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# On $p$-schemes with the same degrees of thin radical and thin residue

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**Abstract:** Let $p$ and $n > 1$ be a prime number and an integer, respectively. In this paper, first we show that any $p$-scheme whose thin radical and thin residue are equal is isomorphic to a fission of the wreath product of 2 thin schemes. In addition, we characterize association $p$-schemes whose thin radical and thin residue each have degree equal to $p$. We also characterize association $p$-schemes on $p^n$ points whose thin radical and thin residue each have degree equal to $p^{n-1}$, and whose basis relations each have valency 1 or $p^{n-1}$. Moreover, we show that such schemes are Schurian.

**Key words:** Association scheme, $p$-scheme, thin radical, thin residue

## 1. Introduction

Association schemes are related to a variety of combinatorial objects (codes, designs, graphs, etc.). In [6], schemes are presented as a natural generalization of permutation groups. In this direction, $p$-schemes correspond to $p$-groups, where $p$ is a prime number. The concept of $p$-schemes was given in [6] as follows: a scheme $\mathcal{C}$ is called *$p$-scheme* if the cardinality of each basis relation of $\mathcal{C}$ is a power of $p$. Recently some algebraic and combinatorial properties of $p$-schemes were studied in [2, 3, 6, 7, 8].

In this paper we deal with association schemes and refer to them as schemes. Given a scheme $\mathcal{C}$, one can define its *thin radical* $\mathcal{O}_\vartheta(\mathcal{C})$ and *thin residue* $\mathcal{O}^\vartheta(\mathcal{C})$. Suppose that $\mathcal{C}$ is a $p$-scheme; this implies that the thin radical $\mathcal{O}_\vartheta(\mathcal{C})$ of $\mathcal{C}$ is a nontrivial $p$-group [6, Theorem 2.2]. Thus, the degree of its thin radical is a power of $p$. All $p$-schemes of degree $p$ are thin, and they are unique up to isomorphism; we denote this unique $p$-scheme by $T_p$. Moreover, the number of isomorphism classes of $p$-schemes of degree $p^2$ is 3, which are $T_{p^2}, T_p \otimes T_p$, and $T_p \wr T_p$. Thus, any $p$-scheme on $V$ with $|V| \in \{1, p, p^2\}$ is Schurian [2, p. 2]. In [1], non-Schurian $p$-schemes of degree $p^3$ were constructed. In [2], $p$-schemes with $|V| = p^3$ and thin residue of degree $p^2$ were studied. In [7], it was shown that any $p$-scheme on $V$ is isomorphic to a fission of the wreath product of $n$ copies of $T_p$, where $|V| = p^n$.

Our main results show that any $p$-scheme whose thin radical and thin residue are equal is isomorphic to a fission of the wreath product of 2 thin schemes. Moreover, we provide a characterization of $p$-schemes whose degrees of thin radical and thin residue are equal, in terms of the wreath product of thin schemes. The following theorems are the main results of this paper.

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**Theorem 1** *Let $\mathcal{C}$ be a scheme whose thin radical and thin residue are equal. $\mathcal{C}$ is then isomorphic to a fission of the wreath product of 2 thin schemes.*

**Theorem 2** *Let $\mathcal{C}$ be a $p$-scheme of degree $p^n$. The degrees of the thin radical and the thin residue of $\mathcal{C}$ are then equal to $p$ if and only if $\mathcal{C}$ is isomorphic to the wreath product of $T_p$ and a thin scheme of degree $p^{n-1}$.*

**Theorem 3** *Let $\mathcal{C}$ be a $p$-scheme of degree $p^n$. The degrees of the thin radical and the thin residue of $\mathcal{C}$ are then equal to $p^{n-1}$ and the valencies of its basis relations are 1 or $p^{n-1}$ if and only if $\mathcal{C}$ is isomorphic to the wreath product of a thin scheme of degree $p^{n-1}$ and $T_p$.*

This paper is organized as follows. In Section 2, we present some notations and terminology on association schemes and $p$-schemes. In Section 3, we prove our main theorems.

## 2. Preliminaries

In this section, we prepare some notations and results for association schemes and $p$-schemes that will be used throughout the paper. We refer the reader to [5, 7, 10, 11] for more details about association schemes and $p$-schemes.

### 2.1. Association schemes

Let $V$ be a nonempty finite set. Let $\mathcal{R} = \{R_0, R_1, \ldots, R_d\}$ be a set of nonempty binary relations on $V$ that partitions $V \times V$. The pair $\mathcal{C} = (V, \mathcal{R})$ is called an *association scheme* (or shortly a *scheme*) if it satisfies the following conditions:

1) $\Delta(V) = \{(v, v) \mid v \in V\} = R_0$.

