.

Definition 2.3 *Assume that* $u : \mathbb{T} \to \mathbb{R}$ *is a function and let* $t \in \mathbb{T}^\kappa$. *Then we define delta derivative* $u^\triangle(t)$ *at* t *to be the number (provided it exists) with the property that given any* $\epsilon > 0$, *there is a neighborhood* U *of* t *such that*

$$|[u(\sigma(t)) - u(s)] - u^\triangle(t)[\sigma(t) - s]| \leq \epsilon|\sigma(t) - s| \qquad \forall s \in U.$$

Alternatively, one can define

$$u^\triangle(t) := \lim_{s \to t, s \neq \sigma(t)} \frac{u(\sigma(t)) - u(s)}{\sigma(t) - s}.$$

Definition 2.4 *For each* $t \in \mathbb{T}$, *let* N *be a neighborhood of* t. *Then we define the upper right Dini derivative* $D^+u^\triangle(t)$ *by the following condition: for a given* $\epsilon > 0$, *there exists a right neighborhood* $N_\epsilon \subset N$ *of* t *such that*

$$\frac{u(\sigma(t)) - u(s)}{\sigma(t) - s} < D^+u^\triangle(t) + \epsilon \quad \textit{for } s \in N_\epsilon;$$

in case $t \in \mathbb{T}$ *is right-scattered and* $u(t)$ *is continuous at* t, *we have*

$$D^+u^\triangle(t) = \frac{u(\sigma(t)) - u(t)}{\mu(t)}$$

where $\mu(t) = \sigma(t) - t$. *Alternatively, one can define*

$$D^+u^{\triangle}(t) := \limsup_{s \to t, s \neq \sigma(t)} \frac{u(\sigma(t)) - u(s)}{\sigma(t) - s}.$$

Definition 2.5 *A function* $u : \mathbb{T} \to \mathbb{R}$ *is said to be rd-continuous if it is continuous at each right-dense point and if there exists a finite left-sided limit at all left-dense points. The set of all rd-continuous functions is denoted by* $\mathbf{C}_{rd} \equiv \mathbf{C}_{rd}[\mathbb{T}, \mathbb{R}]$.

Definition 2.6 *The mapping* $f : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ *is said to be rd-continuous and denoted by* $f \in \mathbf{C}_{rd}[\mathbb{T} \times \mathbb{R}, \mathbb{R}]$ *if:*

i) it is continuous at each (t, x) *with right-dense or maximal* $t \in \mathbb{T}$*;*

ii) the limits $f(t^-, x) = \lim_{(s,y) \to (t^-, x)} f(s, y)$ *and* $\lim_{y \to x} f(t, y)$ *exist at each* (t, x) *with left-dense* t.

Definition 2.7 *Let* $f : \mathbb{T} \to \mathbb{R}$ *be a function, and* $a, b \in \mathbb{T}$. *If there exists a function* $F : \mathbb{T} \to \mathbb{R}$ *such that* $F^{\triangle}(t) = f(t)$ *for all* $t \in \mathbb{T}^{\kappa}$, F *is said to be an antiderivative of* f. *In this case, the integral is given by the formula*

$$\int_a^b f(\tau) \triangle \tau = F(b) - F(a) \text{ for } a, b \in \mathbb{T}.$$

Remark 2.1 *All right-dense continuous functions are integrable.*

Definition 2.8 *If* $a \in \mathbb{T}$, $\sup \mathbb{T} = \infty$, *and* f *is rd-continuous on* $[a, \infty)$, *then we define the improper integral by*

$$\int_a^{\infty} f(\tau) \triangle \tau = \lim_{b \to \infty} \int_a^b f(\tau) \triangle \tau,$$

provided that this limit exists, and we say that the improper integral converges in this case.

Definition 2.9 *A function* $p : \mathbb{T} \to \mathbb{R}$ *is said to be* regressive*, provided that*

$$1 + \mu(t)p(t) \neq 0$$

for all $t \in \mathbb{T}^{\kappa}$. *The set of all regressive and right-dense continuous functions is denoted by* $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R})$.

Definition 2.10 *The set* $\mathcal{R}^+$ *of all positively regressive elements of* $\mathcal{R}$ *is defined by*

$$\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R}) = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0 \text{ for all } t \in \mathbb{T}\}.$$

Definition 2.11 *If* $p : \mathbb{T} \to \mathbb{R}$ *is regressive and right-dense continuous, then the* exponential function *is defined by*

$$e_p(t, s) = \exp(\int_s^t \frac{Log(1 + \mu(\tau)p(\tau))}{\mu(\tau)} \triangle \tau)$$

for $t, s \in \mathbb{T}$.

Remark 2.2 *Consider the dynamic initial value problem*

$$x^{\triangle} = p(t)x,\ x(t_0) = x_0,$$

where $t_0 \in \mathbb{T}$ *and* $p \in \mathcal{R}$. *Then* $x(t, t_0, x_0) = x_0 e_p(t, t_0)$ *is the unique solution to this initial value problem.*

The following induction principle [3, 2] is sometimes useful for analyzing equations on a time scale $\mathbb{T}$.