2) For each $R_i \in \mathcal{R}$, $R_i^{\mathrm{t}} := \{(u, v) \mid (v, u) \in R_i\} \in \mathcal{R}$. We denote $R_i^{\mathrm{t}}$ by $R_{i'}$.

3) For all $R_i, R_j, R_k \in \mathcal{R}$ there exists an *intersection number* $p_{ij}^k$ such that $p_{ij}^k = |R_i(u) \cap R_{j'}(v)|$ for all $(u, v) \in R_k$, where $R(u) := \{v \in V \mid (u, v) \in R\}$.

The elements of $V$ and $\mathcal{R}$ are called *points* and *basis relations* of $\mathcal{C}$, respectively.

The numbers $|V|$ and $|\mathcal{R}|$ are called the *degree* and the *rank* of $\mathcal{C}$ and are denoted by $\deg(\mathcal{C})$ and $\mathrm{rk}(\mathcal{C})$, respectively. We define the *valency* of $R_i$ as $d(R_i) = p_{ii'}^0$. We can described each $R_i$ by its $\{0,1\}$-*adjacency matrix* $A_i$ defined by $(A_i)_{uv} = 1$ if $(u, v) \in R_i$, and 0 otherwise. We denote $\mathcal{R}^{\cup}$ as the set of all unions of the elements of $\mathcal{R}$.

Let $G$ be a transitive permutation group acting on a set $V$; then $G$ acts on $V \times V$ by the componentwise action. An orbit of this action is called an *orbital.* The set of orbitals of $G$ is denoted by $\mathrm{Orb}(G)$. It is well known that $\mathrm{Orb}(G)$ forms an association scheme on $V$, denoted by $\mathrm{Inv}(G)$. A given scheme $\mathcal{C}$ is said to be *Schurian* if $\mathcal{C} = \mathrm{Inv}(G)$ for some permutation group $G$.

For a given scheme $\mathcal{C} = (V, \mathcal{R})$, an *equivalence* of $\mathcal{C}$ is an equivalence relation $E$ on $V$ such that $E$ is a union of some basis relations of $\mathcal{C}$. Denote by $\mathcal{E}(\mathcal{C})$ the set of all equivalences of $\mathcal{C}$. For each $E \in \mathcal{E}(\mathcal{C})$ denote by $d(E)$ the degree of $E$, which is defined as the sum of the valencies of all basis relations of $\mathcal{C}$ that lie in $E$.

Let $E$ be an equivalence of the scheme $\mathcal{C}$. Denote by $V/E$ the set of equivalence classes modulo $E$. For any $X,Y \in V/E$ and $R \in \mathcal{R}$ define

$$R_{X,Y} = R \cap (X \times Y), \quad R_X = R_{X,X}.$$

Moreover, we define

$$\mathcal{R}_X = \{R_X | R_X \neq \emptyset\},$$

$$R_{V/E} := \{(X,Y) \in (V/E) \times (V/E) \mid R_{X,Y} \neq \emptyset\},$$

$$\mathcal{R}_{X/E} := \{R_{V/E} \mid R \in \mathcal{R}\}.$$

It is well known that

$$\mathcal{C}_{V/E} = (V/E, \mathcal{R}_{V/E})$$

is a scheme and is called the *factor scheme* of $\mathcal{C}$ modulo $E$. It is also clear that $\mathcal{C}_X = (X, \mathcal{R}_X)$ is a scheme and is called the *restriction of* $\mathcal{C}$ with respect to $X$.

Let $\mathcal{C} = (V, \mathcal{R})$ be an association scheme, and $E \in \mathcal{E}(\mathcal{C})$. According to [9, Definition 2.2] we make an order on the elements of $V$ as follows: consider an ordering on the elements of $V/E$ and also suppose that for each $X \in V/E$ we have an ordering on $X$. First, we order the classes of the equivalence $E$ on the elements of $V$ using the ordering of $V/E$. Then, in each class $X \in V/E$ and for any $u, v \in V$ such that $u, v \in X$, we order $u$ and $v$ exactly in the same way as in $X$; in this case, we say that the elements of $V$ are ordered according to the equivalence $E$.

Let $\mathcal{C} = (V, \mathcal{R})$ be a scheme. The set

$$\mathcal{O}_\vartheta(\mathcal{C}) = \{R \in \mathcal{R} : d(R) = 1\}$$

is called the *thin radical* of $\mathcal{C}$. We say that $\mathcal{C}$ is a *thin scheme* if $\mathcal{O}_\vartheta(\mathcal{C}) = \mathcal{R}$. It is well known that any thin scheme is Schurian.

The *thin residue* of $\mathcal{C}$ is the smallest equivalence of $\mathcal{C}$ containing basis relations $R_k$ such that $p^k_{ii'} \neq 0$ for some $R_i \in \mathcal{R}$, and denoted by $\mathcal{O}^\vartheta(\mathcal{C})$. From [11, Lemma 4.2.7], we know that the thin residue of $\mathcal{C}$ is the uniquely defined smallest equivalence of $\mathcal{C}$ having a thin quotient scheme.

If there exists a bijection between the point sets of 2 schemes that induces a bijection between their sets of basis relations, then these 2 schemes are *isomorphic*.

For 2 schemes $\mathcal{C} = (V, \mathcal{R})$ and $\mathcal{C}' = (V, \mathcal{R}')$, we define $\mathcal{C} \leq \mathcal{C}'$ if and only if $\mathcal{R}^\cup \subseteq (\mathcal{R}')^\cup$. Then $\mathcal{C}$ is a fusion of $\mathcal{C}'$ and $\mathcal{C}'$ is a fission of $\mathcal{C}$.

Given 2 schemes $\mathcal{C}_1 = (V_1, \mathcal{R}_1)$ and $\mathcal{C}_2 = (V_2, \mathcal{R}_2)$, we put

$$\mathcal{R}_1 \wr \mathcal{R}_2 = \{\Delta(V_2) \otimes R : \ R \in \mathcal{R}_1\} \cup \{S \otimes V_1 \times V_1 : \ S \in \mathcal{R}_2 \setminus \{\Delta(V_2)\}\}.$$

Define the *wreath product* of $\mathcal{C}_1$ and $\mathcal{C}_2$, denoted $\mathcal{C}_1 \wr \mathcal{C}_2$, as the scheme on $V_1 \times V_2$ with the set of basis relations $\mathcal{R}_1 \wr \mathcal{R}_2$. Moreover, by considering $A_0, A_1, \ldots, A_d$ and $B_0, B_1, \ldots, B_e$ as the adjacency matrices of basis relations of $\mathcal{R}_1$ and $\mathcal{R}_2$, respectively, the elements of $V_1 \times V_2$ can be ordered such that the adjacency matrices of $\mathcal{C}_1 \wr \mathcal{C}_2$ are given by

$$C_0 = B_0 \otimes A_0, C_1 = B_0 \otimes A_1, \ldots, C_d = B_0 \otimes A_d,$$

$$C_{d+1} = B_1 \otimes J_{|V_1|}, \ldots, C_{d+e} = B_e \otimes J_{|V_1|},$$

where $J_n$ is the $n \times n$ matrix whose entries are all equal to 1. It is well known that $\mathcal{C}_1 \wr \mathcal{C}_2$ is Schurian if and only if $\mathcal{C}_1$ and $\mathcal{C}_2$ are Schurian. Clearly, the degree of the wreath product of 2 schemes is equal to the product of their degrees.

A scheme $\mathcal{C}$ is called a *$p$-scheme* if the cardinality of each basis relation of $\mathcal{C}$ is a power of $p$, where $p$ is a prime number. This implies that the degree of any $p$-scheme is a power of $p$, and so $\mathcal{O}_\vartheta(\mathcal{C})$ is a nontrivial $p$-group with respect to products of basis relations. Therefore, the order of the thin radical of $\mathcal{C}$ is a power of $p$. In [8], it was shown that the class of $p$-schemes is closed with respect to taking quotients and restrictions.

## 3. Main theorems

In this section, we study the relationship between the degree of the thin residue of a scheme and the valency of its basis relations. Then we prove Theorems 1, 2, and 3.

**Lemma 1** *Let $\mathcal{C}$ be an association scheme. Then, for each basis relation $R$ of the scheme $\mathcal{C}$, we have $d(R) \leq d(\mathcal{O}^\vartheta(\mathcal{C}))$.*

**Proof** Let $\mathcal{C} = (V, \mathcal{R})$ and $R \in \mathcal{R}$. From [11, Lemma 4.2.7], the scheme $\mathcal{C}_{V/\mathcal{O}^\vartheta(\mathcal{C})}$ is a thin scheme. Thus, $d(R_{V/\mathcal{O}^\vartheta(\mathcal{C})}) = 1$. It follows that for each $X \in V/\mathcal{O}^\vartheta(\mathcal{C})$ there exists exactly one block $Y \in V/\mathcal{O}^\vartheta(\mathcal{C})$, such that $R \cap (X \times Y) \neq \emptyset$. It follows that

$$d(R) \leq |Y| = d(\mathcal{O}^\vartheta(\mathcal{C})),$$

as desired. □

**Lemma 2** *Let $\mathcal{C}$ be a scheme that is isomorphic to the wreath product of 2 nontrivial thin schemes, $\mathcal{C}_1$ and $\mathcal{C}_2$. We then have $\mathcal{O}^\vartheta(\mathcal{C}) = \mathcal{O}_\vartheta(\mathcal{C})$. Moreover,*

$$d(\mathcal{O}^\vartheta(\mathcal{C})) = d(\mathcal{O}_\vartheta(\mathcal{C})) = \deg(\mathcal{C}_1).$$