Theorem 2.1 *(Induction principle) Let* $t_0 \in \mathbb{T}$ *and assume that*

$$\{A(t) : t \in [t_0, \infty)\}$$

is a family of statements satisfying the following:

i) The statement $A(t_0)$ *is true.*

ii) If $t \in [t_0, \infty)$ *is right-scattered and* $A(t)$ *is true, then* $A(\sigma(t))$ *is also true.*

iii) If $t \in [t_0, \infty)$ *is right-dense and* $A(t)$ *is true, then there is a neighborhood* U *of* t *such that* $A(s)$ *is true for all* $s \in U \cap (t, \infty)$.

iv) If $t \in (t_0, \infty)$ *is left-dense and* $A(s)$ *is true for all* $s \in [t_0, t)$, *then* $A(t)$ *is true.*

Then $A(t)$ *is true for all* $t \in [t_0, \infty)$.

Proof See [3, 2]. □

Theorem 2.2 *(Gronwall's inequality). Let* $y, f \in \mathbf{C}_{rd}$ *and* $p \in \mathcal{R}^+$, $p \geq 0$. *Then*

$$y(t) \leq f(t) + \int_{t_0}^{t} y(\tau)p(\tau)\triangle\tau \ \textit{ for all } t \in \mathbb{T}$$

implies

$$y(t) \leq f(t) + \int_{t_0}^{t} e_p(t, \sigma(\tau))f(\tau)p(\tau)\triangle\tau \ \textit{ for all } t \in \mathbb{T}.$$

Proof See [3, 2]. □

Corollary 2.1 *Let* $y \in \mathbf{C}_{rd}$, $p \in \mathcal{R}^+$, $p \geq 0$ *and* $\alpha \in \mathbb{R}$. *Then*

$$y(t) \leq \alpha + \int_{t_0}^{t} y(\tau)p(\tau)\triangle\tau \ \textit{ for all } t \in \mathbb{T}$$

implies

$$y(t) \leq \alpha e_p(t, t_0) \ \textit{ for all } t \in \mathbb{T}.$$

Proof See [3, 2]. □

We will consider the dynamic system

$$x^{\triangle} = f(t,x) \qquad x(t_0) = x_0, \tag{1}$$

where $f \in \mathbf{C}_{rd}[\mathbb{T} \times S(\rho), \mathbb{R}^n]$, $S(\rho) = \{x \in \mathbb{R}^n : \|x\| \leq \rho\}$ where $\rho > 0$. Here $\|x\|$ denotes any n-dimensional norm of the vector x. Generally, the bound ρ will be considered finite. Once we consider the instability of solution of the dynamic equation, we let ρ be infinite. In that case, the solutions are unbounded and therefore the region under consideration must hold them.

In addition to dynamical system (1), we also consider the associated perturbed dynamical system with different initial conditions

$$y^{\triangle} = f(t,y) + R(t,y) \qquad y(\tau_0) = y_0, \tag{2}$$

where $R \in \mathbf{C}_{rd}[\mathbb{T} \times S(\rho), \mathbb{R}^n]$ is called the perturbation term.

We assume that $f,\, R \in \mathbf{C}_{rd}[\mathbb{T} \times S(\rho), \mathbb{R}^n]$ are smooth enough to guarantee the existence, uniqueness, and rd-continuous dependence of solutions of (1) and (2).

In the course of the investigation, we need the following class of functions.

Definition 2.12 *A function $\varphi(r)$ is said to be class $\mathcal{K}$ if $\varphi \in \mathbf{C}_{rd}[\mathbb{T}, \mathbb{R}_+]$, $\varphi(0) = 0$, $\varphi(r) \to \infty$ as $r \to \infty$, and $\varphi(r)$ is strictly monotone increasing in r.*

Definition 2.13 *A function $V(t,x) \in \mathbf{C}_{rd}[\mathbb{T} \times S(\rho), \mathbb{R}_+]$ is said to be positive definite if there exists a function $\varphi \in \mathcal{K}$ such that*

$$V(t,x) \geq \varphi(\|x\|) \qquad \text{for } (t,x) \in \mathbb{T} \times S(\rho).$$

Definition 2.14 *For a real valued function $V(t,x) \in \mathbf{C}_{rd}[\mathbb{T} \times S(\rho), \mathbb{R}_+]$ we define the Dini derivative as follows: for a given $\epsilon > 0$ there exists a neighborhood $\mathcal{N}_\epsilon$ of $t \in \mathbb{T}$ such that*

$$\frac{V(\sigma(t),x(\sigma(t))) - V(s,x(\sigma(t)) - (\sigma(t)-s)f(t,x(t)))}{\sigma(t)-s} < D^+V^{\triangle}(t,x) + \epsilon, \quad s \in \mathcal{N}_\epsilon, \quad s > t$$

for $(t,x) \in \mathbb{T} \times S(\rho)$.

In case $t \in \mathbb{T}$ is right-scattered and $V(t,x(t))$ is continuous at t, we have

$$D^+V^{\triangle}(t,x(t)) = \frac{V(\sigma(t),x(\sigma(t))) - V(t,x(t))}{\mu(t)}$$

where $\mu(t) = \sigma(t) - t$. Alternatively, one can define

$$D^+V^{\triangle}(t,x) := \limsup_{s \to t, s \neq \sigma(t)} \frac{V(t+\mu(t), x(t) + \mu(t)f(t,x(t))) - V(s,x(t))}{\sigma(t)-s},$$

$\mu(t) = \sigma(t) - t$.

Considering the dynamical systems (1) and (2), for $V(t,x) \in \mathbf{C}_{rd}[\mathbb{T} \times S(\rho), \mathbb{R}_+]$, we define the Dini-like derivative with respect to systems (1) and (2).