**Proof** Let $\mathcal{C}_1 = (V_1, \mathcal{R}_1)$ and $\mathcal{C}_2 = (V_2, \mathcal{R}_2)$ be 2 nontrivial thin schemes. Let $\mathcal{C}$ be the wreath product of $\mathcal{C}_1$ and $\mathcal{C}_2$. The basis relations of $\mathcal{C}$ are then

$$\mathcal{R} := \{\Delta(V_2) \otimes R : \ R \in \mathcal{R}_1\} \cup \{S \otimes V_1 \times V_1 : \ S \in \mathcal{R}_2 \setminus \{\Delta(V_2)\}\}.$$

Let $R \in \mathcal{R}_1$. Then

$$d(\Delta(V_2) \otimes R) = d(R). \tag{1}$$

On the other hand, for each $S \in \mathcal{R}_2 \setminus \{\Delta(V_2)\}$, we have

$$d(S \otimes (V_1 \times V_1)) = \deg(\mathcal{C}_1) d(S). \tag{2}$$

Now, since $\mathcal{C}_1$ is a nontrivial thin scheme, all basis relations of $\mathcal{C}_1$ are of degree 1, and we also have $\deg(\mathcal{C}_1) > 1$. Thus, from (1) and (2) we conclude that

$$\mathcal{O}_\vartheta(\mathcal{C}) = \{\Delta(V_2) \otimes R : \ R \in \mathcal{R}_1\}.$$

It follows that $d(\mathcal{O}_\vartheta(\mathcal{C})) = \deg(\mathcal{C}_1)$.

To complete the proof of the lemma, it is enough to show that

$$\mathcal{O}^{\vartheta}(\mathcal{C}) = \{\Delta(V_2) \otimes R : \ R \in \mathcal{R}_1\}. \quad (3)$$

Let $R \in \mathcal{R}_1$. Then, for each $S \in \mathcal{R}_2$, we have

$$\Delta(V_2) \otimes R \in \widehat{S}^t \widehat{S},$$

where $\widehat{S} = S \otimes (V_1 \times V_1)$. From the definition of $\mathcal{O}^{\vartheta}(\mathcal{C})$ we get

$$\Delta(V_2) \otimes R \in \mathcal{O}^{\vartheta}(\mathcal{C}). \quad (4)$$

On the other hand, since $\mathcal{C}_2$ is a nontrivial scheme, we have $\deg(\mathcal{C}_2) > 1$. Let $S \in \mathcal{R}_2 \setminus \{\Delta(V_2)\}$. Then we have

$$S \otimes (V_1 \times V_1) \notin \mathcal{O}^{\vartheta}(\mathcal{C}). \quad (5)$$

Otherwise, $S \in \mathcal{O}^{\vartheta}(\mathcal{C}_2)$. Since $\mathcal{C}_2$ is a thin scheme, it follows that $\mathcal{O}^{\vartheta}(\mathcal{C}_2) = \{\Delta(V_2)\}$. Thus, $S = \{\Delta(V_2)\}$, which is a contradiction. Therefore, from (4) and (5) we get (3), as desired. □

For a given scheme $\mathcal{C}$, it is not necessary to have the equality $\mathcal{O}^{\vartheta}(\mathcal{C}) = \mathcal{O}_{\vartheta}(\mathcal{C})$ even if $d(\mathcal{O}^{\vartheta}(\mathcal{C})) = d(\mathcal{O}_{\vartheta}(\mathcal{C}))$. Considering this condition, we show that $\mathcal{C}$ is isomorphic to a fission of the wreath product of 2 thin schemes.

To prove the main theorems, we need the following remarks:

**Remark 1** *Since each thin scheme is Schurian, using the corresponding permutation group, one can choose an ordering in the set of points of this scheme.*

**Remark 2** *Let $\mathcal{C}$ be a scheme on $V$ and $E \in \mathcal{E}(\mathcal{C})$. If the scheme $\mathcal{C}_X$ is a thin scheme for $X \in V/E$, then from separability of thin schemes (see [4, Theorem 2.1] and [9, p. 1908]), for any $X, Y \in V/E$ the schemes $\mathcal{C}_X$ and $\mathcal{C}_Y$ are isomorphic.*

**Remark 3** *Given a bijection $\varphi$ from a set $V$ to a set $W$, and a binary relation $R$ on the set $V$, we will let $R^{\varphi}$ denote the induced binary relation on the set $W$.*

**Proof of Theorem 1**

Let $\mathcal{C}$ be a scheme on $V$ with the set of basis relations $\mathcal{R}$. By assumption, suppose that $\mathcal{O}^{\vartheta}(\mathcal{C}) = \mathcal{O}_{\vartheta}(\mathcal{C})$. If the thin radical and the thin residue are both trivial, then the scheme itself must be trivial, and the proof is complete. Thus, we assume that $|\mathcal{O}_{\vartheta}(\mathcal{C})| > 1$.