Definition 2.15 *Let* $V \in \mathbf{C}_{rd}[\mathbb{T} \times S(\rho), \mathbb{R}_+]$, $V(t,u)$ *is locally Lipschitzian in* u.

$$D^+V^{\triangle}(t,u,\eta) := \limsup_{s \to t, s \neq \sigma(t)} \frac{V(t+\mu(t), u(t)+\mu(t)\widetilde{f}(t,u(t);\eta)) - V(s,u(t))}{\sigma(t)-s}$$

for $(t,u,\eta) \in \mathbb{T} \times S(\rho) \times \mathbb{R}$, *where* $\mu(t) = \sigma(t) - t$, $u(t,\tau_0,u_0) = y(t,\tau_0,y_0) - x(t-\eta,t_0,x_0)$ *for* $t \geq \tau_0$, *and* $\widetilde{f}(t,u(t);\eta) = f(t,u(t)+x(t-\eta)) + R(t,u(t)+x(t-\eta)) - f(t-\eta,x(t-\eta))$.

We now give the definitions of stability. These definitions identify the possible behavior for solution of perturbed dynamic system (2).

Definition 2.16 *The solutions of the perturbed dynamic system* (2) *are said to be stable with respect to unperturbed dynamic system* (1) *with initial time difference if, given any* $\epsilon > 0$ *and* $\tau_0 \in \mathbb{T}$, *there exist* $\delta(\epsilon,\tau_0) > 0$ *and* $\widetilde{\delta}(\epsilon,\tau_0) > 0$ *such that* $\|y_0 - x_0\| < \delta$ *and* $\eta < \widetilde{\delta}$ *implies*

$$\|y(t,\tau_0,y_0) - x(t-\eta,t_0,x_0)\| < \epsilon$$

for $t \geq \tau_0$, *for every solution* $y(t,\tau_0,y_0)$ *of the perturbed dynamic system* (2), *where* $\eta = \tau_0 - t_0 > 0$.

Definition 2.17 *The solutions of perturbed dynamic system* (2) *are said to be asymptotically stable with respect to unperturbed dynamic system* (1) *with initial time difference if they are stable with respect to equation* (1) *with initial time difference and if, given any* $\epsilon > 0$ *and* $\tau_0 \in \mathbb{T}$, *there exist* $\delta_0(\tau_0) > 0$, $\widetilde{\delta}_0(\tau_0) > 0$ *and* $T = T(\epsilon,\tau_0) > 0$ *such that* $\|y_0 - x_0\| < \delta_0$ *and* $\eta < \widetilde{\delta}_0$ *implies*

$$\|y(t,\tau_0,y_0) - x(t-\eta,t_0,x_0)\| < \epsilon$$

for $t \geq \tau_0 + T$, *for every solution* $y(t,\tau_0,y_0)$ *of perturbed dynamic system* (2).

Definition 2.18 *The solutions of perturbed dynamic system* (2) *are said to be unstable with respect to unperturbed dynamic system* (1) *with initial time difference if they are not stable with respect to unperturbed dynamic system* (1) *with initial time difference.*

Definition 2.16 and *Definition 2.17 are equivalent to the statement that all solutions of perturbed dynamic system* (2) *that start sufficiently close to the initial conditions of the unperturbed solution respectively remain close to it or eventually approach it. Definition 2.18 requires that for each solution of the unperturbed equation* (1), *a solution of the perturbed equation* (2) *can be found that starts arbitrarily close to the unperturbed solution and eventually diverges from it.*

We stress that all of the above definitions are independent of the behavior of the solutions of the unperturbed dynamic system. Indeed, we particularly show that the equilibria of the original dynamic equations may be stable, asymptotically stable, or even unstable. We illustrate these situations as follows on different time scales.

Example 2.1 *Let* $\mathbb{T} = \mathbb{Z}$. *Consider the dynamic equation*

$$x^{\triangle} = \triangle x = c \qquad x(t_0) = x_0$$

where c is any constant, whose solution is given by $x(t,t_0,x_0) = x_0 + c(t-t_0)$, which is unstable.

In addition, consider the associated perturbed equation

$$\triangle y = c + g(t+1) \qquad y(\tau_0) = y_0,$$

where $\{g(t)\}$ is any sequence for which $\sum^{\infty} g(t) = 0$. The corresponding solution is then given by $y(t,\tau_0,y_0) = y_0 + c(t-\tau_0) + \sum_{k=\tau_0}^{t} g(k)$. Then the difference is

$$y(t,\tau_0,y_0) - x(t-\eta,t_0,x_0) = y_0 - x_0 + \sum_{k=\tau_0}^{t} g(k),$$

which can be made arbitrarily small. Therefore, the solution of the perturbed equation is stable with respect to the unperturbed equation with initial time difference.