Clearly, $\mathcal{O}^{\vartheta}(\mathcal{C})$ is an equivalence of $\mathcal{C}$; thus, for each $X \in V/\mathcal{O}^{\vartheta}(\mathcal{C})$, the scheme $\mathcal{C}_X$ is a thin scheme of degree $|X|$. Moreover, the scheme $\mathcal{C}_{V/\mathcal{O}^{\vartheta}(\mathcal{C})}$ is a thin scheme of degree $|V|/|X|$. Let $V/\mathcal{O}^{\vartheta}(\mathcal{C}) = \{X_1, \ldots, X_{|V|/|X|}\}$ be the set of points of the scheme $\mathcal{C}_{V/\mathcal{O}^{\vartheta}(\mathcal{C})}$. Thus, by Remark 1, we have an ordering on $V/\mathcal{O}^{\vartheta}(\mathcal{C})$; also, for each $X_i \in V/\mathcal{O}^{\vartheta}(\mathcal{C})$, $1 \le i \le |V|/|X|$, we have an ordering on $X_i$. On the other hand, by Remark 2, for $X_i, X_j \in V/\mathcal{O}^{\vartheta}(\mathcal{C})$, $1 \le i, j \le |V|/|X|$, the schemes $\mathcal{C}_{X_i}$ and $\mathcal{C}_{X_j}$ are isomorphic, and so there is a bijection between the elements of $X_i$ and $X_j$. Therefore, we can assume the corresponding order of $X_1$ for each $X_i$, $1 \le i \le |V|/|X|$.

Now, suppose that the elements of the set $V$ are ordered according to the equivalence $\mathcal{O}^\vartheta(\mathcal{C})$. Thus, we have a bijection, $\varphi$, from $V$ to the Cartesian product of the sets $X_1$ and $V/\mathcal{O}^\vartheta(\mathcal{C})$.

We claim that $\mathcal{C}$ is isomorphic to a fission of the wreath product of $\mathcal{C}_{X_1}$ and $\mathcal{C}_{V/\mathcal{O}^\vartheta(\mathcal{C})}$. Suppose that $\mathcal{C}_{X_1} \wr \mathcal{C}_{V/\mathcal{O}^\vartheta(\mathcal{C})} = (X_1 \times V/\mathcal{O}^\vartheta(\mathcal{C}), \mathcal{S})$. It is sufficient to show that $\varphi$ induces a bijection such that each element of $\mathcal{S}$ corresponds to a union of some elements of $\mathcal{R}$.

Let $S \in \mathcal{S}$ and $d(S) = 1$. Then, since $|X_1| > 1$, it follows from the definition of the wreath product that $S \cap (X_1 \times X_1) \neq \emptyset$. Thus, there is a relation $R \in \mathcal{R}$ such that

$$S = R^\varphi. \tag{6}$$

Now let $S \in \mathcal{S}$ and $d(S) \neq 1$. We show that $S$ corresponds to a union of some basis relations of $\mathcal{R}$. Let $\sim$ be the equivalence relation on $\mathcal{R}$ induced by the canonical map to $\mathcal{R}_{V/\mathcal{O}^\vartheta(\mathcal{C})}$. For each $R \in \mathcal{R}$, define $\hat{R} := R_{V/\mathcal{O}^\vartheta(\mathcal{C})} \in \mathcal{R}_{V/\mathcal{O}^\vartheta(\mathcal{C})}$. Then $\hat{R}$ corresponds to a permutation $g_{\hat{R}}$ on the set $V/\mathcal{O}^\vartheta(\mathcal{C})$ as follows:

$$X_i^{g_{\hat{R}}} = X_j \Longleftrightarrow R_{X_i, X_j} \neq \emptyset.$$

Thus, for each $R, T \in \mathcal{R}$, we have $R \sim T$ if and only if $\hat{R} = \hat{T}$. Let $[R]$ be the equivalence class of $R$. Then $T \in [R]$ if and only if $g_{\hat{T}} = g_{\hat{R}}$. Suppose that $[R_0]$ contains the diagonal relation. Let $X_i, X_j \in V/\mathcal{O}^\vartheta(\mathcal{C})$, considering $[R_k]$ such that $R_k \cap (X_i \times X_j) \neq \emptyset$. For each $(x, y) \in X_i \times X_j$, there exists a basis relation $R \in \mathcal{R}$ such that $(x, y) \in R$, so $X_i^{g_{\hat{R}}} = X_j$. Therefore, $R \in [R_k]$. This shows that $X_i \times X_j \subseteq \bigcup_{R \in [R_k]} R^\varphi$. It then follows that the union of relations $R$ such that $R \in [R_k]$ corresponds to the union of $X_i \times X_j$ such that $X_i^{g_{\hat{R}}} = X_j$. This implies that

$$\bigcup_{R \in [R_k]} R^\varphi = P_{g_{\hat{R_k}}} \otimes J_{|X_1|}, \tag{7}$$

where $P_{g_{\hat{R_k}}}$ is a permutation matrix corresponding to $g_{\hat{R}_k}$.

From Eq. (7) we conclude that for each $S \in \mathcal{S}$ there exists an equivalence class $[R_k]$ such that

$$S = \bigcup_{R \in [R_k]} R^\varphi. \tag{8}$$

This completes the proof of the theorem. □

**Proof of Theorem 2.**

We first prove the necessity condition of the theorem. Let $\mathcal{C} = (V, \mathcal{R})$ be a $p$-scheme and $|V| = p^n$, such that

$$d(\mathcal{O}^\vartheta(\mathcal{C})) = d(\mathcal{O}_\vartheta(\mathcal{C})) = p. \tag{9}$$

By Lemma 1, for each basis relation $R \in \mathcal{R}$ we have $d(R) \leq d(\mathcal{O}^\vartheta(\mathcal{C})) = p$. Moreover, since $\mathcal{C}$ is a $p$-scheme, the valency of each basis relation of $\mathcal{C}$ is a power of $p$. It follows that, for each basis relation $R$ in $\mathcal{R}$, we have

$$d(R) \in \{1, p\}. \tag{10}$$

Now we claim that

$$\mathcal{O}^\vartheta(\mathcal{C}) = \mathcal{O}_\vartheta(\mathcal{C}). \tag{11}$$

Indeed, since $\Delta(V) \in \mathcal{O}^{\vartheta}(\mathcal{C})$ and $d(\mathcal{O}^{\vartheta}(\mathcal{C})) = \sum_{R\in\mathcal{O}^{\vartheta}(\mathcal{C})} d(R)$, we get

$$d(\mathcal{O}^{\vartheta}(\mathcal{C})) = 1 + \sum_{R\in\mathcal{O}^{\vartheta}(\mathcal{C})\setminus\Delta(V)} d(R).$$

From (9), we have

$$1 + \sum_{R\in\mathcal{O}^{\vartheta}(\mathcal{C})\setminus\Delta(V)} d(R) = p.$$

Now from (10) it is clear that the valency of each basis relation of $\mathcal{C}$ is equal to 1 or $p$. It follows that the basis relation $R$ belongs to $\mathcal{O}^{\vartheta}(\mathcal{C})$ if and only if $d(R) = 1$. Thus, Eq. (11) holds as claimed.

On the other hand, $|V| = p^n$ and $d(\mathcal{O}^{\vartheta}(\mathcal{C})) = p$ imply that $|V/\mathcal{O}^{\vartheta}(\mathcal{C})| = p^{n-1}$. Let $V/\mathcal{O}^{\vartheta}(\mathcal{C}) = \{X_0, X_1, \ldots, X_{p^{n-1}-1}\}$. Thus, by Remark 1, we have an ordering on $V/\mathcal{O}^{\vartheta}(\mathcal{C})$; also, for each $X_i \in V/\mathcal{O}^{\vartheta}(\mathcal{C})$, $0 \le i \le p^{n-1}-1$, we have an ordering on $X_i$. On the other hand, by Remark 2, for $X_i, X_j \in V/\mathcal{O}^{\vartheta}(\mathcal{C})$, $0 \le i, j \le p^{n-1}-1$, the schemes $\mathcal{C}_{X_i}$ and $\mathcal{C}_{X_j}$ are isomorphic, and so there is a bijection between the elements of $X_i$ and $X_j$. Therefore, we can assume the corresponding order of $X_0$ for each $X_i$, $0 \le i \le p^{n-1}-1$. Thus, we may assume that the elements of $V$ are ordered according to the equivalence $\mathcal{O}^{\vartheta}(\mathcal{C})$.

Thus, we have a bijection, $\varphi$, from $V$ to the Cartesian product of the sets $X_0$ and $V/\mathcal{O}^{\vartheta}(\mathcal{C})$. Now we claim that $\mathcal{C}$ is isomorphic to the wreath product of $\mathcal{C}_{X_0}$ and $\mathcal{C}_{V/\mathcal{O}^{\vartheta}(\mathcal{C})}$.