Example 2.2 *Let $\mathbb{T} = \mathbb{R}$. Consider the dynamic equation*

$$x^{\triangle} = x' = -ax \qquad x(t_0) = x_0,$$

where $a > 0$, whose asymptotically stable solution is given by

$$x(t,t_0,x_0) = x_0 e^{-a(t-t_0)}.$$

Further, consider the associated perturbed equation

$$y' = -(a+b)y \qquad y(\tau_0) = y_0,$$

whose solution is $y(t,\tau_0,y_0) = y_0 e^{-(a+b)(t-\tau_0)}$. As a consequence,

$$y(t,\tau_0,y_0) - x(t-\eta,t_0,x_0) = e^{-a(t-\tau_0)}[y_0 e^{-b(t-\tau_0)} - x_0].$$

- *If $b > -a$, this difference approaches 0 as $t \to \infty$ and thus the perturbed solutions are asymptotically stable with respect to the unperturbed equation with ITD.*
- *If $b < -a$, then the perturbed solutions are unstable with respect to the unperturbed equation with ITD.*
- *If $b = -a$, then the perturbed solutions are stable with respect to the unperturbed equation with ITD.*

3. Main results

In this section, we prove some theorems about stability properties of solutions of a perturbed dynamic system. First of all, we prove the comparison result in terms of Lyapunov-like functions.

Theorem 3.1 *Assume that:*

i) $V \in \mathbf{C}_{rd}[\mathbb{T} \times S(\rho), \mathbb{R}_+]$, $V(t,u)$ is locally Lipschitzian in u and

$$D^+V^{\triangle}(t,u,\eta) \le g(t,V(t,u),\eta), \quad \text{for } (t,u,\eta) \in \mathbb{T} \times S(\rho) \times \mathbb{R},$$

where $u = u(t) = y(t,\tau_0,y_0) - x(t-\eta,t_0,x_0)$, $\eta = \tau_0 - t_0 > 0$, $g \in \mathbf{C}_{rd}[\mathbb{T} \times \mathbb{R}^2, \mathbb{R}_+]$;

ii) the maximal solution $r(t,\tau_0,w_0,\eta)$ *of* $w^{\triangle}=g(t,w,\eta)$ $w(\tau_0)=w_0\geq 0$ *exists for* $t\geq\tau_0\geq 0$*;*

iii) $g(t,w,\eta)\mu(t)$ *is nondecreasing in* $w\in\mathbb{R}$ *for each* $\eta\in\mathbb{R}^+$ *and* $t\in\mathbb{T}$*.*

Then $V(\tau_0,y_0-x_0)\leq w_0$ *implies*

$$V(t,y(t,\tau_0,y_0)-x(t-\eta,t_0,x_0))\leq r(t,\tau_0,w_0,\eta),\ t\geq\tau_0,\ t,\tau_0\in\mathbb{T}.$$

Proof

Set $u(t)=y(t,\tau_0,y_0)-x(t-\eta,t_0,x_0)$ for $t\geq\tau_0$ so that $u(\tau_0)=y_0-x_0$ and

$$\begin{aligned}u^{\triangle}(t)&=f(t,y(t,\tau_0,y_0))+R(t,y(t,\tau_0,y_0))-f(t-\eta,x(t-\eta,t_0,x_0))\\&=f(t,u(t)+x(t-\eta,t_0,x_0))+R(t,u(t)+x(t-\eta,t_0,x_0))-f(t-\eta,x(t-\eta,t_0,x_0))\\&=\widetilde{f}(t,u(t);\eta),\quad \text{for } t\geq\tau_0.\end{aligned}$$

We apply the induction principle to the statement

$$A(t):V(t,u(t,\tau_0,u_0))\leq r(t,\tau_0,w_0,\eta),\qquad t\geq\tau_0,\quad t,\tau_0\in\mathbb{T}.$$

(I) Let $t=\tau_0$. Since $V(\tau_0,y_0-x_0)\leq w_0$, it follows that $A(\tau_0)$ is true.

(II) Let t be right-scattered and $A(t)$ is true. We shall show that $A(\sigma(t))$ is true. Set $m(t)=V(t,u(t))$. Then, using the definition of the derivative for a right-scattered point, we have the inequality

$$\begin{aligned}m(\sigma(t))-r(\sigma(t))&=(D^+m^{\triangle}(t)-r^{\triangle}(t))\mu(t)+(m(t)-r(t))\\&\leq(g(t,m(t))-g(t,r(t)))\mu(t)+(m(t)-r(t)).\end{aligned}$$

Then, since $A(t)$ is true, by assumption (iii) it follows that

$$m(\sigma(t))-r(\sigma(t))\leq 0.$$

In view of the fact that

$$\frac{m(\sigma(t))-m(t)}{\mu(t)}=\frac{V(\sigma(t),u(\sigma(t)))-V(t,u(t))}{\mu(t)},$$

we see that $A(\sigma(t))$ is true.

(III) Let t be right-dense and U be a neighborhood of t. Assume that $A(t)$ is true. We need to show that $A(s)$ is true for $s>t$, $s\in U$. Since the right neighborhood of t is an interval, we consider it as in continuous case. Therefore, let h be a positive number.

$$\begin{aligned}m(s+h)-m(s)&=V(s+h,u(s+h))-V(s+h,u(s)+h\widetilde{f}(s,u(s);\eta))\\&\quad+V(s+h,u(s)+h\widetilde{f}(s,u(s);\eta))-V(s,u(s))\\&=V(s+h,u(s)+h\widetilde{f}(s,u(s);\eta)+h\epsilon(h))-V(s+h,u(s)+h\widetilde{f}(s,u(s);\eta))\\&\quad+V(s+h,u(s)+h\widetilde{f}(s,u(s);\eta))-V(s,u(s))\end{aligned}$$

Since V is locally Lipschitzian in u and $L > 0$ is the Lipschitz constant and ϵ is the error term, we have

$$\begin{aligned} D^+m^{\triangle}(s) &\leq \lim_{h\to 0^+} L\|\epsilon(h)\| + \limsup_{h\to 0^+, s+h\in\mathbb{T}} \frac{V(s+h, u(s)+h\widetilde{f}(s,u(s);\eta)) - V(s,u(s))}{h} \\ &= D^+V^{\triangle}(s,u(s)). \\ &\leq g(s,m(s)) \end{aligned}$$

Since A(t) is true, by Theorem 3.1.1 in [9] we obtain that

$$m(s) = V(s, u(s,\tau_0,u_0)) \leq r(s,\tau_0,w_0,\eta), \text{ for } s \geq t, \ s \in U.$$

(IV) Let t be left-dense and $A(s)$ is true for $s < t$. We need to show that $A(t)$ is true. This follows by rd-continuity of $V(t,u)$ and $r(t)$.

Thus, by induction principle, we conclude that

$$V(t, u(t,\tau_0,u_0)) \leq r(t,\tau_0,w_0,\eta), \ t \in \mathbb{T}, \ t \geq \tau_0.$$

□

Remark 3.1 *If the inequality (i) is reversed and* $V(\tau_0, y_0 - x_0) \geq w_0$*, then we have to replace the conclusion by* $V(t,u(t,\tau_0,u_0)) \geq r_*(t,\tau_0,w_0,\eta), \ t\in\mathbb{T}, \ t \geq \tau_0$*, where* $r_*(t,\tau_0,w_0,\eta)$ *is the minimal solution of the comparison equation.*

The following theorem ensures both that the function $V(t, y(t) - x(t-\eta))$ remains well defined and that the difference $y(t,\tau_0,y_0) - x(t-\eta,t_0,x_0)$ of the solution of the perturbed dynamic system (2) with shifted solution of the original unperturbed dynamic system (1) remains on $S(\rho)$.

Theorem 3.2 *Assume that*

i) $f \in \mathbf{C}_{rd}[\mathbb{T}\times\mathbb{R}^n, \mathbb{R}^n]$ *is Lipschitzian in time and space such that*

$$\|f(t, u(t,\tau_0,u_0) + x(t-\eta,t_0,x_0)) - f(t-\eta, x(t-\eta,t_0,x_0))\| \leq L(t)\|u(t)\| + N(t)\eta$$

where $u(t,\tau_0,u_0) = y(t,\tau_0,y_0) - x(t-\eta,t_0,x_0)$ *for* $t \geq \tau_0$*,* $u_0 = y_0 - x_0$ *and* $\eta = \tau_0 - t_0 > 0$*;*

ii) $L(t) \in \mathcal{R}^+$*,* $L(t) > 0$ *and* $N(t) > 0$ *for* $t \in \mathbb{T}$*;*

iii) The perturbation term $R(t,y)$ *satisfies* $\|R(t,y)\| \leq a\|h(t)\|$ *for sufficiently small positive constant* a *and for some function* $h(t)$ *that is absolutely integrable on* $[\tau_0,\infty)\cap\mathbb{T}$*, where* $\|\cdot\|$ *is the* n*-dimensional vector norm;*

iv) There exist constants M_1*,* M_2*, and* M_3 *such that*

$$\int_{\tau_0}^{t} L(s)\triangle s \leq M_1, \ \int_{\tau_0}^{t} N(s)\triangle s \leq M_2, \ \int_{\tau_0}^{t} \|h(s)\|\triangle s \leq M_3, \ for \ t \geq \tau_0.$$

Then

$$\|y(t,\tau_0,y_0)-x(t-\eta,t_0,x_0)\| \le \rho \text{ for all } t \ge \tau_0$$

provided that y_0 *and* τ_0 *are chosen sufficiently close to* x_0 *and* t_0*, respectively.*

Proof We have

$$y(t,\tau_0,y_0)=y_0+\int_{\tau_0}^{t} f(s,y(s))\triangle s+\int_{\tau_0}^{t} R(s,y(s))\triangle s$$

and

$$x(t-\eta,t_0,x_0)=x_0+\int_{\tau_0}^{t} f(s-\eta,x(s-\eta))\triangle s.$$

As a consequence,

$$\begin{aligned}\|y(t,\tau_0,y_0)-x(t-\eta,t_0,x_0)\| \le\ & \|y_0-x_0\|+\int_{\tau_0}^{t}\|R(s,y(s))\|\triangle s\\ &+\int_{\tau_0}^{t}\|f(s,y(s,\tau_0,y_0))-f(s-\eta,x(s-\eta,t_0,x_0))\|\triangle s\\ \le\ & \|y_0-x_0\|+a\int_{\tau_0}^{t}\|h(s)\|\triangle s+\eta\int_{\tau_0}^{t}N(s)\triangle s\\ &+\int_{\tau_0}^{t}L(s)\|y(s)-x(s-\eta)\|\triangle s\\ \le\ & \|y_0-x_0\|+aM_3+\eta M_2+\int_{\tau_0}^{t}L(s)\|y(s)-x(s-\eta)\|\triangle s,\end{aligned}$$

where $\|\cdot\|$ is the n-dimensional vector norm.