Suppose that $R \in \mathcal{O}^{\vartheta}(\mathcal{C})$. From (11), the scheme $\mathcal{C}_{X_i}$ is a thin scheme of degree $p$ for each $0 \le i \le p^{n-1}-1$, and so it is isomorphic to $T_p$. Let $A_R$ and $A_{R_{X_0}}$ be the adjacency matrices of $R$ and $R_{X_0}$, respectively. Then we have $A_R = I_{p^{n-1}} \otimes A_{R_{X_0}}$. It follows that

$$R^{\varphi} = \Delta(V/\mathcal{O}^{\vartheta}(\mathcal{C})) \otimes R_{X_0}. \tag{12}$$

Now suppose that $R \notin \mathcal{O}^{\vartheta}(\mathcal{C})$. From (10) and (11) we conclude that $d(R) = p$. On the other hand, the scheme $\mathcal{C}_{V/\mathcal{O}^{\vartheta}(\mathcal{C})}$ is a thin scheme. Thus, we have

$$d(R_{V/\mathcal{O}^{\vartheta}(\mathcal{C})}) = 1. \tag{13}$$

Moreover, since $R \notin \mathcal{O}^{\vartheta}(\mathcal{C})$, we have

$$R_{V/\mathcal{O}^{\vartheta}(\mathcal{C})} \neq \Delta(V/\mathcal{O}^{\vartheta}(\mathcal{C})). \tag{14}$$

Let $X_i \in V/\mathcal{O}^{\vartheta}(\mathcal{C})$. From (13), there exists exactly one element $X_j \in V/\mathcal{O}^{\vartheta}(\mathcal{C})$, such that

$$R \cap (X_i \times X_j) \neq \emptyset. \tag{15}$$

It is a well-known fact that all classes of an equivalence relation of an association scheme have the same size; thus, we get $|X_i| = |X_j| = d(\mathcal{O}^{\vartheta}(\mathcal{C}))$. From (9) we have

$$|X_i| = |X_j| = p. \tag{16}$$

On the other hand, since $d(R) = p$, from (15) and (16) we conclude that

$$X_i \times X_j \subseteq R. \tag{17}$$

Moreover, from (14) we have $X_i \neq X_j$. Let $A_R$ and $\overline{A_R}$ be the adjacency matrices of $R$ and $R_{V/\mathcal{O}^\vartheta(\mathcal{C})}$, respectively. Then, from (17) and by the above ordering on $V$, we obtain $A_R = \overline{A_R} \otimes J_p$. Thus, we have

$$R^\varphi = R_{V/\mathcal{O}^\vartheta(\mathcal{C})} \otimes (X_0 \times X_0). \tag{18}$$

Therefore, from (12) and (18) we conclude that $\mathcal{C}$ is isomorphic to the wreath product of $\mathcal{C}_{X_0}$ and $\mathcal{C}_{V/\mathcal{O}^\vartheta(\mathcal{C})}$. On the other hand, $\mathcal{C}_{X_0}$ is isomorphic to $T_p$ and the scheme $\mathcal{C}_{V/\mathcal{O}^\vartheta(\mathcal{C})}$ is a thin scheme on $p^{n-1}$ points. Thus,

$$\mathcal{C} \cong T_p \wr \mathcal{C}_{V/\mathcal{O}^\vartheta(\mathcal{C})}.$$

Conversely, let $T_p = (V_1, \mathcal{R}_1)$ and $\mathcal{C}' = (V_2, \mathcal{R}_2)$ be a thin scheme of degree $p^{n-1}$. Let $\mathcal{C}$ be the wreath product of $T_p$ and $\mathcal{C}'$. From Lemma 2, we have

$$d(\mathcal{O}^\vartheta(\mathcal{C})) = d(\mathcal{O}_\vartheta(\mathcal{C})) = \deg(T_p).$$

Since $T_p$ is a thin $p$-scheme on $p$ points, we have $\deg(T_p) = p$, and the proof is complete. □

Since each thin scheme is Schurian and the wreath product of 2 Schurian schemes is Schurian, the following corollary is a direct consequence of Theorem 2:

**Corollary 1** *Any $p$-scheme whose degrees of thin radical and thin residue are equal to $p$ is Schurian.*

**Proof of Theorem 3.**
We first assume that $\mathcal{C} = (V, \mathcal{R})$ is a $p$-scheme and $|V| = p^n$, and that

$$d(\mathcal{O}^\vartheta(\mathcal{C})) = d(\mathcal{O}_\vartheta(\mathcal{C})) = p^{n-1}, \tag{19}$$

and for each $R \in \mathcal{R}$ we have $d(R) \in \{1, p^{n-1}\}$. Since $\Delta(V) \in \mathcal{O}^\vartheta(\mathcal{C})$ and $d(\mathcal{O}^\vartheta(\mathcal{C})) = \sum_{R \in \mathcal{O}^\vartheta(\mathcal{C})} d(R)$, it is easy to check that $R \in \mathcal{O}^\vartheta(\mathcal{C})$ if and only if $d(R) = 1$. Hence,

$$\mathcal{O}^\vartheta(\mathcal{C}) = \mathcal{O}_\vartheta(\mathcal{C}). \tag{20}$$