Set $m(t)=\|y(t)-x(t-\eta)\|$ and $A=\|y_0-x_0\|+aM_3+\eta M_2$. Then

$$m(t)\le A+\int_{\tau_0}^{t}L(s)m(s)\triangle s.$$

Then, by Corollary 2.1, we obtain the following inequality:

$$\begin{aligned}\|y(t)-x(t-\eta)\| \le\ & A\exp\Big(\int_{\tau_0}^{t}\frac{Log(1+\mu(s)L(s))}{\mu(s)}\triangle s\Big)\\ \le\ & A\exp\Big(\int_{\tau_0}^{t}L(s)\triangle s\Big)\\ \le\ & A\exp(M_1),\end{aligned}$$

which can be made smaller than any given ρ by choosing the constant a sufficiently small and by choosing y_0 and τ_0 sufficiently close to x_0 and t_0, respectively. □

Theorem 3.3 *Assume that:*

i) $V \in \mathbf{C}_{rd}[\mathbb{T} \times S(\rho), \mathbb{R}_+]$, *$V(t,u)$ is locally Lipschitzian in u and*

$$D^+V^{\triangle}(t,u,\eta) \leq g(t, V(t,u), \eta), \quad \textit{for } (t,u,\eta) \in \mathbb{T} \times S(\rho) \times \mathbb{R}$$

where $u = u(t) = y(t,\tau_0,y_0) - x(t-\eta, t_0, x_0)$, $\eta = \tau_0 - t_0 > 0$, $g \in \mathbf{C}_{rd}[\mathbb{T} \times \mathbb{R}^2, \mathbb{R}_+]$;

ii) $g(t,w,\eta)\mu(t)$ *is nondecreasing in* $w \in \mathbb{R}$ *for each* $\eta \in \mathbb{R}^+$ *and* $t \in \mathbb{T}$;

iii) There exists a function $b \in \mathcal{K}$ *such that*

$$V(t,u) \geq b(\|u\|) \qquad \textit{for } (t,u) \in \mathbb{T} \times S(\rho);$$

iv) The maximal solution $r(t,\tau_0,w_0,\eta)$ *of* $w^{\triangle} = g(t,w,\eta)$ $w(\tau_0) = w_0 \geq 0$ *exists for* $t \geq \tau_0 \geq 0$;

v) The scalar equation $w^{\triangle} = g(t,w,\eta)$ $w(\tau_0) = w_0 \geq 0$ $t \geq \tau_0$, $t, \tau_0 \in \mathbb{T}$ *is stable.*

The solutions of the perturbed dynamic system are then stable with respect to the unperturbed dynamic system with ITD, provided that

$$\|y(t,\tau_0,y_0) - x(t-\eta,t_0,x_0)\| \leq \rho \textit{ for all } t \geq \tau_0,$$

where $x(t,t_0,x_0)$ *and* $y(t,\tau_0,y_0)$ *are solutions of* (1) *and* (2), *respectively.*

Proof Let $0 < \epsilon < \rho$ and $\tau_0 \in \mathbb{T}$ be given. Since the scalar dynamic equation is equistable, we have for a given $b(\epsilon) > 0$ that there exists a $\delta_1 = \delta_1(\epsilon,\tau_0) > 0$ and $\widetilde{\delta} = \widetilde{\delta}(\epsilon,\tau_0) > 0$ such that

$$w_0 < \delta_1 \text{ and } \eta < \widetilde{\delta} \text{ implies } w(t,\tau_0,w_0,\eta) < b(\epsilon) \text{ for } t \geq \tau_0. \tag{3}$$

Choose $w_0 = V(\tau_0, y_0 - x_0)$. Since $V(t,u)$ is rd-continuous and $V(t,0) = 0$, it is possible to find a positive function $\delta = \delta(\epsilon,\tau_0)$ that is rd-continuous in τ_0 for each $\epsilon > 0$, satisfying the inequalities

$$\|y_0 - x_0\| < \delta, \qquad V(\tau_0, y_0 - x_0) < \delta_1$$

simultaneously. We claim that

$$\|y_0 - x_0\| < \delta \text{ and } \eta < \widetilde{\delta} \text{ implies } \|y(t,\tau_0,y_0) - x(t-\eta,t_0,x_0)\| < \epsilon \text{ for } t \geq \tau_0.$$

Suppose that this is not true. Then there would exist a solution $y(t,\tau_0,y_0)$ of (2) with $\|y_0 - x_0\| < \delta$, $\eta < \widetilde{\delta}$ and $t_1 > \tau_0$ such that

$$\|y(t_1,\tau_0,y_0) - x(t_1-\eta,t_0,x_0)\| = \epsilon, \; \|y(t,\tau_0,y_0) - x(t-\eta,t_0,x_0)\| \leq \epsilon, \text{ for } t \in [\tau_0, t_1]. \tag{4}$$

The choice $w_0 = V(\tau_0, y_0 - x_0)$ and condition (i) give, as a consequence of Theorem 3.1, the estimate

$$V(t, y(t,\tau_0,y_0) - x(t-\eta,t_0,x_0)) \leq r(t,\tau_0,w_0,\eta), \;\; t \in [\tau_0, t_1], \tag{5}$$

where $r(t,\tau_0,w_0,\eta)$ is the maximal solution of the comparison equation. Then condition (iii) and the relations (3), (4), and (5) lead to the contradiction

$$b(\epsilon) \leq V(t_1, y(t_1,\tau_0,y_0) - x(t_1-\eta,t_0,x_0)) \leq r(t_1,\tau_0,w_0,\eta) < b(\epsilon).$$

This proves that the solutions of the perturbed system are stable with respect to the unperturbed system with ITD. □

Theorem 3.4 *Let assumptions (i)–(iv) of Theorem 3.3 be satisfied. If*

v^*) The scalar equation $w^{\triangle} = g(t,w,\eta)$ $w(\tau_0) = w_0 \geq 0$ $t \geq \tau_0$, $t, \tau_0 \in \mathbb{T}$ is asymptotically stable,

then the solutions of the perturbed dynamic system are asymptotically stable with respect to the unperturbed dynamic system with ITD, provided that

$$\|y(t,\tau_0,y_0) - x(t-\eta,t_0,x_0)\| \leq \rho \text{ for all } t \geq \tau_0,$$

where $x(t,t_0,x_0)$ and $y(t,\tau_0,y_0)$ are solutions of (1) and (2), respectively.