By the same argument as the proof of Theorem 2, we have a bijection, $\varphi$, from $V$ to the Cartesian product of the sets $X_0$ and $V/\mathcal{O}^\vartheta(\mathcal{C})$, where $V/\mathcal{O}^\vartheta(\mathcal{C}) = \{X_0, X_1, \ldots, X_{p-1}\}$.

From (20), it follows that $R \notin \mathcal{O}^\vartheta(\mathcal{C})$ if and only if $d(R) = p^{n-1}$. For such a basis relation $R$, the valency of $R_{V/\mathcal{O}^\vartheta(\mathcal{C})}$ is equal to 1, because $\mathcal{C}_{V/\mathcal{O}^\vartheta(\mathcal{C})}$ is a thin scheme. Moreover, from (20) the relation $R_{V/\mathcal{O}^\vartheta(\mathcal{C})}$ is a nondiagonal basis relation. Thus, for each $X_i \in V/\mathcal{O}^\vartheta(\mathcal{C})$ there exists exactly one element $X_j \in V/\mathcal{O}^\vartheta(\mathcal{C})$, $i \neq j$, such that $R \cap (X_i \times X_j) \neq \emptyset$. Since $|X_i| = |X_j| = p^{n-1}$ and $d(R) = p^{n-1}$, we conclude that $X_i \times X_j \subseteq R$. Thus, we have

$$R^\varphi = R_{V/\mathcal{O}^\vartheta(\mathcal{C})} \otimes (X_0 \times X_0). \tag{21}$$

Now suppose $R \in \mathcal{O}^\vartheta(\mathcal{C})$. From (20), the scheme $\mathcal{C}_{X_i}$ is a thin scheme for each $0 \leq i \leq p-1$. Moreover, for each $i$ and $j$ the scheme $\mathcal{C}_{X_i}$ is isomorphic to $\mathcal{C}_{X_j}$. It follows that

$$R^\varphi = \Delta(V/\mathcal{O}^\vartheta(\mathcal{C})) \otimes R_{X_0}. \tag{22}$$

Therefore, from (21) and (22) we conclude that

$$\mathcal{C} \cong \mathcal{C}_{X_0} \wr \mathcal{C}_{V/\mathcal{O}^{\vartheta}(\mathcal{C})},$$

where $\mathcal{C}_{X_0}$ is a thin scheme on $p^{n-1}$ points and $\mathcal{C}_{V/\mathcal{O}^{\vartheta}(\mathcal{C})}$ is a thin scheme on $p$ points. Since any thin scheme of degree $p$ is uniquely isomorphic to $T_p$:

$$\mathcal{C} \cong \mathcal{C}_{X_0} \wr T_p.$$

Conversely, let $\mathcal{C}' = (V_1, \mathcal{R}_1)$ be a thin scheme of degree $p^{n-1}$ and $T_p = (V_2, \mathcal{R}_2)$. Define $\mathcal{C} = \mathcal{C}' \wr T_p$. From Lemma 2, we have

$$d(\mathcal{O}^{\vartheta}(\mathcal{C})) = d(\mathcal{O}_{\vartheta}(\mathcal{C})) = \deg(\mathcal{C}') = p^{n-1}.$$

Now let $R$ be a basis relation of $\mathcal{C}$. Then, from (1) and (2) in the proof of Lemma 2, we have $d(R) = 1$ or $d(R) = \deg(\mathcal{C}') = p^{n-1}$. This completes the proof of the theorem. □

The following corollary is a direct consequence of Theorem 3:

**Corollary 2** *Any $p$-scheme of degree $p^n$ whose degrees of thin radical and thin residue are equal to $p^{n-1}$ and the valency of each basis relation is either $1$ or $p^{n-1}$ is Schurian.*

Note that if $\mathcal{C}$ is a $p$-scheme of degree $p^n$ whose degrees of thin radical and thin residue are equal to $p^i$, $1 < i < n-1$, then $\mathcal{C}$ is not necessarily Schurian. For example, consider the scheme of degree 16, No. 173 in Hanaki's classification of association schemes (http://math.shinshu-u.ac.jp/ hanaki/as/). The degree of the thin radical and the thin residue of this 2-scheme is 4, but it is not Schurian. One can study some conditions on basis relations of such $p$-schemes to ensure that these be Schurian.

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