Proof Since the scalar system is asymptotically stable, it is also stable. Hence, by Theorem 3.3, the solution of the perturbed system is stable with respect to the unperturbed system with ITD. Therefore, we can choose that $\epsilon = \rho > 0$, $\delta_0 = \delta_0(\rho,\tau_0) > 0$ and $\widetilde{\delta_0} = \widetilde{\delta_0}(\rho,\tau_0) > 0$ such that

$$\|y_0 - x_0\| < \delta_0 \text{ and } \eta < \widetilde{\delta_0} \text{ implies } \|y(t,\tau_0,y_0) - x(t-\eta,t_0,x_0)\| < \rho \text{ for } t \geq \tau_0. \tag{6}$$

To prove quasiasymptotic stability, let $0 < \epsilon < \rho$ and $\tau_0 \in \mathbb{T}$ be given. Then it follows from the quasiasymptotic stability of the scalar equation that, given $b(\epsilon) > 0$, $\tau_0 \in \mathbb{T}$, there exist positive numbers $\delta_1 = \delta_1(\tau_0)$, $\widetilde{\delta_1} = \widetilde{\delta_1}(\tau_0)$, and $T = T(\epsilon,\tau_0)$ such that

$$w_0 < \delta_1 \text{ and } \eta < \widetilde{\delta_1} \text{ implies } w(t,\tau_0,w_0,\eta) < b(\epsilon) \text{ for } t \geq \tau_0 + T. \tag{7}$$

Since $V(t,u)$ is rd-continuous and $V(t,0) = 0$, we can find a positive number $\delta_2 = \delta_2(\epsilon,\tau_0)$ satisfying the inequalities

$$\|y_0 - x_0\| < \delta_2, \quad V(\tau_0, y_0 - x_0) < \delta_1$$

simultaneously. The choice $w_0 = V(\tau_0, y_0 - x_0)$, assumption (I), and relation (6) give, as a consequence of Theorem 3.1, the estimate

$$V(t, y(t,\tau_0,y_0) - x(t-\eta,t_0,x_0)) \leq r(t,\tau_0,w_0,\eta) \text{ for } t \geq \tau_0. \tag{8}$$

Set $\delta = \min\{\delta_0,\delta_2\}$ and $\widetilde{\delta} = \min\{\widetilde{\delta_0},\widetilde{\delta_1}\}$. Suppose that there exists a sequence $\{t_k\} \in \mathbb{T}$, $t_k \geq \tau_0 + T$, $t_k \to \infty$ as $k \to \infty$ and a solution $y(t,\tau_0,y_0)$ of perturbed system with $\|y_0 - x_0\| < \delta$ and $|\eta| < \widetilde{\delta}$ such that

$$\|y(t_k,\tau_0,y_0) - x(t_k-\eta,t_0,x_0)\| \geq \epsilon.$$

This leads to the contradiction

$$b(\epsilon) \leq V(t_k, y(t_k,\tau_0,y_0) - x(t_k-\eta,t_0,x_0)) \leq r(t_k,\tau_0,w_0,\eta) < b(\epsilon)$$

because of (6), (7), (8), and (iii). Thus, the solutions of the perturbed system are asymptotically stable with respect to the unperturbed system with ITD. □

Finally, we conclude this section with a criterion for the solution of the perturbed dynamic system (2) to be unstable with respect to the original unperturbed dynamic system (1) with ITD.

Theorem 3.5 *Assume that there exist functions* $V(t,u)$ *and* $g(t,w,\eta)$ *satisfying the following properties:*

i) $V \in \mathbf{C}_{rd}[\bar{G}, \mathbb{R}_+]$, $V(t,u)$ *is locally Lipschitzian in* u *on* $\bar{G}$, $V(t,u) = 0$ *for all* $(t,u) \in \bar{G} - G$ *and* $V(t,u)$ *is positive and bounded on* G, *where* $G \subset \mathbb{T} \times S(\rho)$ *is some open set such that* G *has at least 1 boundary point* $(T, 0)$, $T > 0$;

ii) $D^+V^{\triangle}(t,u,\eta) \geq g(t, V(t,u), \eta) \geq 0,$ *for* $(t,u,\eta) \in G \times \mathbb{R}$, *where* $u = u(t) = y(t,\tau_0,y_0) - x(t-\eta,t_0,x_0)$, $\eta = \tau_0 - t_0 > 0$, $g \in \mathbf{C}_{rd}[\mathbb{T} \times \mathbb{R}^2, \mathbb{R}_+]$;

iii) $g(t,w,\eta)\mu(t)$ *is nondecreasing in* $w \in \mathbb{R}$ *for each* $\eta \in \mathbb{R}$ *and* $t \in \mathbb{T}$;

iv) For $\tau_0 > T$, *the solutions* $w(t,\tau_0,w_0,\eta)$ *of the comparison equation, for arbitrarily small* $w_0 > 0$, *are either unbounded or indeterminate, for* $t \geq \tau_0$.

Then the solutions of the perturbed dynamic system are unstable with respect to the unperturbed dynamic system with ITD.

Proof Let $x(t,t_0,x_0)$ be any solution of the unperturbed dynamic system. Choose a point $(\tau_0, y_0 - x_0)$ in the vicinity of $(T,0)$. Consider the solution $y(t,\tau_0,y_0)$ of the perturbed system. Then the Lipschitzian nature of $V(t,u)$ and condition (ii) yield

$$V(t, y(t,\tau_0,y_0) - x(t-\eta,t_0,x_0)) \geq V(\tau_0, y_0 - x_0) = w_0 > 0 \tag{9}$$

for all $t \geq 0$, for which $(t, y(t,\tau_0,y_0) - x(t-\eta,t_0,x_0)) \in G$. Since $V(t,u) = 0$ for all $(t,u) \in \bar{G} - G$, it follows from (9) that $(t, y(t,\tau_0,y_0) - x(t-\eta,t_0,x_0)) \in G$ for $t \geq \tau_0$. Moreover, we also have

$$D^+V^{\triangle}(t, y(t) - x(t-\eta)) \geq g(t, V(t, y(t) - x(t-\eta)), \eta),$$

which, in view of Remark 3.1, implies that

$$V(t, y(t) - x(t-\eta)) \geq \rho(t,\tau_0,w_0,\eta) \quad t \geq \tau_0 \tag{10}$$

where $\rho(t,\tau_0,w_0,\eta)$ is the minimal solution of the comparison equation. Since $V(t,u)$ is bounded by assumption, estimate (10) leads to an absurdity, if we assume that the solutions of the perturbed dynamic system are stable with respect to the unperturbed system with ITD. This proves the theorem. □

4. Application

In this section we give an example that illustrates the main results.

Example 4.1 *Consider the following dynamic equation:*

$$x^{\triangle} = -x \qquad x(t_0) = x_0.$$

Further, consider the related perturbed equation with different initial conditions:

$$y^{\triangle} = -y + \exp(-t) \qquad y(\tau_0) = y_0.$$

We claim that this example satisfies the conditions of Theorem 3.3 and Theorem 3.4.

Proof Choose $V(x) = x^2$ and $b(r) = \frac{r^2}{2}$. Then

$$\begin{aligned}\widetilde{f}(t,u(t);\eta) &= f(t,u(t)+x(t-\eta)) + R(t,u(t)+x(t-\eta)) - f(t-\eta,x(t-\eta)) \\ &= -u(t) - x(t-\eta) + \exp(-t) - (-x(t-\eta)) \\ &= -u(t) + \exp(-t)\end{aligned}$$

and

$$\begin{aligned}\dot{V}(t,u,\eta) &= 2u\widetilde{f}(t,u(t);\eta) + \mu(t)\widetilde{f}(t,u(t);\eta)^2 \\ &= 2u(-u+\exp(-t)) + \mu(t)(-u+\exp(-t))^2 \\ &= -2u^2 + 2u\exp(-t) + \mu(t)(u^2 - 2u\exp(-t) + \exp(-2t)) \\ &= (\mu(t)-2)u^2 + 2u(\exp(-t) - \mu(t)\exp(-t)) + \mu(t)\exp(-2t) \\ &\leq (\mu(t)-1)u^2 + (\exp(-t) - \mu(t)\exp(-t))^2 + \mu(t)\exp(-2t),\end{aligned}$$

where we have made use of Young's inequality. Thus, we obtain the comparison equation as follows:

$$w^{\triangle} = g(t,w,\eta), \qquad w(\tau_0) = w_0,$$

where $g(t,w,\eta) = (\mu(t)-1)w + (\exp(-t) - \mu(t)\exp(-t))^2 + \mu(t)\exp(-2t)$.

Case 1: Let $\mathbb{T} = \mathbb{R}$. Then the comparison equation reduces to

$$w' = -w + \exp(-2t), \qquad w(\tau_0) = w_0,$$

which has solution $w(t,\tau_0,w_0) = w_0\exp(\tau_0 - t) + \exp(-\tau_0 - t) - \exp(-2t)$. Thus, by Theorem 3.4, solutions of perturbed dynamic equations are asymptotically stable with respect to the original unperturbed dynamic equation.

Case 2: Let $\mathbb{T} = \mathbb{N}$. Then the comparison equation reduces to

$$\begin{aligned}\triangle w &= \exp(-2n), \qquad w(\tau_0) = w_0 \\ w(n+1) &= w(n) + \exp(-2n), \qquad w(\tau_0) = w_0,\end{aligned}$$

which has solution $w(n,\tau_0,w_0) = w_0 + \sum_{k=\tau_0}^{n-1}\exp(-2n)$. Thus, by Theorem 3.3, solutions of perturbed dynamic equations are stable with respect to the original unperturbed dynamic equation.

□

5. Concluding remarks

In this paper, we develop a new approach to determine behavior of solutions of a perturbed dynamic system relative to an original unperturbed dynamic system, which have different initial times on arbitrary time scales. We give some stability properties.

It is obvious that the notions introduced here can be extended to also include all of the various refinements of the stability properties, such as uniform stability with ITD, uniform asymptotic stability with ITD, and so forth.

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